



# Small oscillations of the pendulum, Euler’s method, and adequality

Vladimir Kanovei · Karin U. Katz ·  
Mikhail G. Katz · Tahl Nowik

Received: 22 February 2016 / Accepted: 6 April 2016 / Published online: 5 May 2016  
© Chapman University 2016

**Abstract** Small oscillations evolved a great deal from Klein to Robinson. We propose a concept of solution of differential equation based on Euler’s method with infinitesimal mesh, with well-posedness based on a relation of adequality following Fermat and Leibniz. The result is that the period of infinitesimal oscillations is independent of their amplitude.

**Keywords** Harmonic motion · Infinitesimal · Pendulum · Small oscillations

## Contents

1 Small oscillations of a pendulum . . . . .	231
2 Vector fields, walks, and integral curves . . . . .	232
3 Adequality . . . . .	233
4 Infinitesimal oscillations . . . . .	234
5 Adjusting linear prevector field . . . . .	234
6 Conclusion . . . . .	235
References . . . . .	236

## 1 Small oscillations of a pendulum

The breakdown of infinite divisibility at quantum scales makes irrelevant the mathematical definitions of derivatives and integrals in terms of limits as  $x$  tends to zero. Rather, quotients like  $\frac{\Delta y}{\Delta x}$  need to be taken in a certain range, or level. The work [15] developed a general framework for differential geometry at level  $\lambda$ , where  $\lambda$  is an infinitesimal

---

V. Kanovei  
IPPI, Moscow, Russia

V. Kanovei  
MIIT, Moscow, Russia

K. U. Katz · M. G. Katz (✉) · T. Nowik  
Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel  
e-mail: katzmik@macs.biu.ac.il

but the formalism is a better match for a situation where infinite divisibility fails and a scale for calculations needs to be fixed accordingly. In this paper, we implement such an approach to give a first rigorous account “at level  $\lambda$ ” for small oscillations of the pendulum.

In his 1908 book *Elementary Mathematics from an Advanced Standpoint*, Felix Klein advocated the introduction of calculus into the high school curriculum. One of his arguments was based on the problem of small oscillations of the pendulum. The problem had been treated until then using a somewhat mysterious *superposition principle* involving a hypothetical circular motion of the pendulum. Klein advocated what he felt was a better approach, involving the differential equation of the pendulum; see [13], p. 187.

The classical problem of the pendulum translates into the second-order nonlinear differential equation  $\ddot{x} = -\frac{g}{\ell} \sin x$  for the variable angle  $x$  with the vertical direction, where  $g$  is the constant of gravity and  $\ell$  is the length of the (massless) rod or string. The problem of small oscillations deals with the case of small amplitude, i.e.,  $x$  is small, so that  $\sin x$  is approximately  $x$ .

Then the equation is boldly replaced by the linear one  $\ddot{x} = -\frac{g}{\ell}x$ , whose solution is harmonic motion with period  $2\pi\sqrt{\ell/g}$ .

This suggests that the period of small oscillations should be independent of their amplitude. The intuitive solution outlined above may be acceptable to a physicist, or at least to the mathematicians’ proverbial physicist. The solution Klein outlined in his book does not go beyond the physicist’s solution.

The Hartman–Grobman theorem [6,7] provides a criterion for the flow of the nonlinear system to be conjugate to that of the linearized system, under the hypothesis that the linearized matrix has no eigenvalue with vanishing real part. However, the hypothesis is not satisfied for the pendulum problem.

To give a rigorous mathematical treatment, it is tempting to exploit a hyperreal framework following [15]. Here the notion of small oscillation can be given a precise sense, namely oscillation with infinitesimal amplitude.

However, even for infinitesimal  $x$  one cannot boldly replace  $\sin x$  by  $x$ . Therefore, additional arguments are required.

The linearization of the pendulum is treated in [18] using Dieners’ “Short Shadow” Theorem; see Theorem 5.3.3 and Example 5.3.4 there. This text can be viewed as a self-contained treatment of Stroyan’s Example 5.3.4.

The traditional setting exploiting the real continuum is only able to make sense of the claim that *the period of small oscillations is independent of the amplitude* by means of a paraphrase in terms of limits. In the context of an infinitesimal-enriched continuum, such a claim can be formalized more literally; see Corollary 6.1. What enables us to make such distinctions is the richer syntax available in Robinson’s framework.

Terence Tao has recently authored a number of works exploiting ultraproducts in general, and Robinson’s infinitesimals in particular, as a fundamental tool; see e.g., [19,20]. In the present text, we apply such an approach to small oscillations.

Related techniques were exploited in [14]. See also [16].

## 2 Vector fields, walks, and integral curves

The framework developed in [17] involves a proper extension  ${}^*\mathbb{R} \supseteq \mathbb{R}$  preserving the properties of  $\mathbb{R}$  to a large extent discussed in Remark 2.2. Elements of  ${}^*\mathbb{R}$  are called hyperreal numbers. A positive hyperreal number is called *infinitesimal* if it is smaller than every positive real number.

We choose a fixed positive infinitesimal  $\lambda \in {}^*\mathbb{R}$  (a restriction on the choice of  $\lambda$  appears in Sect. 6). Given a classical vector field  $V = V(z)$  where  $z \in \mathbb{C}$ , one forms an infinitesimal displacement  $\delta_F(z) = \lambda V(z)$  with the aim of constructing the integral curves of the corresponding flow  $F_t$  in the plane. Note that a zero of  $\delta_F$  corresponds to a fixed point (i.e., a constant integral “curve”) of the flow. The infinitesimal generator is the function  $F : {}^*\mathbb{C} \rightarrow {}^*\mathbb{C}$ , also called a *prevector field*, defined by

$$F(z) = z + \delta_F(z), \tag{1}$$

where  $\delta_F(z) = \lambda V(z)$  in the case of a displacement generated by a classical vector field as above, but could be a more general internal function  $F$  as discussed in [15].

We propose a concept of solution of differential equation based on Euler’s method with infinitesimal step size, with well-posedness based on a property of adequality (see Sect. 3), as follows.

**Definition 2.1** The *hyperreal flow*, or *walk*,  $F_t(z)$  is a  $t$ -parametrized map  ${}^*\mathbb{C} \rightarrow {}^*\mathbb{C}$  defined whenever  $t$  is a hypernatural multiple  $t = N\lambda$  of  $\lambda$ , by setting

$$F_t(z) = F_{N\lambda}(z) = F^{\circ N}(z), \tag{2}$$

where  $F^{\circ N}$  is the  $N$ -fold composition.

The fact that the infinitesimal generator  $F$  given by (1) is invariant under the flow  $F_t$  of (2) receives a transparent meaning in this framework, expressed by the commutation relation  $F \circ F^{\circ N} = F^{\circ N} \circ F$  due to *transfer* (see Remark 2.2) of associativity of composition of maps.

*Remark 2.2* The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are *transferred*) to an extended number system. Thus, the familiar extension  $\mathbb{Q} \subseteq \mathbb{R}$  preserves the property of being an ordered field. To give a negative example, the extension  $\mathbb{R} \subseteq \mathbb{R} \cup \{\pm\infty\}$  of the real numbers to the so-called *extended reals* does not preserve the property of being an ordered field. The hyperreal extension  $\mathbb{R} \subseteq {}^*\mathbb{R}$  preserves *all* first-order properties, including the identity  $\sin^2 x + \cos^2 x = 1$  (valid for all hyperreal  $x$ , including infinitesimal and infinite values of  $x \in {}^*\mathbb{R}$ ). The natural numbers  $\mathbb{N} \subseteq \mathbb{R}$  are naturally extended to the hypernaturals  ${}^*\mathbb{N} \subseteq {}^*\mathbb{R}$ . For a more detailed discussion, see the textbook *Elementary Calculus* [12].

**Definition 2.3** The *real flow*  $f_t$  on  $\mathbb{C}$  for  $t \in \mathbb{R}$  when it exists is constructed as the shadow (i.e., standard part) of the hyperreal walk  $F_t$  by setting  $f_t(z) = \text{st}(F_{N\lambda}(z))$  where  $N = \lfloor \frac{t}{\lambda} \rfloor$ , while  $\lfloor x \rfloor$  rounds off the number  $x$  to the nearest hyperinteger no greater than  $x$ , and “st” (standard part or shadow) rounds off each finite hyperreal to its nearest real number.

For  $t$  sufficiently small, suitable regularity conditions ensure that the point  $F_{N\lambda}(z)$  is finite so that the shadow is defined.

The usual relation of being infinitely close is denoted  $\approx$ . Thus  $z, w$  satisfy  $z \approx w$  if and only if  $\text{st}(z - w) = 0$ . This relation is an additive one (i.e., invariant under addition of a constant).

The appropriate relation for working with small prevector fields is not additive but rather multiplicative (i.e., invariant under multiplication by a constant), as detailed in Sect. 3.

### 3 Adequality

We will use Leibniz’s notation  $\sqcap$  to denote the relation of adequality (see below). Leibniz actually used a symbol that looks more like  $\sqcap$  but the latter is commonly used to denote a product. Leibniz used the symbol to denote a generalized notion of equality “up to” (though he did not distinguish it from the usual symbol  $=$  which he also used in the same sense). A prototype of such a relation (though not the notation) appeared already in Fermat under the name *adequality*. For a re-appraisal of Fermat’s contribution to the calculus see [10]; for Leibniz’s, see [4, 11]; for Euler see [1, 2]; for Cauchy’s contribution, see [3, 5, 8, 9]. We will use the sign  $\sqcap$  for a multiplicatively invariant relation among (pre)vectors defined as follows.

**Definition 3.1** Let  $z, w \in {}^*\mathbb{C}$ . We say that  $z$  and  $w$  are *adequal* and write  $z \sqcap w$  if either  $\frac{z}{w} \approx 1$  (i.e.,  $\frac{z}{w} - 1$  is infinitesimal) or  $z = w = 0$ .

This implies in particular that the angle between  $z, w$  (when they are nonzero) is infinitesimal, but  $\sqcap$  is a stronger condition. If one of the numbers is appreciable, then so is the other and the relation  $z \sqcap w$  is equivalent to  $z \approx w$ .

If one of  $z, w$  is infinitesimal then so is the other, and the difference  $|z - w|$  is not merely infinitesimal, but so small that the quotients  $|z - w|/z$  and  $|z - w|/w$  are infinitesimal, as well.

We are interested in the behavior of orbits in a neighborhood of a fixed point  $0$ , under the assumption that the infinitesimal displacement satisfies the Lipschitz condition. In such a situation, we have the following theorem.

**Theorem 3.2** *Assume that for some finite  $K$ , we have  $\delta_F(z) - \delta_F(w) < K\lambda|z - w|$ . Then prevector fields defined by adequate infinitesimal displacements produce hyperreal walks that are adequate at each finite time, or in formulas: if  $\delta_F \sqcap \delta_G$  then  $F_t \sqcap G_t$  when  $t$  is finite.*

This was shown in [15, Example 5.12].

### 4 Infinitesimal oscillations

Let  $x$  denote the variable angle between an oscillating pendulum and the downward vertical direction. By considering the projection of the force of gravity in the direction of motion, one obtains the equation of motion  $m\ell\ddot{x} = -mg \sin x$  where  $m$  is the mass of the bob of the pendulum,  $\ell$  is the length of its massless rod, and  $g$  is the constant of gravity. Thus we have a second-order nonlinear differential equation

$$\ddot{x} = -\frac{g}{\ell} \sin x. \tag{3}$$

The initial condition of releasing the pendulum at angle  $a$  (for *amplitude*) is

$$\begin{cases} x(0) = a, \\ \dot{x}(0) = 0. \end{cases}$$

We replace (3) by the pair of first-order equations

$$\begin{cases} \dot{x} = \sqrt{\frac{g}{\ell}} y, \\ \dot{y} = -\sqrt{\frac{g}{\ell}} \sin x, \end{cases}$$

and initial condition  $(x, y) = (a, 0)$ . We identify  $(x, y)$  with  $z = x + iy$  and  $(a, 0)$  with  $a + i0$  as in Sect. 2. The classical vector field corresponding to this system is then

$$X(x, y) = \sqrt{\frac{g}{\ell}} y - i\sqrt{\frac{g}{\ell}} \sin x. \tag{4}$$

The corresponding prevector field  $F$  is defined by the infinitesimal displacement  $\delta_F(z) = \lambda\sqrt{\frac{g}{\ell}} y - i\lambda\sqrt{\frac{g}{\ell}} \sin x$  so that  $F(z) = z + \delta_F(z)$ . We are interested in the flow of  $F$ , with initial condition  $a + 0i$ , generated by hyperfinite iteration of  $F$ .

Consider also the linearization, i.e., prevector field  $E(z) = z + \delta_E(z)$  defined by the displacement

$$\delta_E(z) = \lambda\sqrt{\frac{g}{\ell}} y - i\lambda\sqrt{\frac{g}{\ell}} x = -i\lambda\sqrt{\frac{g}{\ell}} z$$

where as before  $z = x + iy$ . We are interested in small oscillations, i.e., the case of infinitesimal amplitude  $a$ . Since sine is asymptotic to the identity function for infinitesimal inputs, we have  $\delta_E \sqcap \delta_F$ . Due to the multiplicative nature of this relation, the rescalings of  $E$  and  $F$  by change of variable  $z = aZ$  remain adequate and therefore define adequate walks and identical real flows by Theorem 3.2.

### 5 Adjusting linear prevector field

We will compare  $E$  to another linear prevector field

$$\begin{aligned} H(x + iy) &= e^{-i\lambda\sqrt{\frac{g}{\ell}}}(x + iy) \\ &= \left(x \cos \lambda\sqrt{\frac{g}{\ell}} + y \sin \lambda\sqrt{\frac{g}{\ell}}\right) + \left(-x \sin \lambda\sqrt{\frac{g}{\ell}} + y \cos \lambda\sqrt{\frac{g}{\ell}}\right) i \end{aligned}$$

given by clockwise rotation of the  $x, y$  plane by infinitesimal angle  $\lambda\sqrt{\frac{g}{\ell}}$ , so that

$$\delta_H(z) = \left( e^{-i\lambda\sqrt{\frac{g}{\ell}}} - 1 \right) z.$$

The corresponding hyperreal walk, defined by hyperfinite iteration of  $H$ , satisfies the exact equality

$$H_t(a, 0) = \left( a \cos \sqrt{\frac{g}{\ell}} t, -a \sin \sqrt{\frac{g}{\ell}} t \right) \tag{5}$$

whenever  $t$  is a hypernatural multiple of  $\lambda$ . In particular, we have the periodicity property  $H_{\frac{2\pi}{\sqrt{g/\ell}}}(z) = z$  and therefore

$$H_{t+\frac{2\pi}{\sqrt{g/\ell}}} = H_t \tag{6}$$

whenever both  $t$  and  $\frac{2\pi}{\sqrt{g/\ell}}$  are hypernatural multiples of  $\lambda$ . Note that we have  $\delta_E(z) = -i\lambda\sqrt{\frac{g}{\ell}}z$  and  $\delta_H(z) = (e^{-i\lambda\sqrt{\frac{g}{\ell}}} - 1)z$ . Therefore

$$\frac{\delta_H(z)}{\delta_E(z)} = \frac{e^{-i\lambda\sqrt{\frac{g}{\ell}}} - 1}{-i\lambda\sqrt{\frac{g}{\ell}}} = \frac{1}{\lambda\sqrt{\frac{g}{\ell}}} \left( \sin \lambda\sqrt{\frac{g}{\ell}} + (\cos \lambda\sqrt{\frac{g}{\ell}} - 1)i \right)$$

The usual estimates give

$$\frac{\sin \lambda\sqrt{\frac{g}{\ell}}}{\lambda\sqrt{\frac{g}{\ell}}} \approx 1, \quad \frac{\cos \lambda\sqrt{\frac{g}{\ell}} - 1}{\lambda\sqrt{\frac{g}{\ell}}} \approx 0 \tag{7}$$

so  $\frac{\delta_H(z)}{\delta_E(z)} \approx 1 + 0i = 1$ , that is  $\delta_H(z) \sqcap \delta_E(z)$ . By Theorem 3.2 the hyperfinite walks of  $E$  and  $H$  satisfy  $E_t(a, 0) \sqcap H_t(a, 0)$  for each finite initial amplitude  $a$  and for all finite time  $t$  which is a hypernatural multiple of  $\lambda$ .

### 6 Conclusion

The advantage of the prevector field  $H$  is that its hyperreal walk is given by an explicit formula (5) and is therefore periodic with period precisely  $\frac{2\pi}{\sqrt{\frac{g}{\ell}}}$ , provided we choose our base infinitesimal  $\lambda$  in such a way that  $\frac{2\pi}{\lambda\sqrt{\frac{g}{\ell}}}$  is hypernatural.

We obtain the following consequence of (6): modulo a suitable choice of a representing prevector field (namely,  $H$ ) in the adequality class, the hyperreal walk is periodic with period  $2\pi\sqrt{\ell/g}$ . This can be summarized as follows.

**Corollary 6.1** *The period of infinitesimal oscillations of the pendulum is independent of their amplitude.*

If one rescales such an infinitesimal oscillation to appreciable size by a change of variable  $z = aZ$  where  $a$  is the amplitude, and takes standard part, one obtains a standard harmonic oscillation with period  $2\pi\sqrt{\ell/g}$ . The formulation contained in Corollary 6.1 has the advantage of involving neither rescaling nor shadow-taking.

**Acknowledgments** We are grateful to Jeremy Schiff for drawing our attention to the Hartman–Grobman theorem, and to Semen Kutateladze and Dalibor Pražák for some helpful suggestions. M. Katz was partially supported by the Israel Science Foundation Grant No. 1517/12.

## References

1. Bair, J., Błaszczyk, P., Ely, R., Henry, V., Kanovei, V., Katz, K., Katz, M., Kutateladze, S., McGaffey, T., Schaps, D., Sherry, D., Shnider, S.: Is mathematical history written by the victors? *Not. Am. Math. Soc.* **60**(7), 886–904 (2013). <http://www.ams.org/notices/201307/moti-p886.pdf>. [arXiv:1306.5973](https://arxiv.org/abs/1306.5973)
2. Bair, J., Błaszczyk, P., Ely, R., Henry, V., Kanovei, V., Katz, K., Katz, M., Kutateladze, S., McGaffey, T., Reeder, P., Schaps, D., Sherry, D., Shnider, S.: Interpreting the infinitesimal mathematics of Leibniz and Euler. *J. Gen. Philos. Sci.* (2016). doi:[10.1007/s10838-016-9334-z](https://doi.org/10.1007/s10838-016-9334-z)
3. Bascelli, T., Bottazzi, E., Herzberg, F., Kanovei, V., Katz, K., Katz, M., Nowik, T., Sherry, D., Shnider, S.: Fermat, Leibniz, Euler, and the gang: the true history of the concepts of limit and shadow. *Not. Am. Math. Soc.* **61**(8), 848–864 (2014)
4. Bascelli, T., Błaszczyk, P., Kanovei, V., Katz, K., Katz, M., Schaps, D., Sherry, D.: Leibniz vs Ishiguro: closing a quarter-century of syncategoremania. *HOPOS J. Int. Soc. Hist. Philos. Sci.* **6**(1) (2016). doi:[10.1086/685645](https://doi.org/10.1086/685645). [arXiv:1603.07209](https://arxiv.org/abs/1603.07209)
5. Borovik, A., Katz, M.: Who gave you the Cauchy–Weierstrass tale? The dual history of rigorous calculus. *Found. Sci.* **17**(3), 245–276 (2012). doi:[10.1007/s10699-011-9235-x](https://doi.org/10.1007/s10699-011-9235-x)
6. Grobman, D.: Homeomorphisms of systems of differential equations. *Doklady Akademii Nauk SSSR* **128**, 880–881 (1959)
7. Hartman, P.: A lemma in the theory of structural stability of differential equations. *Proc. Am. Math. Soc.* **11**(4), 610–620 (1960)
8. Katz, K., Katz, M.: Cauchy’s continuum. *Perspect. Sci.* **19**(4), 426–452 (2011). [arXiv:1108.4201](https://arxiv.org/abs/1108.4201). [http://www.mitpressjournals.org/doi/abs/10.1162/POSC\\_a\\_00047](http://www.mitpressjournals.org/doi/abs/10.1162/POSC_a_00047)
9. Katz, K., Katz, M.: A Burgessian critique of nominalistic tendencies in contemporary mathematics and its historiography. *Found. Sci.* **17**(1), 51–89 (2012). doi:[10.1007/s10699-011-9223-1](https://doi.org/10.1007/s10699-011-9223-1). [arXiv:1104.0375](https://arxiv.org/abs/1104.0375)
10. Katz, M., Schaps, D., Shnider, S.: Almost equal: the method of adequity from Diophantus to Fermat and beyond. *Perspect. Sci.* **21**(3), 283–324 (2013)
11. Katz, M., Sherry, D.: Leibniz’s infinitesimals: their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond. *Erkenntnis* **78**(3), 571–625 (2013). doi:[10.1007/s10670-012-9370-y](https://doi.org/10.1007/s10670-012-9370-y). [arXiv:1205.0174](https://arxiv.org/abs/1205.0174)
12. Keisler, H.J.: *Elementary Calculus: An Infinitesimal Approach*, 2d edn. Prindle, Weber & Schmidt, Boston (1986). <http://www.math.wisc.edu/~keisler/calc.html>
13. Klein, F.: *Elementary Mathematics from an Advanced Standpoint*. Vol. I. Arithmetic, Algebra, Analysis. Translation by E. R. Hedrick and C. A. Noble [Macmillan, New York, 1932] from the third German edition [Springer, Berlin, 1924]. Originally published as *Elementarmathematik vom höheren Standpunkte aus* (Leipzig, 1908)
14. Lobry, C., Sari, T.: Non-standard analysis and representation of reality. *Internat. J. Control* **81**(3), 517–534 (2008)
15. Nowik, T., Katz, M.: Differential geometry via infinitesimal displacements. *J. Logic Anal.* **7**(5), 1–44 (2015). <http://www.logicandanalysis.org/index.php/jla/article/view/237/106>. [arXiv:1405.0984](https://arxiv.org/abs/1405.0984)
16. Pražák, D., Rajagopal, K., Slavík, J.: A non-standard approach to a constrained forced oscillator. Preprint (2016)
17. Robinson, A.: *Non-Standard Analysis*. North-Holland Publishing, Amsterdam (1966)
18. Stroyan, K.: *Advanced Calculus Using Mathematica: Notebook Edition* (2015)
19. Tao, T.: *Hilbert’s Fifth Problem and Related Topics*. Graduate Studies in Mathematics, vol. 153. American Mathematical Society, Providence (2014)
20. Tao, T., Van Vu, V.: Sum-avoiding sets in groups. [arXiv:1603.03068](https://arxiv.org/abs/1603.03068) (2016)