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The development of the descriptive theory of sets under the influence of the work of Luzin

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Introduction

In the science of mathematics one cannot find too many examples of discoveries, ideas, and investigations made by a single scholar that turned out in the course of decades to have a definite influence on the establishment and development of an entire branch of mathematics. This can only be the product of enormous talent and intuition, and that is just how one speaks with full justification about the founder of the descriptive theory of sets Nikolai Nikolaevich Luzin.

Luzin came to the descriptive theory in the second decade of the 20th century, a young, already known mathematician, the author of recognized papers on the theory of functions. At that time the descriptive theory was essentially still in its infancy. In 1905 Henri Lebesgue proposed the study of "definable" point sets, that is, sets that can be determined unambiguously, without recourse to the axiom of choice. The original systematic investigations of the "definable" mathematical objects was limited to the Borel sets of a Euclidean space and to Baire functions of one or several real variables. (These sets and functions were discovered by Émile Borel and René Baire at the very end of the last century in connection with the development of mathematical analysis and function theory.) Despite some successes (which manifested themselves specifically in the construction of a transfinite classification of the Baire functions and the Borel sets and the proof by Lebesgue in [28] of the fact that this classification can be extended through the countable ordinals), for a long time the cardinality problem for Borel sets evaded solution.

This problem was posed by Luzin before the participants of the seminar on the theory of functions, which he organised until the first world war at the University of Moscow. The solution was found in 1916 by one of the participants, the great topologist P.S. Aleksandrov. Aleksandrov showed that an uncountable Borel set in a Euclidean space necessarily contains a perfect subset, and so has the cardinality of the continuum. In other words, Cantor's continuum hypothesis holds for Borel sets.

The A-operation introduced by Aleksandrov for the solution of the cardinality problem for Borel sets allowed Suslin, also a participant of the Luzin seminar, to construct a new class of "definable" sets containing all Borel sets but not exhausted by them. Suslin called these new sets "the sets (A)", but they are also known as A-sets, analytic sets (Luzin's term), Suslin sets (Hausdorff's term). In essence, it was only after the discovery of A-sets and the early research of Suslin and Luzin on A-sets (the results of these investigations were set forth in the notes [31] and [2] and in more detail in the article [24]) that the descriptive theory became an independent branch of mathematics. From that moment on and throughout more than two decades the development of the descriptive theory progressed under the conceptual leadership and the direct involvement of Luzin. Luzin's creative activity was the definitive pivot of the classical period of the development of the descriptive theory, in the course of which the object of study (Borel sets, A-sets, projective sets, and certain other types of "definable" set) was defined, specific methods of investigating these sets were created (such as sieves), deep results about them were obtained and, finally, problems were formulated that did not yield to a solution, the limits of the techniques of classical mathematics were reached, and prerequisites for the modern development of the descriptive theory were established.

Luzin's pupils Novikov and Keldysh, who compiled and edited in 1958 the second volume of Luzin's collected works ([23]), distinguished three cycles of Luzin papers on the descriptive theory. To the first main cycle they referred studies of concrete types of "definable" sets. Within this cycle one can indicate several basic trends.

Firstly, the study of the A-sets discovered by Suslin, where Luzin obtained important results such as the theorem on their measurability and their Baire property (see the end of §2 of our survey) and the separation theorem (§3).

Secondly, the search for various types of plane sets (or in other terminology: implicit functions), particularly in connection with the uniformization problem. Here Luzin and Novikov obtained a number of fundamental results (see §4), which later were further developed in works of other experts.

Thirdly, the development of the sieve operation and the decomposition into constituents—the main technical apparatus of the theory of A-sets and their complementary CA-sets. §3 of our survey is devoted to sieves and constituents.

The fourth trend we would entitle the deeper study of the structure of the classes of the Borel hierarchy, where Luzin also obtained important results such as the separation theorems. Much was also done by Luzin's students Lavrent'ev and Keldysh. We say more about this in §1.

Finally, a fifth trend, the discovery of projective sets and the construction of the projective hierarchy (see §2). The A-sets and CA-sets, which were already fairly well understood at the moment of the discovery of projective sets (1924–1925) form only the first level of the projective hierarchy. Naturally, Luzin attempted to study sets on higher levels. This was totally unsuccessful: the methods that "worked" well in the domain of A-sets and CA-sets gave nothing at all for "arbitrary" projective sets. In particular, there remained open the questions of measurability, the Baire property, and the cardinality of projective sets. However, it undoubtedly took the intuition of a great mathematician to look as Luzin did at this phenomenon not as a consequence of the inadequacy of the technical apparatus of the theory, but as the significantly deeper phenomenon of the incompleteness in principle and the insufficiency of the traditional mathematical tools and methods of argument as applied to projective sets and to predict that "it is impossible and never will be possible" to reach a solution of the problem of Lebesgue measurability, the Baire property, and the cardinality of projective sets. This prediction, which Luzin made in 1925, was confirmed by the subsequent development of the descriptive theory; a strict proof of the undecidability of these problems was obtained by Novikov (1951) and Cohen (1970) only after methods of proof for undecidability within the framework of axiomatic set theory had been worked out.

To the present day projective sets remain at the centre of attention of experts in descriptive theory. Of the main directions of current research on projective sets we shall talk in §6.

That was the first cycle of Luzin papers on the descriptive theory. Bordering on these are papers by Selivanovskii, Novikov, Kolmogorov, Lyapunov on C-sets and R-sets—a distinctive category of "definable" set lying between the first and second level of the projective hierarchy, and investigations of Novikov on the second projective level (§5).

In the 30's, when the ideas of the classical descriptive theory of sets were near exhaustion and the limits of their applicability were fairly well recognized, Luzin gradually moved away from the direct investigation of distinct types of projective sets and turned his attention to the most important problems of the descriptive theory such as the problem of the cardinality of CA-sets and the problem of the effective existence of a point set of cardinality $\aleph_1$. Work in this direction forms the second cycle of Luzin's investigations. Attempting to define the nature of the obstacle on the path to a solution of these problems and of the famous continuum problem, Luzin introduced into mathematics the fundamental concept of a collection of Borel sets of bounded class (nowadays one says: bounded in rank); he raised some problems on the construction of such collections, among them some formulated with the help of sieves and constituents. Luzin regarded problems in this series as "weakened forms" of the continuum problem; in fact they turned out to be significantly more difficult and deeper than the former and have found their solution only in papers of the last few years. We consider these problems and results connected with them in § 7.

Recently in the descriptive theory a new direction has been intensively developed, which is concerned with the study of equivalence relations; here the fundamental role of Borel sets of bounded classes becomes manifest. This topic is treated in § 8. Luzin himself did not seriously work on equivalence relations, however, one might say that he did stand at the source of this trend, because as early as 1927 he selected such main problems as the evaluation of the number of equivalence classes and the effective existence of a set that chooses from each equivalence class just one point. An adequate technical apparatus for the study of these problems was not developed until the 70's.

Finally, the third cycle of Luzin's papers in the descriptive theory is connected with the application of the axiom of choice. On the whole, Luzin was critical in the use of this axiom in mathematical reasonings on account of its extremely non-constructive character, which becomes manifest in the fact that existence proofs obtained by means of the axiom of choice do not give concrete "definable" sets with the required properties. However, he also paid a certain respect to the axiom of choice in the derivation of consequences of the existence of point sets with such unexpected properties that it would be hopeless to realize by more effective constructions. Of Luzin's work in connection with the axiom of choice and the much later response to it we talk in the concluding § 9.

Speaking of Luzin's role in the development of the descriptive theory it would be wrong not to mention that Luzin paid much attention to mathematical-philosophical problems in the foundation of mathematics and contributed a number of deep ideas on the nature of the difficulties in this area. Unfortunately, this side of Luzin's creativity has escaped the attention of our national philosophy of science. It is impossible not to recall also

Luzin's extremely important and fruitful didactic activity which is well documented in articles published in Uspekhi Mat. Nauk (1974), 29:5 (= Russian Math. Surveys 29:5 (1974), 173–208), and also in the gazette of the Kemerovsk State University "Progress in science", 7 September 1983 and in the Vestnik Akad. Nauk SSSR, 1984, No. 11.

Concluding this preface, the author believes it would be only right to point out the great influence on the presentation and selection of the material for the present article (particularly, as regards § § 1–5, which are devoted to the classical Luzin period in the descriptive theory) which the papers [49], [50], [51], [59], [61], [71], [74], [77], written by students and successors of Luzin in the descriptive theory, had and also discussions with Uspenskii and Taimanov.

The bulk of our paper is preceded by a §0, in which we are concerned with the interrelationships between the various spaces which are considered in work on the descriptive theory.

§0. Euclidean and Baire spaces

Initially the descriptive theory of sets was developed almost exclusively in the Euclidean spaces $\mathbf{R}^m$. But even towards the end of the 20's it became clear that the results, methods, and arguments specific to the descriptive theory in Euclidean spaces, go over quite easily without change (or in individual cases with changes that usually simplify matters) to sets in the Baire space $\mathcal{N}$ and the spaces $\mathcal{N}^m$, where $m \geqslant 1$.

The Baire space $\mathcal{N}$ (of countable weight) is formed by all infinite sequences $\langle a_1, a_2, a_3, \ldots\rangle$ of natural numbers a_i and thus is the topological product of countably many copies of the set of natural numbers (see [30], 155, where this space is denoted by $B_{\aleph_0}$). The space $\mathcal{N}$ is homeomorphic to the set of all irrational points on the real line $\mathbf{R}$ (or on any interval of $\mathbf{R}$) ([30], 155).

In what follows we agree to call all the $\mathcal{N}^m$, $m \geqslant 1$, *Baire spaces* (they are all homeomorphic to $\mathcal{N}$).

The development of the descriptive theory showed that the Baire spaces are more appropriate to the principal descriptive constructions than the Euclidean spaces (see, for example, [50], Remark 42). Already in Luzin's "Leçons sur les ensembles analytiques" [11] the space $\mathcal{N}$ (in the form of the set of irrational points) plays the role of the basic descriptive space (Luzin's "fundamental domain"). The account in the surveys [61] and [77], which in a certain sense summarize the classical period of the descriptive theory, is also constructed around the Baire space. Somewhat later Addison [80] showed that the Baire spaces, in contrast to the Euclidean spaces, allow the use of formulae of a special fairly simple language to describe point sets and to formalize computations by means of which one succeeds in shortening quite substantially the classical "geometric" proofs, achieving at the same time a greater clarity of the substance of the matter.

Of course, the Baire spaces lose out to the Euclidean spaces from the point of view of geometric intuitivity, however, this feature is not essential because one can identify the points of $\mathcal{N}$ with the irrational points of the real line. Thanks to this, the structure of the descriptive theory in Euclidean and Baire spaces leads to essentially identical theorems (except in individual cases, say, closedness corresponding to compactness, continuity to countable discontinuity, etc).

The exposition in the present survey is arranged so that it covers both the Baire and the Euclidean spaces, and we point out differences in definitions and results for the two types of space in those few cases where they occur. As regards the system of references, following a tradition in the descriptive theory going back to Luzin's times, we refer to publications in which theorems under discussion were first proved (or stated), irrespectively of what spaces (that is, Euclidean or Baire) were in fact considered in these original papers.

§1. Research on the structure of Borel classes

Uspenskii's article in the current issue of the Uspekhi Mat. Nauk makes it unnecessary to dwell here on the history of the discovery of Borel sets and the important investigations of Borel, Baire, and Lebesgue. We begin straightaway with an account of the modern concept of a Borel set.

The *Borel sets* (or *B*-sets) in a given space form the smallest class of sets in that space that contains all open sets and is closed under the operations of complementation, countable union, and countable intersection. The Borel sets can be organized into *Borel classes*, which form the *Borel hierarchy*. At present, in papers on the descriptive theory it is customary to adopt the following construction of the Borel hierarchy.

The Borel classes are denoted by $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, where $1 \leqslant \xi < \omega_1$, and ω_1 is the first uncountable transfinite ordinal. The classes are constructed by induction on ξ. To the first class $\mathbf{\Sigma}^0_1$ there belong all the open sets of the given space. If the class $\mathbf{\Sigma}^0_\xi$ has already been constructed, then to the class $\mathbf{\Pi}^0_\xi$ there belong all sets that are complements to sets in $\mathbf{\Sigma}^0_\xi$, and to $\mathbf{\Delta}^0_\xi$ all sets that themselves as well as their complements belong to $\mathbf{\Sigma}^0_\xi$, that is, $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$. Finally, if $\xi \geqslant 2$, the class $\mathbf{\Sigma}^0_\xi$ is formed by all countable unions of sets belonging to the classes $\mathbf{\Pi}^0_\eta$, where $1 \leqslant \eta < \xi$.

Clearly, the classes $\mathbf{\Pi}^0_1$ and $\mathbf{\Delta}^0_1$ consist, respectively, of the closed and open-and-closed sets, while the classes $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ are the same as the classes F_σ and G_δ.

Every Borel set belongs to one of the classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, or $\mathbf{\Delta}^0_\xi$. When this is so, let us note, it also belongs to every Borel class with an index greater than ξ, since the Borel classes satisfy the condition of increase

$$\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subsetneqq \mathbf{\Delta}^0_\zeta \quad \text{for} \quad 1 \leqslant \xi < \zeta < \omega_1.$$

It is known that the inclusion here is strict, so that none of the Borel classes exhausts the totality of all Borel sets.

In papers on the descriptive theory in the 20's to the 40's it was customary to consider another system of classification of Borel sets, which was introduced by de la Vallé-Poussin and worked out in detail by Luzin. Without dwelling on the construction of the de la Vallée-Poussin-Luzin hierarchy (see [77]), we remark only that every class K_ξ of this hierarchy coincides with the class $\mathbf{\Delta}^0_{\xi+1}$, and that sets to be called elements of the class K_ξ (they were often considered in classical studies) are identical with the $\mathbf{\Pi}^0_\xi$-sets ([77], §10).

In the theory of Borel sets one can distinguish two trends. Firstly, a deep study of the classification, and secondly, a study of Borel sets in relation to certain other notions of the descriptive theory such as projective sets and sieves. In this section we limit ourselves to a survey of the achievements in the first direction, leaving the other one to §§3 and 4, where we consider these important concepts and where the corresponding theorems about Borel sets connected with them turn out to be better placed.

1.1. Theorems of separation and reduction.
Let X and Y be a pair of disjoint sets. If some third set U contains all the points of X and has no points in common with Y, then we say that the set U *separates* X from Y. The concept of separation was introduced in descriptive theory by Luzin in [7].

Usually in connection with separation one considers the following main problem: to clarify which of the following three theorems (which are also called principles) are satisfied in a given class K of point sets (for example, the Borel class $\mathbf{\Pi}^0_{17}$).

First separation theorem. *Any two disjoint sets of the class K can be separated from one another by a set that belongs to K itself as well as its complement.*

Second separation theorem. *If from two arbitrary sets X and Y of the class K we remove their common part, then the resulting remainder sets $X-Y$ and $Y-X$ can be included in disjoint sets that are complementary to sets of the class K.*

Inseparability theorem. *There exists a pair of disjoint sets of the class K that cannot be included in pairwise disjoint sets complementary to sets of the class K.*

Luzin's investigation in [10] and [11], Ch. II of the laws of separation for the Borel classes in Baire spaces showed that for any ordinal ξ with $1 \leqslant \xi < \omega_1$ the class $\mathbf{\Pi}^0_\xi$ satisfies the first separation theorem (the separating set being in $\mathbf{\Delta}^0_\xi$) and the second separation theorem (with separating sets in $\mathbf{\Sigma}^0_\xi$), while $\mathbf{\Sigma}^0_\xi$ satisfies the inseparability theorem. Almost the same holds in

Euclidean spaces, except that the class $\mathbf{\Pi}_1^0$ does not satisfy the separation theorems. We remark that the Borel classes $\mathbf{\Delta}_\xi^0$ being closed under the operations of complementation and taking the difference between two sets automatically satisfy the separation theorems.

Closely connected with separation is the reduction theorem in Kuratowski's [62], which reads as follows:

For any pair of sets X and Y in the class K we can find a pair of disjoint sets $X' \subseteq X$ and $Y' \subseteq Y$ in K whose union is the same as that of X and Y.

The reduction theorem holds for the Borel classes $\mathbf{\Sigma}_\xi^0$ (and, of course, for $\mathbf{\Delta}_\xi^0$), but not for $\mathbf{\Pi}_\xi^0$. This inversion by comparison with the separation theorems is however not unexpected: the fact of the matter is that when the reduction theorem holds in some class K, then so do both separation theorems for the class of complements and (for the classes $\mathbf{\Sigma}_\xi^0$ and the projective classes $\mathbf{\Sigma}_n^1$ and $\mathbf{\Pi}_n^1$) the inseparability theorem for K itself.

1.2. Subclasses.

Lavrent'ev's note [35] uncovered a very interesting structure of the classes $K_\xi = \mathbf{\Delta}_{\xi+1}^0$, which allow a decomposition into subclasses formed on the basis of the least possible length of a transfinite or finite chain of $\mathbf{\Pi}_\xi^0$-sets that are distinguished from one another in a peculiar way and reach in their union a given set of class $\mathbf{\Delta}_{\xi+1}^0$. Later these subclasses were studied by Luzin in [10], [11], Lyapunov [55], Sierpiński and others.

In the book [63] §37.IV the following construction is presented, which reduces to the Lavrent'ev classes but differs somewhat from their original definition. It turns out that for every $\mathbf{\Delta}_{\xi+1}^0$-set X one can choose a number $\theta < \omega_1$ and a $\subseteq$-decreasing sequence of sets X_ν, where $\nu < \theta$, of class $\mathbf{\Pi}_\xi^0$ such that $X = \bigcup_\nu(X_\nu - X_{\nu+1})$, where the union is over all *even* numbers $\nu < \theta$ (and if θ itself is odd, then one must define additionally $X_\theta = \varnothing$). All the sets that can be obtained in this way for fixed θ and ξ form the θ-th subclass (the small class θ in [63]) of $\mathbf{\Delta}_{\xi+1}^0$.

A systematic study of various methods of forming subclasses was quite recently made by Louveau [119]. Burgess [107] discovered an interesting application of the subclasses to the theory of C-sets and R-sets.

1.3. Invariance of classes.

Lavrent'ev proved in [34] the following theorem on the topological invariance of Borel classes. Let K be one of the Borel classes (other than $\mathbf{\Sigma}_1^0$, $\mathbf{\Pi}_1^0$, $\mathbf{\Delta}_1^0$, $\mathbf{\Sigma}_2^0$, $\mathbf{\Delta}_2^0$). Then every point set homeomorphic to any set in K itself belongs to K. An analogous theorem is valid for the subclasses of Borel classes, and also for the projective and some other classes of point sets (see, for example, [66], §36).

This gives rise to the problem of extending the result to maps of a more general kind than homeomorphisms. By considering open continuous transformations (that is, the image of any open set must be open) Keldysh [69] established that every Borel set is a continuous open image of a suitable set of class $\mathbf{\Delta}^0_3$, so that Borel classes of level three and above are not preserved under continuous open maps.

A different picture emerges for *closed compact* maps, which are characterized by the following two conditions: 1) the image of every closed set is closed, and 2) the inverse image of every point is compact. Taimanov (for $\xi \geqslant \omega$, see [94]) and Saint Raymond [100] (for finite ξ) established that for $\xi \geqslant 3$ the classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, and also the class $\mathbf{\Pi}^0_2$ are preserved under continuous closed compact maps.

In Euclidean spaces the theorems of Lavrent'ev and Taimanov-Saint Raymond remain in force for the classes $\mathbf{\Sigma}^0_2$ and $\mathbf{\Delta}^0_2$.

1.4. Canonical sets and the problem of universality.

The work of Lavrent'ev, Keldysh, and other mathematicians drew attention to the study of the topological properties of Borel sets. Considerable interest was raised by the problem proposed by Luzin in [10] and [11] of selecting in each class $\mathbf{\Pi}^0_\xi$ a special family of *canonical* sets, restrictive enough so that all sets of this family are pairwise homeomorphic, but rich enough so that every $\mathbf{\Pi}^0_\xi$-set can be obtained in a simple manner from the canonical sets in $\mathbf{\Pi}^0_\xi$.

In the class $\mathbf{\Pi}^0_1$ of closed sets there are three types of canonical sets: points, homeomorphic images of the Cantor discontinuum, and closed sets homeomorphic to a Baire space (and for the real line intervals in place of the latter). Every closed set is a union of countably many canonical closed sets.

In the class $\mathbf{\Pi}^0_2$ $(=G_\delta)$ Aleksandrov and Uryson singled out one type of canonical sets: the homeomorphic images of a Baire space. Every set in $\mathbf{\Pi}^0_2$ is the union of a single such canonical set and countably many $\mathbf{\Delta}^0_2$-sets.

A definition of a canonical $\mathbf{\Pi}^0_\xi$-set for $\xi \geqslant 3$ was given by Keldysh in [70]. This definition contains two items: 1) the given $\mathbf{\Pi}^0_\xi$-set X is of the first category in its closure, and 2) every non-empty intersection $X \cap B$ of X with a basic open-and-closed set B of the relevant space $\mathcal{N}^m$ is a *universal* $\mathbf{\Pi}^0_\xi$-set. (The requirement of universality means that for every $\mathbf{\Pi}^0_\xi$-set Y there is a perfect set P such that Y is homeomorphic to the intersection $X \cap B \cap P$.) Keldysh found that for $\xi \geqslant 3$ the canonical sets of class $\mathbf{\Pi}^0_\xi$ are pairwise homeomorphic and every $\mathbf{\Pi}^0_\xi$-set is the union of a single canonical set and countably many sets of class $\mathbf{\Pi}^0_\eta$, where $1 \leqslant \eta < \xi$.

So far the following problem which was stated in [70] (and also in [50], Remark 50) has remained unsolved; is every strictly $\mathbf{\Pi}^0_\xi$-set universal? (*Strictly* $\mathbf{\Pi}^0_\xi$-sets are those $\mathbf{\Pi}^0_\xi$ sets that do not belong to the dual class $\mathbf{\Sigma}^0_\xi$. Every universal $\mathbf{\Pi}^0_\xi$-set, and such sets exist in every class $\mathbf{\Pi}^0_\xi$, is a strictly $\mathbf{\Pi}^0_\xi$-set.) For an affirmative answer it would be sufficient to show that every strictly $\mathbf{\Pi}^0_\xi$-set X contains a subset that is closed in X and homeomorphic to one of the universal $\mathbf{\Pi}^0_\xi$-sets. In connection with this problem we mention an interesting result of Steel [121]: in a Cantor discontinuum for $\xi \geqslant 3$ any two sets of the first category that are strictly $\mathbf{\Pi}^0_\xi$-sets in the intersection with a basic interval are homeomorphic to each other.

§2. Projective sets. Construction of the hierarchy

The principal role in the construction of projective sets is played by the operation of *projection*: we have in mind a projection onto a subspace with one axis less, when every point $\langle x, y, \ldots, u, v\rangle$ is carried into the point $\langle x, y, \ldots, u\rangle$. Thus, the projection of a set in the Baire space $\mathcal{N}^{m+1}$ (or the Euclidean space $\mathbf{R}^{m+1}$) is situated in $\mathcal{N}^m$ (or in $\mathbf{R}^m$).

Projective sets in the Baire spaces $\mathcal{N}^m$, where $m \geqslant 1$, form the smallest class of sets in these spaces that is closed under the operations of projection and complementation and contains all open sets. Projective sets in Euclidean spaces are defined in exactly the same way only instead of open sets one must take sets of the class F_σ.

Projective sets can be organised in a hierarchy of projective classes on the basis of the least number of operations of complementation and projection that are necessary to construct the given set, starting out from the open sets (or the F_σ-sets in the Euclidean case). *The projective classes* are denoted in modern work on the descriptive theory by $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$, where $n \geqslant 0$ is any natural number. (These symbols were introduced by Addison in [80].) In contrast to the Borel hierarchy, here it is not necessary to resort to transfinite indices, because the construction breaks off at the stage ω.

In the Baire spaces the initial class $\mathbf{\Sigma}^1_0$ consists of all open sets, that is, it coincides with the Borel class $\mathbf{\Sigma}^0_1$. In Euclidean spaces one has to include in $\mathbf{\Sigma}^1_0$ all sets F_σ, that is, $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_2$.

Next, for any n the class $\mathbf{\Pi}^1_n$ is formed by the complements to all sets in $\mathbf{\Sigma}^1_n$, and then $\mathbf{\Sigma}^1_{n+1}$ by the projections of sets in $\mathbf{\Pi}^1_n$. Finally, as in the definition of the Borel hierarchy, the class $\mathbf{\Delta}^1_n$ is defined to be the intersection of $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

In the more traditional system of notation introduced by Luzin the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ were denoted, respectively, by A_n, CA_n, and B_n. For details on Luzin's construction of the projective classes, see the article by Uspenskii [132].

Luzin's discovery of projective sets in 1925 (the definition of a projective set first appeared in the note [4]) was one of the most significant events in the development of the descriptive theory of sets. These sets have remained up to the present at the centre of attention of experts in descriptive theory, arousing interest by their unusual properties, which in many respects differ from the properties of the simpler "definable" sets, such as Borel sets. If theorems, say, on Borel sets as a rule are valid equally at all levels of the Borel hierarchy, as regards projective sets (except some simple propositions such as the relations $\mathbf{\Sigma}^1_n \not\subseteq \mathbf{\Pi}^1_n$, $\mathbf{\Pi}^1_n \not\subseteq \mathbf{\Sigma}^1_n$, $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subsetneqq \mathbf{\Delta}^1_{n+1}$, which are already noted by Luzin in [7], §72), substantial results within the framework of the classical descriptive theory can be obtained only for sets of the first, and rarely the second projective level. (The zero level, the classes $\mathbf{\Sigma}^1_0$, $\mathbf{\Pi}^1_0$, $\mathbf{\Delta}^1_0$, is in its properties more akin to the Borel hierarchy and we do not specially speak of them.)

Sets of the first projective level, that is, the sets $\mathbf{\Sigma}_1^1$, $\mathbf{\Pi}_1^1$, $\mathbf{\Delta}_1^1$. have been studied most thoroughly. Investigation of sets of these classes began, strictly speaking, even before Luzin discovered the general notion of a projective set. At that time the sets of class $\mathbf{\Sigma}_1^1$ were known as A-sets, or analytic sets, or Suslin sets. For the sets of class $\mathbf{\Pi}_1^1$ the following names were also used: CA-sets, analytic complements, co-analytic sets, co-Suslin sets. For some supplementary information on the names and the history of their discovery, see §1 of the article by Uspenskii [122]. Finally, we mention that by a theorem of Suslin [21] $\mathbf{\Delta}_1^1$ coincides with the class of Borel sets.

In surveying the classical theory of the first projective level we single out three groups of results. Two of these, which are particularly rich in interesting theorems, are presented in the separate sections §3 (direct applications of the sieve operation) and §4 (sets with special sections). The third group contains results concerning regularity properties: the perfect kernel property, the Baire property, and the measurability property (see the article by Uspenskii [132], §3, or [125], 252). Here are the main theorems concerning these properties:

1) Every $\mathbf{\Sigma}_1^1$-set (in Baire or Euclidean spaces) has the perfect kernel property. We refer to this assertion as the Aleksandrov-Hausdorff-Suslin theorem. (For historical information, see the article by Uspenskii [132]).

2) Every $\mathbf{\Sigma}_1^1$-set has the Baire property—Luzin [2].

3) Every $\mathbf{\Sigma}_1^1$-set in a Euclidean space is Lebesgue-measurable—Luzin [2].

The last theorem can be carried over to sets in Baire spaces by means of the concept of absolute measurability, that is, measurability with respect to the entire class of measures. A *Borel σ-finite measure* on a given space $\mathscr{X}$ is any countably additive measure that is defined on all Borel sets of that space (and no others) and satisfies the condition that $\mathscr{X}$ is a union of countably many sets having finite values of the measure (generally, $+\infty$ is allowed as a value of the measure). A set $X \subseteq \mathscr{X}$ is said to be measurable in the sense of such a measure m if there exists a pair of Borel sets Y and Z such that $Y \subseteq X \subseteq Z$ and $m(Y) = m(Z)$. Finally, a set X is absolutely measurable when it is measurable in the sense of any Borel σ-finite measure on the given space.

An example of a Borel σ-finite measure on the real line $\mathbf{R}$ is the usual Lebesgue measure, restricted to Borel sets of $\mathbf{R}$. Thus, an absolutely measurable set of real numbers is Lebesgue-measurable.

The concept of absolute measurability allows us to restate Luzin's theorem on the measurability of $\mathbf{\Sigma}_1^1$-sets in the following form: every $\mathbf{\Sigma}_1^1$-set (in any Euclidean or Baire space) is absolutely measurable. This result, like that on the Baire property, obviously remains in force for $\mathbf{\Pi}_1^1$-sets.

§3. The sieve operation and its applications to projective sets of the first level

The operation of passing through a sieve (briefly: the sieve operation), introduced by Luzin, became the basic technical tool in the construction of the classical theory of the first, and later of the second, level of the projective hierarchy. Moreover, the value of this operation goes beyond the framework of a technical apparatus: it itself became the object of a profound study in works of Luzin and other mathematicians and the source of important problems posed by Luzin (on this, see §7).

In the interests of economy of space we do not repeat the definition of a sieve, nor of the exterior and interior sets and constituents, presented in §5 of Uspenskii's article [132]. The exterior and interior sets defined by a sieve C are denoted in what follows by $[C]$ and $[C]_*$, respectively, and the exterior and interior constituents corresponding to a given index $\nu < \omega_1$ by $[C]_\nu$ and $[C]_{*\nu}$. In addition, we often consider the following sets, which we call *approximations*:

$$[C]_{<\nu} = \bigcup_{\mu<\nu} [C]_\mu, \quad [C]_{*<\nu} = \bigcup_{\mu<\nu} [C]_{*\mu}.$$

A few words about the important concept of an *index*. The *exterior* index $\operatorname{Ind}_x C$ of a sieve C at any point x of the exterior set $[C]$ is defined to be the unique number $\nu < \omega_1$ for which $x \in [C]_\nu$. At points x of the interior set $[C]_*$ the exterior index $\operatorname{Ind}_x C$ is taken to be ω_1. Moreover, at points of the interior set $[C]_*$ the *interior* index $\operatorname{Ind}_{*x} C$ is defined to be the unique number $\nu < \omega_1$ for which $x \in [C]_{*\nu}$. At points x of the exterior set $[C]$ the interior index is not defined.

Concluding these remarks and definitions we now present the basic theorems on sieves, constituents, and indices, stopping on the way for the most important applications of sieves to the theory of the first projective level.

The sieving theorem [7]. *The class $\mathbf{\Sigma}^1_1$ coincides with that of all sets passed through an open sieve, that is, the class of all interior sets $[C]_*$, where C is an open sieve. Correspondingly, the class $\mathbf{\Pi}^1_1$ coincides with the totality of all exterior sets $[C]$ given by open sieves.*

A sieve $C = \langle C_q : q \in \mathbf{Q}\rangle$ is *open* when all the elements C_q of the sieve are open sets. The sieving theorem remains valid when instead of open sets we consider *Borel* sieves (that is, with Borel elements).

We recall that by $\mathbf{Q}$ we denote the set of all rational numbers.

Theorem on the Borel property of constituents ([7], [11], Ch. III). *If C is a Borel sieve, then $[C]_\nu, [C]_{*\nu}, [C]_{<\nu}$, and $[C]_{*<\nu}$ are Borel sets for any $\nu < \omega_1$.*

In the 30's Luzin and his student Keldysh made a detailed study of the position of the constituents of open sieves in the Borel hierarchy. Luzin considered a special class of so-called universal open sieves and proved [13] that the "complexity" of a constituent $[C]_\nu$ of such sieves C increases monotonically as ν tends to ω_1: strictly speaking, for each $\xi < \omega_1$ there are only countably many indices ν such that the constituents $[C]_\nu$ belong to the Borel class $\mathbf{\Delta}^0_\xi$, and all the rest appear only in higher Borel classes.

Keldysh obtained in [67] the following upper estimate for the classes of a constituent $[C]_\nu$ of an open sieve C. Let $\nu < \omega_1$. One can uniquely choose an ordinal number $\lambda < \omega_1$ and a natural number $n \geqslant 1$ such that $\omega^\lambda \cdot n \leqslant \nu < \omega^\lambda(n+1)$. Now, if $n = 1$, then $[C]_\nu$ belongs to the class $\mathbf{\Pi}^0_{2\lambda+1}$ and for $n > 1$ this constituent is the difference of two $\mathbf{\Pi}^0_{2\lambda+1}$-sets.

Recently Miller was able to prove that this estimate is exact, that is, it is attained as a special sieve, the binary (or canonical) Lebesgue sieve, constructed by Luzin in [7] §2. Regarding this sieve Miller discovered that for any $\nu < \omega_1$ if λ and n are defined as above, then for $n = 1$ the constituent $[C]_\nu$ is strictly a $\mathbf{\Pi}_{2\lambda+1}^0$-set (this means that $[C]_\nu$ does not belong to the class $\mathbf{\Sigma}_{2\lambda+1}^0$ of complementary sets), and for $n > 1$ this constituent is strictly the difference of two $\mathbf{\Pi}_{2\lambda+1}^0$-sets in the sense that it does not belong to the class of sets that are complementary to such sets (see [126]).

In [68] Keldysh studied the question: which is the smallest value for the index $\nu < \omega_1$ a given Borel set X can have so that there is an open sieve for which $X = [C] = [C]_{<\nu}$.

We move on to some other theorems on sieves.

Criterion for an exterior set to be Borel [7]. *Let C be a Borel sieve. Then for the exterior set $[C]$ to be Borel it is necessary and sufficient that $[C] = [C]_{<\nu}$ for some $\nu < \omega_1$.*

The sufficiency follows immediately from the theorem on the Borel property of constituents. For interior sets and approximations there is no analogous criterion: one can construct (this was, in fact, done in [25]) an open sieve C such that $[C]_* = \mathcal{N}$ and each constituent $[C]_{*\nu}$ is non-empty (which prevents $[C]_* = [C]_{*<\nu}$ no matter what $\nu < \omega_1$).

Boundedness principle ([11], Ch. III). *Again, let C be a Borel sieve. Then for every $\mathbf{\Sigma}^1_1$-set $Y \subseteq [C]$ there exists an ordinal number $\nu < \omega_1$ such that $Y \subseteq [C]_\nu$.*

The criterion for an exterior set to have the Borel property follows immediately from this principle, as does Suslin's theorem [31] on the coincidence of the class $\mathbf{\Delta}^1_1$ with the class of Borel sets, though originally both of these propositions were proved by alternative means. But the main application of the boundedness principle consists in Luzin's analysis ([11], Ch. III) of the cardinality-kernel problem for $\mathbf{\Pi}^1_1$-sets. We recall that in $\mathbf{\Sigma}^1_1$ this problem was conclusively solved by the theorem of Aleksandrov-Hausdorff-Suslin referred to in §2.

Let $X = [C]$ be any $\mathbf{\Pi}^1_1$-set given by an open sieve C. From the point of view of the number of non-empty constituents $[C]_\nu$ and the presence of uncountable constituents, three cases are possible.

1) There are only countably many non-empty constituents $[C]_\nu$, and each of them contains at most countably many points. In this case $X = \bigcup_{\nu<\omega_1} [C]_\nu$ is *at most countable.*

2) The number of non-empty constituents $[C]_\nu$ is uncountable (by the criterion for being Borel this is equivalent to X not being Borel) and as before each constituent $[C]_\nu$ is at most countable. In this case X *has the cardinality* $\aleph_1$ as the union of $\aleph_1$ countable sets. Moreover, X cannot contain a perfect subset, because such a set would have to be contained in the union of a certain countable number of constituents, which leads to a contradiction to the assumption that each constituent is at most countable. (Perfect sets in Baire and Euclidean spaces have the cardinality of the continuum.)

3) At least one of the constituents $[C]_\nu$ is uncountable. By the theorem that constituents are Borel, $[C]_\nu$ is Borel, hence, a $\mathbf{\Sigma}^1_1$-set. Consequently, the constituent $[C]_\nu$ contains a perfect subset, by the Aleksandrov-Hausdorff-Suslin theorem. But then the set $X = [C]$ contains this perfect subset, hence, *it has the cardinality of the continuum.*

Of special interest is the second of these three possibilities—when a $\mathbf{\Pi}^1_1$-set X has the cardinality $\aleph_1$ and does not contain a perfect subset, but has uncountably many non-empty constituents $[C]_\nu$, each being at most countable. Can such a possibility be realized by a suitable $\mathbf{\Pi}^1_1$-set (or, what is equivalent in view of the sieving theorem, by an open sieve C) or are there no such sets (and sieves)? This problem, which was posed by Luzin in the early note [2], was perhaps regarded as central in the classical descriptive theory. We turn to this problem in §6, where we report on the much later investigations of Novikov and Solovay, which established that it is undecidable.

For the interior constituents a proposition analogous to the boundedness principle does not hold, and the case similar to 2) is simply excluded by the following theorem of Selivanovskii [38]: if a Borel sieve C has uncountably many non-empty interior constituents $[C]_{*\nu}$, then at least one of these is uncountable.

Yet another application of the boundedness principle was noticed by Luzin in [11], Ch. III, namely, the *regularity* of the decomposition of the exterior set into constituents. This goes as follows. Let C be a Borel sieve, as before. Then there exists a transfinite ordinal number $\theta < \omega_1$ for which the difference set

$$[C]-[C]_{<\theta}=\bigcup_{\nu\geqslant\theta}[C]_\nu$$

is of the first category. In other words, the exterior set $[C]$ agrees to within a set of the first category with the union of a certain countable family of constituents $[C]_\nu$. This means that the decomposition into exterior constituents is regular with respect to *category.* A similar regularity holds for *measures*: if a Borel σ-finite measure (which is σ-additive, see §2) is given on the space in question, then the set $[C]$ coincides to within a set of measure zero with the union of a certain countable family (dependent on the measure) of constituents $[C]_\nu$.

It is remarkable that the decomposition of the interior set into interior constituents is also regular relative to category and measure; this was established by Selivanovskii in [38].

Comparison principle for indices (Novikov, see [51]). *Let C_1 and C_2 be a pair of Borel sieves for passing sets in a single space. Then the set of all points x in this space for which* $\mathrm{Ind}_x\, C_1 \leqslant \mathrm{Ind}_x\, C_2$ *belongs to the class* $\mathbf{\Sigma}^1_1$.

This important result allows us to investigate exhaustively the laws of separation and reduction on the first projective level. The content of these laws is as follows:

a) in the class $\mathbf{\Sigma}^1_1$ both separation theorems hold (with Borel sets as separating sets in the first theorem)—this was proved by Luzin in [7] and [11], Ch. III;

b) in the class of $\mathbf{\Pi}^1_1$-sets the inseparability theorem (Novikov [40]) and the reduction theorem hold (Kuratowski [62]).

The separation and reduction theorems have received various extensions to the case of arbitrarily finitely and countably many separating or reducing sets (multiple separation and reduction theorems). On this, see [18], [41], [53], [61], and [63], 358.

§4. The first projective level: sets with special sections

This branch of the descriptive theory is especially rich in interesting results. The main problem here concerns the clarification of the relationships between linear and plane sets. Sets are regarded as *linear* if they are subsets of the real line $\mathbf{R}$, realized in the form of the OX-axis, or subsets of the Baire space $\mathscr{N}$ understood as a "horizontal axis". *Plane* sets are those situated in the Euclidean plane OXY or in the Baire plane $\mathscr{N}^2$.

Below we talk only of linear or plane *Baire* sets but indicate the modifications that arise in the passage to the Euclidean case.

Linear and plane sets are related by the operation of projection understood by analogy with the general case of §2 (that is, every point of the $\langle x, y\rangle$-plane projects to x), but with an element of geometric intuitivity: projection onto the horizontal axis.

Turning now to sections, let P be a plane set; thus, $P \subseteq \mathscr{N}^2$. Every point $x \in \mathscr{N}$ determines a vertical *section* $P_x = \{y: \langle x, y\rangle \in P\}$ of P. Now P_x is contained in $\mathscr{N}$, consequently, is a linear set; however, we must imagine it as situated on the vertical axis $\mathscr{N}$, that is, on the second axis in the decomposition $\mathscr{N}^2 = \mathscr{N} \times \mathscr{N}$.

Depending on the character of the sections P_x we distinguish various types of plane sets. For example, if every section P_x of a given set P consists of at most one point, then the set P is called *single-valued* (or uniform). Such sets are connected with one of the most important problems in the descriptive theory, namely, uniformization (on this, see §4 in the article of Uspenskii [132]).

4.1. Uniformization on the first projective level.
We recall that a plane set P uniformizes a plane set Q when $P \subseteq Q$, P is single-valued, and its projection (in the sense just explained, that is, projection onto the horizontal axis) coincides with the projection of Q. *The uniformization theorem* for a given class K (for example, the projective class) can be stated as follows: every plane set of the class K can be uniformized by a (single-valued) set of the same class. Investigations on uniformization can be grouped, in general, around the central problem: to prove or disprove the uniformization theorem for the projective class in question.

The term "uniformization" was introduced in the descriptive theory by Luzin in [12], however, some interesting theorems had already been obtained in the second half of the 20's (in a somewhat different context: the selection of a single-valued branch in a given multi-valued function). We begin our account of the classical results with the "negative" theorem of Novikov [40]: there exists a closed plane set (in the Euclidean case it is G_δ) that does not allow a uniformization by sets of the class $\mathbf{\Sigma}^1_1$. Thus, the uniformization theorem does not hold for the classes $\mathbf{\Pi}^1_0$ (of closed sets), $\mathbf{\Delta}^1_1$ (of Borel sets), and $\mathbf{\Sigma}^1_1$ (analytic or A-sets).

Conversely, as was shown by the Japanese mathematician Kondô, the class $\mathbf{\Pi}^1_1$ satisfies the uniformization theorem. Kondô's contributions [72] were based on a method devised by Novikov (see [27]) for the effective selection of a point from a non-empty $\mathbf{\Pi}^1_1$-set. Even before Kondô's result became known, Novikov [43], [44] and Lyapunov [54], [56] obtained some important theorems on the uniformization of $\mathbf{\Pi}^1_1$-sets and on the nature of the projection of single-valued $\mathbf{\Pi}^1_1$-sets.

The question of uniformization of $\mathbf{\Sigma}^1_1$-sets was studied by Luzin and Yankov. Luzin established that every $\mathbf{\Sigma}^1_1$-set can be uniformized by a set that is effectively constructible, that is, without recourse to the axiom of choice ([5], Ch. V, [11], Ch. IV). Analysing Luzin's proposition, Yankov showed in [75] that every $\mathbf{\Sigma}^1_1$-set can be uniformized by a set that is a countable intersection of countable unions of differences of $\mathbf{\Sigma}^1_1$-sets.

4.2. Single-valued and countable-valued sets.
Special properties became apparent already in the first papers on uniformization in the case of plane sets P satisfying the condition that every vertical section P_x is at most countable; such sets are called *countable-valued.* Novikov discovered [40] that every countable-valued set of class $\mathbf{\Delta}^1_1$ (that is, Borel set) can be uniformized by a $\mathbf{\Delta}^1_1$-set, although, as we have said, the uniformization theorem does not hold for the class $\mathbf{\Delta}^1_1$ in general.

The peculiarity of countable-valued and also single-valued sets manifested itself in connection with the problem of the *class of a projection.* It was discovered that the projection of every single-valued (Luzin [7]) and even countable-valued (Novikov [40]) set of class $\mathbf{\Delta}_1^1$ is necessarily a $\mathbf{\Delta}_1^1$-set, whereas the projections of $\mathbf{\Delta}_1^1$-sets (and even of closed sets) of general form fill out the class $\mathbf{\Sigma}_1^1$, which is wider than $\mathbf{\Delta}_1^1$. Luzin established that, conversely, every linear $\mathbf{\Delta}_1^1$-set is the projection of an appropriate single-valued closed set [7]. In the Euclidean case in place of closed sets one has to take sets of class $\mathbf{\Pi}_2^0$, that is, G_δ-sets.

Another two groups of interesting results consist of *covering* and *decomposition* theorems. Glivenko showed [39] that any single-valued $\mathbf{\Sigma}_1^1$-set can be covered by a single-valued set of class $\mathbf{\Delta}_1^1$, whereas there is a single-valued $\mathbf{\Pi}_1^1$-set that does not allow such a covering. Luzin found ([11], Ch. IV) that every countable-valued $\mathbf{\Sigma}_1^1$-set can be covered by a countable-valued $\mathbf{\Delta}_1^1$-set. Later several more subtle covering theorems were obtained (see the article [74]).

Finally, Luzin proved in [11], Ch. IV, the decomposition theorem for the class $\mathbf{\Delta}_1^1$: every countable-valued $\mathbf{\Delta}_1^1$-set is the union of countably many single-valued sets of the same class, and a similar theorem for the class $\mathbf{\Sigma}_1^1$.

A treatment and systematic account of all this material were given by Luzin in the fourth chapter of his "Leçons" [11]. We remark that in the proofs of the majority of the theorems quoted in this section the principal role is played by Luzin's first and second separation theorems for the class $\mathbf{\Sigma}_1^1$.

4.3. Other types of section.

At the end of the 30's the attention of experts was drawn to plane sets with *compact* and *σ-compact* sections. (In the Euclidean case closed sections and sections of class F_σ are considered.) Novikov established in [46] that the projections of $\mathbf{\Delta}_1^1$-sets with compact sections are themselves of class $\mathbf{\Delta}_1^1$. Arsenin extended this result to sets with σ-compact sections [73] (see also [61]). The proof uses certain multiple separation theorems.

Subsequent studies showed that plane $\mathbf{\Delta}_1^1$-sets with σ-compact sections permit uniformization by means of a (single-valued) set of class $\mathbf{\Delta}_1^1$ (Shchegolkov [76]) and decomposition into countably many sets of class $\mathbf{\Delta}_1^1$ with compact sections (Saint Raymond [101]).

Recently Louveau [118] obtained a beautiful theorem: if $\xi < \omega_1$ and a plane $\mathbf{\Delta}_1^1$-set P is such that each of its sections P_x belongs to the class $\mathbf{\Sigma}_{\xi+1}^0$, then P can be expressed as the union of countably many $\mathbf{\Delta}_1^1$-sets each section of which is a set of class $\mathbf{\Pi}_\xi^0$.

In the 70's a study was begun of sets with "large" sections, that is, of positive measure or not of the first category. The strongest results in this direction were obtained in [120]. It is shown there, in particular, that a $\mathbf{\Delta}_1^1$-set whose non-empty sections are all not of the first category (or of positive measure, when a certain Borel σ-finite measure is fixed) necessarily has a projection of class $\mathbf{\Delta}_1^1$ and permits uniformization by a means of $\mathbf{\Delta}_1^1$-set.

A few more theorems on uniformization by $\mathbf{\Delta}_1^1$-sets are quoted in [106] (where uniformizing sets are called *selectors*).

§5. The theory of operations on sets. C-sets and R-sets. The second projective level

In the fifth chapter of his book [11], having defined projective sets, Luzin introduced the following general problem: to clarify whether theorems proved at that time for the first level, such as separation theorems or theorems on single-valued or countable-valued sets, remain valid on the second and higher levels of the projective hierarchy. At the beginning of the 30's very little was known even about the second projective level, essentially only the fact that every $\mathbf{\Sigma}^1_2$-set is the union of $\aleph_1$ Borel sets, however, in contrast to the analogous decompositions of $\mathbf{\Sigma}^1_1$-sets and $\mathbf{\Pi}^1_1$-sets, regularity with respect to measure and category (see §3) for the decompositions of $\mathbf{\Sigma}^1_2$-sets had not been established. And the question of measurability, the Baire property, the perfect kernel property, as well as problems concerning separation, special sections, etc, remained open, and the situation as Luzin writes in [19], §23 appeared altogether hopeless.

Problems of regularity properties for sets of the second projective level also remained unsolved while later through the efforts of Novikov and Solovay (see the next section) it was established that a solution is here quite impossible within the framework of the usual mathematical tools. True, even in the 20's two important forms of $\mathbf{\Delta}^1_2$-sets were known for which the problems of measurability and the Baire property turned out to be soluble in the affirmative: these were the C-sets and the R-sets.

C-sets, which were introduced by Selivanovskii (see [37], where Selivanovskii attributes the idea of these sets to Luzin), form the smallest class of sets in the space in question that is closed under the operation of passing through a sieve of that class (or, what is equivalent, under the A-operation) and the operation of complementation and contains all open sets.

Selivanovskii showed that all C-sets are absolutely measurable and possess the Baire property. Just like Borel sets, the C-sets form a hierarchy of increasing classes, indexed by the natural numbers and the countable ordinals. The classes of this hierarchy are denoted by C_ξ, CC_ξ, and BC_ξ, where $1 \leqslant \xi < \omega_1$. The first class C_1 includes all interior sets of sieves formed from the open sets of the space. For any ξ the class CC_ξ consists of the complements of sets of C_ξ, while BC_ξ is the common part of C_ξ and CC_ξ. Finally, for $\xi \geqslant 2$ the class C_ξ contains all interior sets of sieves whose elements are sets in $\bigcup_{1 \leqslant \eta < \xi} CC_\eta$. Selivanovskii established in [37] that the classes of this hierarchy, like the Borel and the projective classes, grow with increasing index ξ and comprise all C-sets.

The separation laws for the hierarchy of C-sets were discovered by Novikov in [45]. It turned out that every class C_ξ satisfies both separation theorems and every class CC_ξ the inseparability theorem. As was shown by Kantorovich and Livenson in [52], all C-sets belong to the projective class

$\mathbf{\Delta}_2^1$ but do not exhaust it. A wider part of $\mathbf{\Delta}_2^1$ is formed by the R-sets, which were discovered by Kolmogorov in the course of working out a theory of operations on sets. We give an account of some propositions of the latter.

Let I be a fixed indexing set (countable, as a rule; in principle we may assume that $I = \omega$, but usually we employ more complicated indexing sets, which reveal better the essence of the relevant operations). The elements of the set I are called indices, sets of indices are called chains and sets of chains bases. Every base B specifies an operation Φ_{IB} that assigns to every I-indexed family $\langle X_i : i \in I\rangle$ the set $\Phi_{IB}\langle X_i : i \in I\rangle = \bigcup_{u\in B} \bigcap_{i\in u} X_i$. Operations of this kind are called δs-operations.

To this category there belong the operations of taking the union $\bigcup_I$ and intersection $\bigcap_I$ for I-indexed families. As a base for the first of these one can take the set of all singleton chains $\{i\}$, where $i \in I$, and a base of the second contains the single chain $u = I$. Aleksandrov's A-operation [29] also belongs here, and an indexing set of it is the set of all sequences $\langle a_1, \ldots, a_m\rangle$ of arbitrary finite length $m \geqslant 1$, formed from natural numbers a_k, while a base consists of chains of the form

$$u = \{\langle a_1\rangle, \ \langle a_1, a_2\rangle, \ \langle a_1, a_2, a_3\rangle, \ \langle a_1, a_2, a_3, a_4\rangle, \ \ldots\},$$

where $a_1, a_2, a_3, a_4, \ldots$ are arbitrary natural numbers.

Kolmogorov indicated the following general methods of constructing δs-operations from operations already obtained:

1) *The complementation operation.* For a given δs-operation $\Phi = \Phi_{IB}$ one introduces the complementary δs-operation Φ^c:

$$\Phi^c\langle X_i\colon i \in I\rangle = C\Phi\langle CX_i\colon i \in I\rangle.$$

(By CX we denote the complement of a set X. We assume that all sets X_i lie in some fixed space relative to which the complements are taken.) A base for the operation Φ^c is provided by the set B^c of all chains $v \subseteq I$ having a non-empty intersection with every chain $u \in B$. The operations $\bigcup_I$ and $\bigcap_I$ are mutually complementary.

2) *Composition.* Suppose that $\Phi = \Phi_{IB}$ is a given δs-operation and that to each $i \in I$ the δs-operation $\Phi_i = \Phi_{I_i B_i}$ is assigned. In this case we can define a new δs-operation Ψ with indexing set $J = \{\langle i, j\rangle\colon i \in I \text{ and } j \in I_i\}$ acting as follows:

$$\Psi\langle X_{ij}\colon \langle i, j\rangle \in J\rangle = \Phi\langle \Phi_i\langle X_{ij}\colon j \in I_i\rangle\colon i \in I\rangle.$$

A base for the operation Ψ consists of all chains $v \subseteq J$ of the form

$$v = \{\langle i, j\rangle\colon i \in u \text{ and } j \in u_i\},$$

where $u_i \in B_i$ for all $i \in u$ and $u \in B$.

When the exterior operation Φ is $\bigcap_I$, the resulting operation Ψ is appropriately denoted by $\bigcap_{i\in I} \Phi_i$.

3) *R-transformation.* From a given δs-operation $\Phi = \Phi_{I B}$ we can construct a new δs-operation $R\Phi$ with indexing set RI, formed by all sequences $\langle i_1, \ldots, i_m \rangle$ of indices $i_k \in I$. A base RB of it consists of all chains that are obtained in the course of a construction, rather like performing a composition ω times. Namely, we take a certain chain $u \in B$ and assign to each sequence $\langle i_1, \ldots, i_m \rangle \in RI$ a chain $u_{i_1 \ldots i_m} \in B$. In the new chain v we include all sequences $\langle i_1, \ldots, i_n \rangle \in RI$ that satisfy the following conditions: $i_1 \in u$ and $i_{m+1} \in u_{i_1 \ldots i_m}$ for any m with $1 \leqslant m < n$. The base RB includes all chains v obtained by this construction.

It can be shown that the action of the operation $R\Phi$ thus defined does not depend at all on the choice of a concrete base B for the initial operation Φ, therefore, the notation $R\Phi$ without indicating the base B is well-defined. The same goes for the notation Φ^c and $\bigcap_{i \in I} \Phi_i$.

The operation $R\Phi$ is considerably stronger than Φ: any set that can be obtained by countably many applications of Φ alternating with complementation from, say, open-and-closed sets of a Baire space (or sets of any other sufficiently "good" family) can be obtained by only a single application of $R\Phi$; and by the second method one can obtain sets that cannot be obtained by the first. The idea of a normal series of R-operations is based on this. The series is formed by operations R_ξ and their complementary operations R_ξ^c, $1 \leqslant \xi < \omega_1$. The initial operation R_1 is taken to be the A-operation, and for $\xi \geqslant 2$ one defines $R_\xi = R(\bigcap_{1 \leqslant \eta < \xi} R_\eta^c)$. We remark that the A-operation itself is identical with the R-transformation of the operation $\bigcup_\omega$. The R_ξ are called R-operations.

By R_ξ we also denote the class of all sets that can be obtained from the open sets of a given space by a single application of the operation R_ξ. Next, CR_ξ is the class of all complements (which is equal to the class of those sets that can be obtained by the operation R_ξ^c on closed sets), while BR_ξ is the common part of the classes R_ξ and CR_ξ. The sets R_ξ, CR_ξ, and BR_ξ thus defined comprise the hierarchy of R-sets; an R-set is one that belongs to one of these classes.

The whole theory of δs-operations stems from two papers of Kolmogorov which he completed at the beginning of the 20's. In one of these papers [36] Kolmogorov introduced the concepts expounded here, except those connected with R-transformations, and showed that by means of any δs-operation one can construct a hierarchy of point sets, similar to the Borel or the C-hierarchies, and when certain requirements hold, then the classes of this hierarchy grow at each stage of the construction. In the other paper, which contains unpublished research of 1922 (on this, see [57], Introduction, or [60], 208) Kolmogorov introduced R-transformations and the R-operations and R-sets derived from it.

Kolmogorov's ideas in the domain of operations on sets were further developed in much detail by other experts. Hausdorff established that for every Borel class $\Gamma = \mathbf{\Sigma}^0_\xi$ or $\mathbf{\Pi}^0_\xi$ there exists a δs-operation Φ that gives after a single application to open-and-closed sets of a given Baire space all sets of the class Γ and only these ([66], §19). As was indicated in the Preface to the fundamental memoir of Kantorovich and Livenson [52], Part I, such operations exist also for the projective classes $\Gamma = \mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ (and also for those classes to which the projective hierarchy extends naturally through transfinite subscripts). In Ch. III of that memoir the definition of the R-transformation was published for the first time (attributed to Kolmogorov) and a result was obtained (Theorem XXX) from which it follows that all R-sets belong to the class $\mathbf{\Delta}^1_2$. Quite recently it has been shown that R-sets form a proper part of the class $\mathbf{\Delta}^1_2$ (see [107]).

A deep investigation of R-sets was undertaken by Lyapunov. Having presented in [57] the construction of the R-operations and R-sets, Lyapunov showed that R-sets can be obtained in another way, by means of the T-operations, which he discovered. In this paper he proved theorems on the absolute measurability and the Baire property of R-sets. (Lyapunov remarks in the Introduction that even earlier these results were obtained by Kolmogorov and mentions the unpublished paper.) He also proved that every R-set allows a decomposition that is regular with respect to measure and category into $\aleph_1$ Borel sets. In [57] Lyapunov constructed a theory of indices for R-operations, which enabled him to prove the separation theorems for the class R_ξ and an inseparability theorem for the class CR_ξ and to consider the action of the R-operations on R-sets of a certain level; in particular, he showed that for any $\xi \geqslant 2$ the class BR_ξ includes all sets of the class $R_{<\xi}$, that is, the smallest class containing all open-and-closed sets that is closed under the operation of complementation and all operations R_η and R^c_η, $1 \leqslant \eta < \xi$, but is not exhausted by them. For details on the study of R-sets, see [57], [59], [60].

C-sets and R-sets occupying a peculiar "intermediate" position between the first and second projective levels gravitate, perhaps, towards the first level from the point of view of the results and techniques applied. A serious study of the sets of the second projective hierarchy in full generality was opened up by the work of Novikov [42] on the problem of separation. The result obtained, as Luzin points out in [19], was entirely unexpected: both separation theorems hold in the class $\mathbf{\Pi}^1_2$ and the inseparability theorem in the class $\mathbf{\Sigma}^1_2$, that is, the other way round compared with the first level. To prove this Novkkov developed an apparatus of the minimal index and then found numerous applications in the study of projective sets.

The Novikov-Kondô uniformization theorem (see §4.1) greatly simplified and even trivialized some properties of sets with special sections on the second projective level. Namely, the projections of single-valued plane $\mathbf{\Pi}^1_1$-sets entirely fill out the class $\mathbf{\Sigma}^1_2$, so that here there is nothing similar to the interesting group of results on projections presented in §4.

The problems of covering and decomposition are solved quite differently. There exists a countable-valued $\mathbf{\Sigma}^1_2$-set that does not permit a covering by any countable-valued set of class $\mathbf{\Pi}^1_2$ and there exists a countable-valued $\mathbf{\Pi}^1_2$-set that is not a countable union of single-valued sets of class $\mathbf{\Sigma}^1_2$ (see [128], §2). The uniformization theorem itself holds for the class $\mathbf{\Sigma}^1_2$, but not for $\mathbf{\Pi}^1_2$ ([125], 258).

§6. Difficulties of the classical theory of projective sets. Search for new paths. The main trends of the contemporary development of the descriptive theory

Thus, the classical investigations of projective sets were limited essentially to sets of the first and second level of the projective hierarchy. More complicated projective sets (and in relation to many problems also sets of the second level and even $\mathbf{\Pi}^1_1$-sets) did not yield to the efforts of researchers. Commenting in the conclusion of his book [11] and summing up the situation of the descriptive set theory at the time, Luzin wrote:

"Only two cases are possible. Either further research will lead one day to precise relations between the projective sets and also to a complete solution of questions concerning the measures, categories, and cardinalities of these sets ... Or the indicated problems ... will always remain unsolved and to them we must add a collection of new problems that are equally natural and equally approachable. In this case it is clear that the day will have come to reform our ideas on the arithmetic continuum."

It is now perfectly clear that the second of Luzin's alternatives has come about. It has been established that many problems of the descriptive theory (in particular, the problems of measurability, the Baire property, and the perfect kernel property for projective sets) cannot, in fact, be solved in the traditional sense of the words "to solve a problem", that is, by means of standard mathematical tools and methods of reasoning it is not possible to give a definitive answer "yes" or "no".

This "reform" of the descriptive theory towards the inevitable and necessary, which Luzin pointed out in 1930, is successfully developed in contemporary research in two closely interconnected main directions: *the use of additional axioms* and *proofs of consistency*.

Experts in mathematical logic and set theory have developed several set-theoretic axiom systems. The widest recognition among them is given to the ZFC-theory of Zermelo-Fraenkel, including the axiom of choice AC, which is adequate as is generally accepted, in formalizing all methods of mathematical reasoning in use. After this axiomatic theory had been worked out, it became possible in a mathematically rigorous sense to raise and solve questions on the deducibility, consistency, and undecidability of some proposition on sets or another. And if we know (or conjecture with sufficient grounds) that by means of ZFC it is impossible to solve a large number of problems in this or another area (say, the problem of projective sets), then it is quite natural to attempt to rectify matters by some new axiom beyond the framework of the Zermelo-Fraenkel system. For such a new axiom three main conditions must be fulfilled: 1) it must be consistent (that is, it must not be inconsistent with the axioms of ZFC); 2) it must be sufficiently acceptable from the point of view of mathematical esthetics and intuition about sets; and 3) it must solve a significant number of problems.

In essence, not that many additional axioms have gained acceptance in the descriptive theory. First of all, this is the case for the axiom of constructibility together with its variants, the axiom of determinacy and its weaker forms, and also to a lesser extent for the axiom of the measurable cardinal and Martin's axiom.

6.1. Constructibility.

Gödel introduced in [78] the concept of a *constructible* set, using this name for any set that allows a transfinite construction of a special form. He stated the *axiom of constructibility*, which postulates the constructibility of every set and is denoted by the equality $V = L$ (where V is the usual symbol for the class of all sets and L denotes the class of all constructible sets).

In [78] he showed that the axiom of constructibility implies the generalized continuum hypothesis. In the proof of this proposition the axiom of choice is not used, moreover, the axiom itself follows from the axiom $V = L$.

Novikov obtained in [47] applications of constructibility to problems of the descriptive theory. It turned out that the axiom $V = L$ implies:

1) the existence of an uncountable $\mathbf{\Pi}^1_1$-set without perfect subsets;

2) the existence of a $\mathbf{\Delta}^1_2$-set of real numbers that does not have the Baire property and is not Lebesgue measurable;

3) the separation theorem for the classes $\mathbf{\Pi}^1_n$ and the inseparability theorem for the class $\mathbf{\Sigma}^1_n$ for any $n \geqslant 3$.

Later Addison showed that $V = L$ implies the reduction theorem [80] and also the uniformization theorem for each class $\mathbf{\Sigma}^1_n$, $n \geqslant 3$. The author of the present survey also obtained some consequences of the axiom of constructibility: in particular, the assertion that for any $n \geqslant 3$ there exists a plane countable-valued $\mathbf{\Pi}^1_{n-1}$-set that is not a countable union of single-valued $\mathbf{\Sigma}^1_n$-sets [128].

On the whole, the axiom of constructibility makes all the projective levels from the third on very similar to the second level—this is very clear from a comparison of the results quoted here with those given in §5.

6.2. The axiom of the measurable cardinal.

This is the name given to the assertion, abbreviated to MC, on the existence of an uncountable set such that on the algebra of all its subsets one can give a non-trivial countably additive two-valued measure ([125], 261).

The axiom MC has several consequences in the descriptive theory that are contrary to consequences of the axiom of constructibility. Thus, Solovay has shown in [86] that the axiom of the measurable cardinal implies absolute measurability, the Baire property, and the perfect kernel property for all sets of the class Σ^1_2. Mansfield [92] deduced from MC the assertion that every plane Π^1_2-set can be uniformized by a set of class Π^1_3.

A set that is referred to in the statement of MC (if it exists—the fact that it exists cannot be proved in ZFC, however, it is regarded as quite likely) must have extremely large cardinality, but this clearly does not stop the axiom of the measurable cardinal from having interesting consequences for such comparatively "small" objects as the set of real numbers. Essentially, all known applications of MC in the descriptive theory can be deduced not from the axiom itself, but from one of the following two propositions, which are consequences of MC:

(*) For every set $u \subseteq \omega$ (where ω is the set of natural numbers) there are only countably many sets $v \subseteq \omega$ that are constructible relative to u.

(**) ("The sharps hypothesis"). For every $u \subseteq \omega$ there exists a set of natural numbers denoted by $u^{\#}$ that codes in a natural manner the truth in the class $L[u]$ of all sets constructible relative to u.

The result of Solovay [86] quoted above was proved precisely by means of (*) via the scheme: MC $\to$ (*) $\to$ absolute measurability, the Baire property, and the perfect set property for all Σ^1_2-sets. It is interesting that (*) is not only sufficient but also necessary for the presence of the perfect kernel property in all Σ^1_2-sets (and even in Π^1_1-sets); this was established by Lyubetskii (see [93]).

6.3. Martin's axiom.

This axiom, which is denoted by MA, is very popular in certain branches of topology (see, for example, the survey [134]), but does very little for projective sets. We mention only that MA plus $\mathfrak{c} > \aleph_1$ (which expresses the negation of the continuum hypothesis) implies absolute measurability and the Baire property for all Σ^1_2-sets [8].

However, Martin's axiom yields a mass of interesting consequences in the specific branch of the descriptive theory where transfinite constructions are used with the help of the axiom of choice (see below §7.4 and §9). In this area there are also applications of the continuum hypothesis $\mathfrak{c} = \aleph_1$, which can also be regarded as a specific additional axiom. Incidentally, the axiom MA (we omit here its rather cumbersome statement and refer the reader to [125] Ch. 6) follows from the continuum hypothesis, therefore, it is usually considered in conjunction with $\mathfrak{c} > \aleph_1$.

6.4. Determinacy.

Axioms connected with this concept have attracted much attention of experts in the descriptive theory during the last 15–20 years.

Let A be some fixed set of points in the Baire space $\mathcal{N}$. Such a set determines a game $G(A)$ between two persons I and II, as follows:

Player I writes down a natural number a_1;

Player II knowing the "move" a_1 writes down his own natural number a_2; again Player I, knowing a_2, writes down a natural number a_3;

Player II, knowing a_3, writes down a_4;

and so on. As a result of this play a point $\alpha = \langle a_1, a_2, a_3, a_4, \ldots\rangle \in \mathcal{N}$ is obtained. If α lies in A the result is regarded as a win for Player I, and otherwise as a win for Player II.

A set A is called *determinate* if one of the two players has a winning strategy in the game just described, that is, a rule for selecting at his turn a move dependent on the preceding moves of the opponent such that the player in question, guided by it, wins irrespective of the moves of the opponent.

Martin [89] established that all Borel sets $A \subseteq \mathcal{N}$ are determinate. And this, perhaps, is a best possible result because the hypothesis $\mathbf{\Sigma}_1^1$-Det that all $\mathbf{\Sigma}_1^1$-sets are determinate (as well as the equivalent $\mathbf{\Pi}_1^1$-Det) cannot be proved in ZFC. However, the hypothesis $\mathbf{\Sigma}_1^1$-Det follows from the axiom MC of the measurable cardinal and is equivalent to the "sharps hypothesis" mentioned in §6.2 and also to the assertion that any two non-Borel $\mathbf{\Sigma}_1^1$-sets are Borel isomorphic. For details on this, see [90], [103], [121].

Of the various "determinacy hypotheses" the greatest interest among experts in the descriptive theory attaches to the *axiom of determinacy* AD, which postulates the determinacy of all sets $A \subseteq \mathcal{N}$, and the *axiom of projective determinacy* PD, which postulates the determinacy of all projective sets $A \subseteq \mathcal{N}$.

The axiom AD implies absolute measurability, the Baire property, and the perfect kernel property for all sets, therefore, it contradicts the axiom of choice AC [83], [85]—in fact, the axiom was proposed as an alternative to the axiom of choice by Mycielski and Steinhaus [82]. The more limited axiom of projective determinacy PD gives the relevant properties for only projective sets and seems not to contradict the axioms of ZFC (see §6.5). However, the axiom PD has remarkable consequences in the domain of the structure theory of projective classes, for example, the separation theorems for the classes $\mathbf{\Sigma}_{2n+1}^1$ and $\mathbf{\Pi}_{2n+2}^1$ and the inseparability theorems for the classes $\mathbf{\Pi}_{2n+1}^1$ and $\mathbf{\Sigma}_{2n+2}^1$ for any n [81], [96].

Generally speaking, the axiom PD makes all odd levels of the hierarchy similar in their properties to the first level and all the even levels to the second level. This phenomenon of extending by steps of length 2 the classical properties of the first and second level to higher projective levels can also be observed in relation to theorems on sets with special sections [97], [131].

However, a number of beautiful results (among them the generalization to odd levels of the theorem of Suslin [31] on the coincidence of the class with the class of all Borel sets, see [97], Ch. 7) could be proved only with the help of the full determinacy axiom AD. In arguments including AD, the role of the "basic" set theory is played not by ZFC, but by ZF + DC, which is obtained by replacing in ZFC the axiom of choice by the axiom of dependent choice DC, a kind of version of the axiom of countable choice, which is sufficient to derive such "positive" consequences of AC as the countability of a countable union of countable sets or the countable additivity of the Lebesgue measure.

All these discoveries were concluded in the second half of the 70's by the construction within the framework of the systems ZF + DC + AD and ZFC + PD of a fairly complete theory of projective sets solving practically all the main problems on projective sets, in front of which the classical descriptive theory had come to a halt. The much later research into determinacy diverged strongly from the classical topics, and in relation to them we limit ourselves to referring the reader to the article [123], the series of papers published in the collections [114], [115], [116], and the survey [136].

6.5. Consistency proofs.

In the theory of sets two principal methods have been worked out for proving the consistency of propositions on sets. The first method consists in deducing the relevant proposition from an additional axiom, whose consistency is an established fact. This is how Gödel proved in [78] the consistency of the generalized continuum-hypothesis GCH: firstly, he showed that the axiom of constructibility $V = L$ does not contradict the axioms ZFC, and then he produced a derivation of GCH from the axiom $V = L$.

By exactly the same means—a derivation from $V = L$—Novikov proved in [47] the consistency of the propositions §6.1.1), 2), and 3).

Consistency can sometimes be proved by a derivation from the conjunction: Martin's axiom MA $+ \mathfrak{c} > \aleph_1$ (the negation of the continuum hypothesis). The fact that MA $+ \mathfrak{c} > \aleph_1$ does not contradict the axioms ZFC was established by Solovay and Tennenbaum (see [125], Ch. 4, §6).

The axiom of the measurable cardinal refers to the group of the so-called *large cardinal axioms*, which postulate the existence of extremely large cardinalities. The consistency of this kind of axioms cannot be proved rigorously—but it can be regarded as a fact, to some extent experimental, since intensive activity around the consequences of the axioms ZF + MC have so far not led to a contradiction. On the basis of this argument the systems ZFC + PD and ZF + DC + AD which contain axioms of determinacy are assumed to be consistent.

Some eminent experts in the descriptive theory are inclined to regard the axiom MC and the axioms of determinacy as propositions that are true in the "real" world of sets or in some parts of it, giving a number of reasonable arguments in favour of such an approach (see [97], Conclusion, [98], [125], Ch. 1 and Ch. 8, §6, [129], and [131]). Consequences of these axioms are treated after the same fashion.

However, the axioms of large cardinals and the axioms AD and PD are regarded as too transcendental means for a "simple" proof of consistency.

Another method for proving consistency was discovered by Cohen [84], the *method of forcing,* which consists in a direct construction of models of set theory. Of the greatest significance for the theory of projective sets are two models constructed by Levy and Solovay by the method of forcing (see [87]). In the first of these models all the axioms of ZFC are satisfied, and it is true that every projective set is absolutely measurable, has the Baire property, and the perfect kernel property.

The existence of this model shows that the assertion of absolute measurability, the Baire property, and the perfect kernel property for projective sets is consistent with the axioms ZFC. But on the strength of the results mentioned above and the results of Novikov in §6.1 the propositions that there exist projective sets that do not have the requisite regularity properties (even in the classes $\mathbf{\Delta}_2^1$ and $\mathbf{\Pi}_1^1$) are also consistent. Thus, the problems of measurability, the Baire property, and the perfect kernel property for projective sets are *undecidable* within the framework of ZFC.

We remark that this undecidability result was predicted by Luzin as early as 1925 (in [4]), almost half a century before it could finally be proved.

In the second Levy-Solovay model all the axioms of ZF + DC are satisfied (but not the full axiom of choice), however, in it every (not necessarily projective) set is absolutely measurable, etc. Hence it is clear that the axiom of choice in its uncountable form is certainly necessary for the construction of such "singular" sets as, for example, sets that are not Lebesgue measurable.

A number of interesting results on consistency and undecidability in the descriptive theory were obtained by means of other models. But some important questions are still open (see [128], §7).

Concluding our survey of the main trends and approaches in the contemporary descriptive theory of sets, let us stress that it was the classical research of Luzin and his pupils and successors that became the point of departure for the subsequent development. The real value of one new concept or another or method in the descriptive theory can be checked first of all by how many classical problems it is able to solve. In the next three sections we consider in more detail some parts of current investigations that are characterized on the one hand by their connection with results obtained by Luzin, and on the other by the great attention they command among experts.

§7. Luzin's problems on the sequence of constituents

One of the main directions of Luzin's creative activity in the 30's consisted in the analysis of the difficulties in the path to a solution of such highly important problems of set theory as the continuum problem, the problem of the cardinality of Π^1_1-sets, the problem of an effective construction of sets that contain exactly $\aleph_1$ points. As an instrument of his analysis Luzin chose the sequence of constituents. He obtained several interesting results on the position of constituents in the Borel hierarchy. But the main success of this cycle of Luzin's investigations was the statement of the problem on the existence of a sieve whose constituents have certain properties and, on a more general level, the problem on the existence of "effective" uncountable sequences of Borel sets satisfying certain conditions. The present section is devoted to these problems.

7.1. Problems on the exterior constituents.
We begin with a series of four problems stated by Luzin in [19], §1 and [20], §1. These problems are given here in the order and numbering that Luzin put them in his papers.

Problem I. Does there exist an open sieve C such that every constituent $[C]_\nu$ contains exactly one point?

Problem II. Does there exist an unbounded open sieve C such that every constituent $[C]_\nu$ is at most countable?

A sieve C is called *unbounded* if it has uncountably many non-empty exterior constituents $[C]_\nu$. For open (and also for Borel) sieves the requirement of unboundedness is equivalent to the exterior set being non-Borel (see §3).

The condition imposed on the sieve in Problem II is weaker than that in Problem I in two respects: firstly, the condition that each exterior constituent is non-empty, which is implied by the requirement of the single element, is replaced by the condition that uncountably many (but not necessarily all) constituents are non-empty, and secondly, the condition "contains not more than one point", which is also implied by the same requirement, is replaced by the condition "contains not more than countably many points". The principal role in defining the status of these problems is played by the first moment whereas the second is of lesser significance. One can state the following "intermediate" problem.

Problem Ia. Does there exist an unbounded open sieve C such that every constituent $[C]_\nu$ contains not more than a single point?

Novikov established in [44] that Problem Ia is equivalent to Problem II in the sense that the existence of a sieve satisfying the requirements of one of these problems logically implies the existence of a sieve satisfying the requirements of the other. (The non-trivial part lies in the step from Problem II to Problem Ia, and this is what Novikov settled.)

Now we turn to the problems raised by Luzin in [19] and [20]. Proposing to weaken further the requirements placed on the sieve, Luzin posed the following two problems.

Problem III. Does there exist an unbounded open sieve C such that the exterior constituents $[C]_\nu$ form a sequence that is bounded in rank?

Problem IV. Does there exist an unbounded open sieve C such that the constituents $[C]_\nu$ can be included in pairwise disjoint Borel sets forming a sequence that is bounded in rank?

The *rank* (or in Luzin's terminology, the class) of a Borel set X is defined to be the least number $\xi < \omega_1$ such that X belongs to the class $\mathbf{\Delta}^0_\xi$ of the Borel hierarchy. Thus, the rank in a certain sense characterizes the position of a given Borel set in the Borel hierarchy. A family consisting of Borel sets is *bounded in rank* when there exists a transfinite ordinal $\zeta < \omega_1$ such that every set in the given family has rank less than ζ. In particular, any family is bounded in rank if it consists of at most countable point sets, because every such set belongs to the class F_σ (that is, $\mathbf{\Sigma}^0_2$) and as a result its rank is not greater than 3.

Problem III which is connected with the "restricted Lebesgue problem" (on this, see §5 of Uspenskii's article [132]), attracted Luzin's special attention. In his papers he stated several versions of Problem III of which we quote here two, numbered III-A and III-B.

Problem III-A ([13], §1). Does there exist an unbounded open sieve C such that from among its non-empty exterior constituents one can select an uncountable family of bounded rank?

Here it is not required that the family of all exterior constituents $[C]_\nu$ is of bounded rank.

Problem III-B ([14], §6). Does there exist an open sieve C such that every exterior constituent $[C]_\nu$ is non-empty and from among the $[C]_\nu$ one can select an uncountable family of bounded rank?

It is quite appropriate to quote yet another version of Problem III.

Problem III-C. Does there exist an open sieve C such that all the constituents $[C]_\nu$ are non-empty and form a sequence that is bounded in rank?

Naturally, similar modifications can be considered also for Problems II and IV. Problem I itself is a "version B" that is weaker than Problem Ia.

7.2. "The basic problem of the theory of analytic sets".

A problem under this heading was stated by Luzin in [17], §5. It consists in the following:

Does there exist an open sieve C such that from among the constituents $[C]_\nu$ and $[C]_{*\nu}$ (taken together) there are uncountably many non-empty ones and all the sets $[C]_\nu$ and $[C]_{*\nu}$ together form a family that is bounded in rank?

Unlike the problems considered in the preceding subsection, this problem is connected with Luzin's "restricted continuum problem", which, we recall, consists in requiring to partition effectively the continuum (understood as the real line or the Baire space) into $\aleph_1$ non-empty Borel sets bounded in rank (see §2 in the article by Uspenskii [132]). A sieve C satisfying the conditions of the "basic problem of the theory of analytic sets" would immediately provide the required partition, since the exterior and interior constituents $[C]_\nu$ and $[C]_{*\nu}$ taken together are pairwise disjoint and their union gives all points of the space in question.

7.3. Solution of the problems.
To begin with we quote some classical results. Problem II is equivalent to the problem whether there exists an uncountable $\mathbf{\Pi}^1_1$-set without a perfect kernel—we discussed in §3 this fact, which was discovered by Luzin ([11], Ch. III).

Novikov showed in [44] that Problem Ia is also equivalent to the existence of an uncountable $\mathbf{\Pi}^1_1$-set without a perfect kernel and so is equivalent to Problem II.

Lyapunov [56] established that a positive solution to Problem II would follow from the assertion that there exists an unbounded open sieve whose exterior constituents are all closed.

Selivanovskii [38] obtained this result: if every interior constituent of an open sieve is at most countable, then there are only countably many non-empty interior constituents. In other words, the analogue to Problem II for interior constituents has a negative solution.

In connection with Problems III and IV Luzin himself undertook a study of a special class of universal open sieves and discovered that the sieves of this class cannot give a positive solution to these problems (see [11], Ch. III, [13], [19], [20]). It turned out that the ranks of the constituents $[C]_\nu$ of such sieves C monotonically tend to the transfinite ordinal ω_1 as ν increases to ω_1. Similar results emerged from Luzin's investigation of the effective construction in [14] of a transfinite sequence of $\aleph_1$ Borel sets containing sets of arbitrary high rank (in fact, all countable ranks). Earlier such sequences were constructed only with the help of the axiom of choice.

Even after Luzin had abandoned this topic, the problems on constituents continued to attract the attention of his students Novikov, Keldysh, and Lyapunov. They discussed them, in particular, in the papers [49], [50], [51], [56], and [71]. However, nothing of substance could be achieved by the tools of the classical descriptive theory, and more recent findings confirmed to a large extent the view Luzin expressed in [19], §6, that "problems of this kind necessitate a departure from the traditional view of the meaning of the words: solution of a problem".

Further progress in research on problems concerning constituents was connected with undecidability proofs. Novikov [47] and Solovay [87] established the undecidability of the problem of an uncountable Π_1^1-set without a perfect kernel (see §6). However, Problems II and Ia, which are equivalent to this, turned out to be undecidable. A study of Luzin's remaining problems and their versions was undertaken by the present author in [130]. The following results were obtained.

Problems III and IV and also the versions A and B of Problems Ia, II, III, and IV (including Luzin's problems III-A and III-B) are equivalent to the problem of an uncountable Π_1^1-set without a perfect kernel (consequently they are equivalent to each other and to II and Ia) and are undecidable in the axiomatic set theory ZFC.

By contrast, the modifications of type B of Problems Ia, II, III, IV, and among them Luzin's Problem I (=Ia-B) turn out to be soluble in the negative: one can prove (in the usual sense of the word) that a sieve of the required type does not exist. The "basic problem in the theory of analytic sets" considered in §7.2 is also soluble in the negative.

If we speak of interior sets, here the picture is somewhat different. The analogues for interior constituents of Problems Ia, II, III, IV, and their modifications of type B (including the analogue of Problem I) are soluble in the negative. But the analogues to the modifications of type A and B of Problems Ia, II, III, IV are insoluble and are equivalent to the problem of an uncountable Π_1^1-set without a perfect kernel.

7.4. Luzin's continuum hypothesis and related problems.

Side-by-side with Cantor's continuum hypothesis, which can be expressed by the equation $\mathfrak{c} = \aleph_1$, another continuum hypothesis is considered in the literature of set theory, which was introduced by Luzin in [19], §9 and [20], §9, and is associated with his name. This hypothesis consists in the equality $\mathfrak{c} = 2^{\aleph_1}$. Indicating that both continuum hypotheses are probably to an extent equally free from contradiction, Luzin states in these papers three hypothetical propositions that are quite compatible with the continuum hypothesis $\mathfrak{c} = 2^{\aleph_1}$, but incompatible with Cantor's continuum hypothesis $\mathfrak{c} = \aleph_1$. Here are these propositions.

***Proposition* I.** *Every point set of cardinality $\aleph_1$ belongs to the class Π_1^1.*

***Proposition* II.** *Let C be an open sieve. Then for every set U consisting of ordinal numbers $\nu < \omega_1$ the union of constituents $[C]_\nu$ with indices $\nu \in U$ is a Π_1^1-set.*

***Proposition* III.** *The union of $\aleph_1$ arbitrarily chosen Borel sets is of class Σ_2^1.*

Luzin associated all three propositions with what he called the "higher continuity" of the continuum, without, however, explaining the meaning of this in more detail.

Each of the propositions I, II, and III implies Luzin's continuum hypothesis. (This is not difficult to deduce from the fact that there are exactly continuously many sets in the classes $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$.) Itself it follows from the axiom of determinacy AD. The latter fact for Proposition I is to a certain extent trivial, since from AD it follows that altogether there are no point sets of cardinality exactly $\aleph_1$ (see [85]). The more complicated deduction of Propositions II and III from the axiom AD is given in [97], Ch. 7.

Thus, if AD is regarded, as we said in the preceding section as in a certain sense a true axiom about "real" sets, then to the same extent the propositions under discussion here become true (and also Luzin's continuum hypothesis). We remark that Luzin himself commenting on these propositions characterized the first two as undoubtedly true, and the third as likely.

Now we consider the relationships of the Propositions I, II, and III and the hypothesis $\mathfrak{c} = 2^{\aleph_1}$ with the axioms ZFC. Cantor's continuum hypothesis implies a negative answer to each of the three propositions and the negation of Luzin's continuum hypothesis, so that neither the propositions nor the hypothesis can be proved by means of the axioms ZFC.

On the other hand, Martin and Solovay established in [88] that the equality $\mathfrak{c} = 2^{\aleph_1}$ follows (in ZFC) from Martin's axiom MA plus $\mathfrak{c} > \aleph_1$, and the Propositions I and III follow from the following complex hypothesis:

$$\mathrm{MA} + (\mathfrak{c} > \aleph_1) + \text{ (the negation of the Proposition (*) in §6.2).}$$

But this hypothesis does not contradict the axioms ZFC (see [88]). Consequently, Propositions I and III and Luzin's continuum hypothesis also do not lead to a contradiction with ZFC, therefore, in view of what we have said above, both propositions and the hypothesis are undecidable.

Matters are different with regard to Proposition II. In [88] it was shown that this proposition implies the measurability of the cardinal $\aleph_1$ in the sense of two-valued countably additive measure. But this measurability is impossible in ZFC, because every uncountable measurable cardinal is strongly inaccessible. Hence, Proposition II is false if we accept the axiom of choice (but, we recall, is true when we accept the axiom of determinacy).

§8. Equivalence relations: Luzin's remarks and contemporary research

The point of departure for the discussion in this section is the continuum hypothesis in the form of the assertion on the absence of intermediate cardinalities between the countable cardinality $\aleph_0$ and that of the continuum $\mathfrak{c} = 2^{\aleph_0}$. The natural "carrier" for such an intermediate cardinality would be an uncountable point set whose cardinality is not equal to that of the continuum, and suitable candidates from the point of view of the descriptive theory would be uncountable sets not containing perfect subsets, that is, sets not having the perfect kernel property (in the sense of §2). Research on these sets formed one of the most interesting directions in the descriptive theory of sets. (Here we should mention the theorem of Aleksandrov and Suslin in §2, Luzin's analysis in §3 of the cardinality of $\mathbf{\Pi}_1^1$-sets, and more recent results in §6.)

Nevertheless, perhaps even richer material for the study of the continuum problem is provided by equivalence relations.

Let E be an equivalence relation on one of the Euclidean or Baire spaces $\mathscr{X}$. (All these spaces are equally appropriate as regards the questions we are about to consider.) *An E-equivalence class* is any set $X \subseteq \mathscr{X}$ satisfying two requirements: firstly, xEy for any pair of points $x, y \in X$ and, secondly, if $x \in X$ and $y \in \mathscr{X}$ and xEy, then y also belongs to X.

Luzin was probably the first in the descriptive theory to turn his attention to the difficulties associated with equivalence relations. The analysis of this topic by Luzin in [7], §§64–65, was concentrated on the following two main problems:

a) How many equivalence classes does a given equivalence relation have?

b) For a given equivalence relation, can one construct effectively a point set containing precisely one point from each equivalence class?

After his memoir [7] Luzin did not return to equivalence relations: the development of the descriptive theory of sets brought other tasks to the forefront. A serious study of this theme did not start until the early 70's.

8.1. Investigation of the number of equivalence classes.
In work in this direction equivalence classes are divided into three categories:

1) *countable* relations—those which have only countably (or finitely) many equivalence classes;

2) *continual* relations—those for which there is a perfect set consisting of pairwise inequivalent elements (such relations necessarily have a continuum of equivalence classes);

3) relations not belonging to the first two categories—these we shall call *singular*.

It is clear that countable equivalence relations are analogues to countable (and finite) point sets, continual relations are analogues to sets that contain a perfect subset, and finally, singular relations are analogues to point sets that, being uncountable, do not contain perfect subsets. And this analogy is by no means purely superficial. To any point set X one can assign an equivalence relation e_X defined as follows: xe_Xy when $x = y$ or both points x and y belong to the complement of X. The equivalence classes of e_X are, firstly, all singletons $\{x\}$, where $x \in X$, and secondly, an additional class formed by the complement of X. Thus, to countable sets X there correspond countable equivalence relations e_X, to sets containing a perfect set continual relations e_X, and finally, to uncountable sets that do not contain perfect subsets the singular relations e_X.

As is clear, the theory of point sets (in connection with questions of cardinality and perfect kernels) forms a special case of the more general theory of equivalence classes. Moreover, the inversion so defined holds:

if a point set X belongs to a class K, then the corresponding relation e_X as a set of pairs lies in the class of complements to sets in K. This means that, for instance, the analogues among equivalence relations of the $\mathbf{\Sigma}_1^1$-point sets are the $\mathbf{\Pi}_1^1$-relations, that is, relations E such that the corresponding set of pairs $\{\langle x, y\rangle: xEy\}$ is of class $\mathbf{\Pi}_1^1$. Conversely, $\mathbf{\Pi}_1^1$-sets are analogues of $\mathbf{\Sigma}_1^1$ equivalence relations.

After these necessary remarks we turn to a survey of results. Of the greatest interest are the singular equivalence relations. Silver [95] obtained a very important result: a $\mathbf{\Pi}_1^1$-relation cannot be singular. In view of what we have just said, this theorem is an analogue and generalization of the Aleksandrov-Hausdorff-Suslin theorem of §3 that no $\mathbf{\Sigma}_1^1$-set fails to have the perfect kernel property.

In the same way the following result of Burgess [105] is an analogue and generalization of Luzin's theorem on the cardinality of $\mathbf{\Pi}_1^1$-sets, which was explained in §3. Burgess found that any singular $\mathbf{\Sigma}_1^1$-equivalence relations necessarily has precisely $\aleph_1$ equivalence classes. However, the analogy here is not quite complete: if the assertion that there is an uncountable $\mathbf{\Pi}_1^1$-set without perfect subsets is undecidable (see §6), then singular Σ_1^1-relations can be constructed directly. To construct one we take an open sieve C whose exterior set $[C]$ is non-Borel. We define xE^*y when either the points x and y belong to the interior set $[C]_*$ or there is an index $\nu < \omega_1$ such that x and y belong to the constituent $[C]_\nu$. It is not difficult to verify that E^* is a $\mathbf{\Sigma}_1^1$-relation whose equivalence classes are all non-empty constituents $[C]_\nu$ (and there are uncountably many of them, since $[C]$ is non-Borel) plus $[C]_*$. Thus, the relation E^* is not countable, but in accordance with the boundedness principle of §3 it cannot be a continual relation. Hence, C is a singular Σ_1^1-relation.

A more exact analogy is obtained if we consider relations *bounded in rank*, that is, equivalence relations E such that all E-equivalence classes are Borel sets and in total are bounded in rank. A study of such relations was made by Stern in [111] and more fully in [112]. Stern established that the assertion on the existence of singular $\mathbf{\Sigma}_1^1$-relations bounded in rank is undecidable in ZFC. In the later of the two papers there are many other interesting properties of Σ_1^1-relations. In particular, for any $\mathbf{\Sigma}_1^1$-relation E that is bounded in rank it is shown in [112] how to construct by certain regular means a $\mathbf{\Pi}_1^1$-set $A_E \subseteq \mathcal{N}$ having the properties that: 1) if player I has a winning strategy in the game $G(A_E)$, then the relation E is countable, 2) if player II has a winning strategy in this game, then E is continual, and 3) if the game $G(A_E)$ is not determinate, then E is a singular relation.

In [111] it was shown that the axiom of projective determinacy PD implies the absence of singular projective relations (of any class) that are bounded in rank, while the full axiom of determinacy AD implies the absence of any singular relations that are bounded in rank. Incidentally, from AD it also follows that there are no collections of $\aleph_1$ Borel sets of bounded rank [103].

8.2. The invariant descriptive theory of sets.
At the basis of this direction lies the same approach as in research on counting the number of equivalence classes, namely, to regard equivalence classes as new "points". Let E be an equivalence relation on one of the Euclidean or Baire spaces $\mathscr{X}$. A set $X \subseteq \mathscr{X}$ is called *invariant* (in the sense of E) if any two points x, y such that xEy both do or do not belong to X. In other words, invariant sets are unions of equivalence classes.

It has been possible to clarify that certain theorems of the descriptive theory remain valid in the domain of invariant sets. For instance, the following invariant analogue to the first separation theorem for $\mathbf{\Sigma}_1^1$-sets holds: any pair of disjoint invariant $\mathbf{\Sigma}_1^1$-sets can be separated by invariant Borel sets (see [108]). Similar results were obtained in [108] for the theorems on the reduction, uniformization, and decomposition into Borel sets. The invariant separability for the classes of the Borel hierarchy was considered in [113]. Closely related investigations are in [105].

The fundamental and frequently quoted paper on invariant sets was published by Vaught [99], however, we refrain from surveying its contents, since this would require an account of special model-theoretic logical symbolism.

8.3. Selector problems.
A *selector* for an equivalence relation E is a point set intersecting every E-equivalence class in exactly one point. Of course, a selector can be obtained by a simple application of the axiom of choice, however, if we wish to construct a "definable", say, a projective, selector (as Luzin suggests in [7], §65), then the task becomes very complicated even for certain very simple equivalence relations. For example, the selector for the equivalence relation of Vitali (which as a set of pairs is Borel) is, as is known, a non-measurable set (and does not have the Baire property), so that this equivalence relation necessarily fails to have a selector of class $\mathbf{\Sigma}_1^1$ or $\mathbf{\Pi}_1^1$, and a search for selectors in higher projective classes can only be done in the sense of consistency.

Thus, if we wish to study selectors of $\mathbf{\Sigma}_1^1$-equivalence relations, then it is appropriate to begin with selectors of class $\mathbf{\Sigma}_2^1$ (or, what is the same in this case, of class $\mathbf{\Delta}_2^1$: it is easy to see that a $\mathbf{\Sigma}_2^1$-selector of a relation of class $\mathbf{\Sigma}_1^1$ belongs to the class $\mathbf{\Delta}_2^1$). Burgess stated in [104] the following selector principle, including in essence a problem:

(SP) every $\mathbf{\Sigma}_1^1$-relation has a selector of class $\mathbf{\Sigma}_2^1$.

To illustrate the relationship between this principle and the axioms ZFC, we make use of two more propositions:

1) there exists a total ordering of a Baire space which as a set of pairs is a $\mathbf{\Sigma}_2^1$-set;

2) there exists a non-measurable set of class $\mathbf{\Delta}_2^1$.

Being applicable to the Vitali relation, the principle (SP) implies (2), but it itself follows, as is not difficult to prove, from (1). However, the proposition (1) does not contradict the axioms ZFC, because it follows from the axiom of constructibility, and proposition (2) cannot be proved in ZFC [86], [87]. Hence, the selector principle is undecidable: it cannot be proved nor disproved.

Budinas [135] showed that the principle (SP) lies "strictly between" (1) and (2); more accurately, (SP) does not follow from (2) and does not imply (1) [135].

§9. Problems and results connected with the axiom of choice and transfinite constructions

The axiom of choice, which was introduced in mathematics by Zermelo in 1904, immediately gave rise to serious criticism from many leading mathematicians of the time (Borel, Baire, Lebesgue, and others) in the first instance because of its ultimately non-effective character, because it does not contain any rule or law by which one can actually carry out the selection of a concrete element x in any given non-empty set X from a given family of sets. Some other eminent mathematicians were of a different opinion and saw no harm in applying the axiom of choice (Hadamard, Hausdorff, later Sierpiński). For certain aspects of the debate that unfolded at the beginning of our century in connection with this axiom, see the book [11], Ch. 1, and also [15], Part III.

On the whole Luzin adopted a critical stance towards using the axiom of choice in mathematical reasoning, however, he did not shirk from considering what the axiom could give in cases where it would be impossible to proceed by purely effective means. To this cycle of Luzin's research there belongs the construction and study of certain singular point sets, the analysis of questions connected with the restricted continuum problem in a non-effective sense, and also the statement and analysis of problems about subsets of the natural numbers. In each of these three directions Luzin obtained important results and posed deep problems, which became points of departure for interesting research at a later time.

9.1. Luzin sets and sets everywhere of the first category.
A *Luzin set* (or a ν-set in the terminology of the book [63], §40) is the name for any point set having the property that its intersection with any nowhere dense set of a given space is at most countable (see [125], 70). Luzin in [16] defined a *set everywhere of the first category* as a point set X such that the intersection $X \cap P$ is of the first category in P for any perfect (non-empty) set P.

Of course, every at most countable set is a Luzin set and is also a set everywhere of the first category. Therefore, only uncountable sets of both types are of interest. A construction of an uncountable Luzin set and an uncountable set everywhere of the first category under the assumption of the continuum hypothesis $\mathfrak{c} = \aleph_1$ was given by Luzin in his note [1]. Later Luzin constructed an uncountable set, more accurately, a set of cardinality precisely $\aleph_1$, that is everywhere of the first category, without recourse to the continuum hypothesis (however, using the axiom of choice, which is assumed throughout this section); see [6], [16].

However, the problem stated in [16] whether there exists a set everywhere of the first category and of the cardinality of the continuum without assuming the continuum hypothesis remained open. It was also not known whether there exist uncountable Luzin sets when $\mathfrak{c} > \aleph_1$.

The name "*Luzin set*" should perhaps be given to sets that are everywhere of the first category, because Luzin dedicated several of his works (including [1], [6], [16]) to precisely these sets. By contrast, the sets that nowadays are called Luzin sets figure only once in Luzin's work, in the note [1].

In spite of a certain similarity in their definition and their identity on countable sets, Luzin sets and sets everywhere of the first category have quite different properties. Thus, an uncountable Luzin set necessarily fails to have the Baire property, and no uncountable subset of such a set can be covered by a set of the first category. At the same time, sets everywhere of the first category are "exceedingly small" in the sense of category. For example, uncountable Luzin sets and uncountable sets everywhere of the first category form disjoint classes. However, the existence of a Luzin set of a certain cardinality follows from the existence of a set everywhere of the first category of the same cardinality (this assertion is, in fact, contained in [1]).

We now turn to current research on these two types of set. As soon as the method of forcing was discovered, it was established that the assertion that Luzin sets of the cardinality of the continuum exist, and then, by what we have just said, the sets everywhere of the first category of the same cardinality exist, is consistent with the axioms ZFC plus $\mathfrak{c} > \aleph_1$. (As was indicated in [134], 89, this result is due to Vopenka and Hrbaček.) On the other hand, the assertion that there are no uncountable Luzin sets follows, as Kunen showed, from Martin's Axiom MA plus $\mathfrak{c} > \aleph_1$, that is, this assertion is also consistent with ZFC+ $\mathfrak{c} > \aleph_1$. Finally, recently Miller [127] established the consistency relative to the axioms ZFC + $\mathfrak{c} > \aleph_1$ of the assertion that there are no sets everywhere of the first category whose cardinality is strictly greater than $\aleph_1$. Thus, the problem in [16] mentioned above and the problem of the existence of uncountable Luzin sets under the assumption that the continuum hypothesis is false, turned out to be *undecidable*.

We have spoken here only of sets in Baire and Euclidean spaces (both, of course, are of equal standing as regards the questions discussed), however, many results remain valid for spaces of a more general form (see [134], §§4.5 and 5.5, [127]).

9.2. Again the restricted continuum problem.

This problem, as we recall, consists in requiring a partition of the continuum (realized as one of the Baire or the Euclidean spaces—they are here completely equivalent to one another) into $\aleph_1$ non-empty Borel sets of bounded rank. Here we consider results connected with this problem in its non-effective interpretation under which we allow not only an effective construction of a specific partition of the required type, but also a direct proof of the existence of such a partition with the help of the axiom of choice.

A positive solution was obtained by Hausdorff in [65]. Hausdorff's result consists in the following: there exists a sequence of sets $X_\xi \subseteq \mathbf{R}$ of class $\mathbf{\Pi}^0_2$ (that is, G_δ), indexed by the ordinals $\xi < \omega_1$, strictly increasing ($X_\eta \subsetneqq X_\xi$ for $\eta < \xi$) and such that the union of all sets X_ξ completely covers the real line $\mathbf{R}$. Deleting from each set X_ξ all points that belong to the union of the preceding sets, we obtain the required partition of the continuum into $\aleph_1$ non-empty sets of class $\mathbf{\Pi}^0_3$, that is, of rank no greater than 4.

Hausdorff's theorem leaves open the question whether the continuum can be partitioned into $\aleph_1$ non-empty sets of type simpler than $\mathbf{\Pi}^0_3$. In particular, does there exist a partition of the continuum into $\aleph_1$ non-empty sets of class $\mathbf{\Pi}^0_2$? This problem was stated by Sierpiński in [33] (see also [63], 495). The analogous problem for closed sets (the class $\mathbf{\Pi}^0_1$) is also of interest, as is the problem of partitioning into $\aleph_1$ nowhere dense sets.

All these variations on Luzin's restricted continuum problem were subjected to intensive study in papers in the 70's. First came an exhaustive study of the problem of a partition into nowhere dense sets. Hechler showed in [91] that the assertion on the possibility of partitioning the continuum into $\aleph_1$ nowhere dense sets and also the negation of this assertion is consistent with the axioms ZFC $+\ \mathfrak{c} > \aleph_1$. A little later Stern [109] obtained the same result for partitionings into $\aleph_1$ non-empty closed sets. Thus, the problems of partitioning the continuum into $\aleph_1$ nowhere dense sets and into $\aleph_1$ non-empty closed sets are undecidable in the theory ZFC $+\ \mathfrak{c} > \aleph_1$. Finally, Fremlin and Shelah [117] established that the problem of partitioning the continuum into $\aleph_1$ non-empty sets of class $\mathbf{\Pi}^0_2$ is equivalent to the first of the two problems just mentioned (in the sense that from the existence of a partition of one type follows the existence of a partition of the other, and vice versa), consequently, this is also undecidable in the theory.

It should be mentioned that the continuum hypothesis $\mathfrak{c} = \aleph_1$ gives trivially a positive solution for all three problems (by means of a partition into $\aleph_1$ singleton sets).

After this excursion into contemporary research we turn again to Hausdorff's results. Luzin's attention was caught by a very interesting mechanism of proof in the course of which the principal difficulty to overcome consisted in the necessity of examining all points of the continuum in $\aleph_1$ stages of a construction, without introducing any restrictions on the cardinality of the continuum, and also some technical difficulties. The serious nature of the latter was emphasized by the following restricted theorem of Luzin [17]: a strictly increasing sequence of length ω_1 consisting of sets of class $\mathbf{\Pi}^0_2$ cannot increase continuously, that is, it is impossible that the equation $X_\xi = \bigcup_{\eta<\xi} X_\eta$ holds for all limit ordinals ξ.

The two papers [21] and [22] contain the result of Luzin's analysis of Hausdorff's construction, both with the same title: "On subsets of the natural numbers", to which our next subsection is devoted.

9.3. Problems and research on subsets of the natural numbers.
We recall that the set of natural numbers $\{1, 2, \ldots\}$ is denoted by ω. Two sets $u, v \subseteq \omega$ (that is, two parts of the series of natural numbers in the terminology of [21] and [22]) were called by Luzin *orthogonal* when the intersection $u \cap v$ is finite. Two families U and V consisting of parts of the series of natural numbers are called *orthogonal* when each $u \in U$ is orthogonal to any $v \in V$. Finally, two such families were called by Luzin *separated* when there exists a set $w \subseteq \omega$ (the separating set) such that all differences $u - w$ for $u \in U$ and all intersections $v \cap w$ for $v \in V$ are finite. It is easy to see that two separated families are necessarily orthogonal.

Is the converse true? This problem was studied in detail by Luzin in [21] and [22], and not only for families of parts of the series of natural numbers, but also for their increasing sequences. A sequence $\langle u_\xi : \xi < \lambda\rangle$ (of any finite or transfinite length λ) of sets $u_\xi \subseteq \omega$ is called increasing if for any pair of indices $\eta < \xi$ the relation $u_\eta \prec u_\xi$ holds, which means that the difference $u_\eta - u_\xi$ is finite whereas the difference $u_\xi - u_\eta$ is infinite.

Two countable sequences, and generally any two countable families of parts of the series of natural numbers, if orthogonal, are separated: this was established by Luzin. However, there exist orthogonal, unseparated, strictly increasing sequences each of length ω_1. The construction of such a pair of sequences was first carried out by Hausdorff in [65] and, in a simplified form, by Luzin in [22]. (Incidentally, the key feature in this construction is Hausdorff's theorem considered above.)

Having the existence of orthogonal unseparated sequences of length ω_1 (and hence of families of cardinality $\aleph_1$), on the one hand, and the separation of any pair of countable (or finite) orthogonal families of parts of the series of natural numbers on the other, Luzin posed in [22] the following two problems "on the important mixed case".

Problem I. Does there exist a pair of orthogonal unseparated families of parts of the series of natural numbers of which one is countable and the other has the cardinality $\aleph_1$?

Problem II. Does there exist a pair of increasing orthogonal unseparated sequences of which one has countable length and the other the length ω_1?

Two more problems, posed in [22], have as their source a certain analogy between the fact that there exist orthogonal unseparated sequences of length

ω_1 and "Pythagoras' phenomenon, which consists in the impossibility of inserting a rational point between two specially selected sequences of rational points that approach each other", as Luzin wrote in [22].

Problem III. Do there exist two increasing orthogonal sequences each of length ω_1, that are separated and allow only one separating set (to within addition or subtraction of a finite collection of natural numbers)?

Problem IV. Does there exist an increasing sequence $\langle u_\xi : \xi < \omega_1 \rangle$ that does not allow any set $u \subseteq \omega$ with infinite complement such that $u_\xi \prec u$ for all $\xi < \omega_1$?

There is a certain interdependence between these four problems. Problem I is equivalent to Problem II, that is, from the existence of a pair of sets satisfying the conditions of Problem I one can deduce the existence of a pair of sequences satisfying Problem II, and vice versa (the converse is obvious). Problem III is equivalent in the same sense with Problem IV. A more subtle result was obtained by Rothberger [79]: from a positive solution of Problem I (or, equivalently, of Problem II) there follows a positive solution of Problem IV (and hence of Problem III). Incidentally, in his article Rothberger considers the interrelationship of the problems discussed here with certain other interesting questions on the structure of the continuum.

A positive solution to all four problems is assured by the assumption of the continuum hypothesis $\mathfrak{c} = \aleph_1$ (this was undoubtedly known to Luzin: the problems with which we are concerned are called in [22] the derivative continuum hypotheses). Consequently, as previously, of interest is only the analysis of the relationship of Problem I–IV with the axioms of the theory $\mathbf{ZFC} + \mathfrak{c} > \aleph_1$.

From Martin's axiom MA plus $\mathfrak{c} > \aleph_1$ there follows a negative solution of all four problems (for Problem I this is mentioned in [91]). A very simple argument leads to this. For let us consider any increasing sequence $\langle u_\xi : \xi < \omega_1 \rangle$ of sets $u_\xi \subseteq \omega$. The collection of complementary sets $v_\xi = \omega - u_\xi$ evidently has the property that the intersection of any finite number of v_ξ is infinite. In this situation we can apply one of the consequences of MA $+ \mathfrak{c} > \aleph_1$, the so-called Booth lemma [134], 77, (or [125], Ch. 6, Corollary 8), which provides an infinite set $v \subseteq \omega$ such that the difference $v - v_\xi$ is finite for every ξ. Defining $u = \omega - v$, we obtain $u_\xi \prec u$ for all ξ. Thus, Problem IV, and then, by what we have said above, the three remaining problems have a negative solution under the assumption of MA $+ \mathfrak{c} > \aleph_1$, that is, a negative solution of all four problems does not contradict the axiom system $\mathbf{ZFC} + \mathfrak{c} > \aleph_1$.

Hechler has shown that a positive solution of these problems also does not contradict $\mathbf{ZFC} + \mathfrak{c} > \aleph_1$ (see [91], [134], 87).

The author of the current survey has proved that a positive solution of Problems II-IV does not imply a positive solution of the Problems I–II.

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