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The correctness of Euler's method for the factorization of the sine function into an infinite product

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CONTENTS

Introduction

One of the most remarkable parts of the first volume of "Introduction to infinitesimal analysis" [1] by Leonhard Euler is §§155-164 of the ninth chapter, where this great mathematician gives the factorization into an infinite product of linear and quadratic factors for a selection of transcendental functions, among which are the hyperbolic and circular sine functions. For these last Euler derives the formulae

$$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\ldots, \tag{1}$$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots \tag{2}$$

It is clear that these formulae are equivalent, as one can be derived from the other by substituting ix for x.

By itself the factorization of the sine functions could hardly be called new. Thus our motivation for this work after the passage of two and a half centuries is the astonishing technique used to obtain them in [1].

Euler's computations on this subject combine simplicity, ease, and intuition with a lack of rigour at almost every step of the argument (arising in freely going over to the infinite), so that if we take into account the beauty of the result, which was remarkable for its time, it would be hard to choose an analogue in mathematical research.

Of course, following Euler, completely rigorous and correct proofs of (1) and (2) were obtained, both in the framework of analysis ([3], 407) and as a consequence of the theory of analytic functions ([4], Ch. VII). More recently a rigorous derivation has been given using non-standard analysis— the theory in which infinitely large and infinitely small quantities in the sense of Leibniz are allowed (these arguments are introduced in §3).

However, all these proofs can be characterized by the fact that, in order to prove the required result, they use methods that are very different from the constructions in [1]. Thus they do not answer the question posed in our title. We give a new derivation of (1) and (2) which, by contrast to those mentioned above, follows Euler's calculations punctiliously. All the ambiguities of Euler's construction are justified and made precise using general principles of classical analysis and non-standard analysis. From this proof we deduce the correctness of the method Euler gave for the factorization of the trancendental functions sine and sinh into infinite products.

§1. Euler's method. The basic statements

The arguments used in Euler's book are so remarkable that we will reproduce them almost word for word, putting extracts from [1] in quotation marks, and making only one change: the symbol i, which Euler used to denote an infinitely large number, will be replaced by ω.

For simplicity we will divide our presentation into items that correspond to the key elements of Euler's construction. These divisions are not made in [1].

Euler begins the demonstration of his method in §155 with an example where the required result is not obtained directly, but this relative failure unexpectedly proves to be an additional argument in the later more successful calculation.

1. We try to factorize the function $e^x - 1$ (this is the example that concerns us), starting from the formula

$$e^x = \left(1 + \frac{x}{\omega}\right)^\omega, \tag{3}$$

where ω is a natural number that is infinitely large.

2. Euler uses the possibility (established in §151 of [1]) of factorizing $a^\omega - b^\omega$ into factors

$$t_k = a^2 + b^2 - 2ab\cos\frac{2k\pi}{\omega}, \quad \text{where} \quad 0 < 2k < \omega, \tag{4}$$

to which we add a factor $a-b$, and if ω is an even number, a factor $a+b$ also.

3. In the case under consideration we take $a = 1+x/\omega$ and $b = 1$, and then the factors (4) take the form

$$t_k = \left(1+\frac{x}{\omega}\right)^2 + 1 - 2\left(1+\frac{x}{\omega}\right)\cos\frac{2k\pi}{\omega}. \tag{5}$$

Later there is a transformation of t_k in which

$$\text{the factor } \cos\frac{2k\pi}{\omega} \text{ is changed to } 1-\frac{2k^2\pi^2}{\omega^2} \tag{6}$$

"to reduce to zero the remaining terms because ω is infinite". Euler intends that "the arc $2k\pi/\omega$ is infinitely small" and so the remaining terms in the Maclaurin series for the cosine "vanish" by comparison with the first two taken in (6). We obtain

$$t_k = \frac{4k^2\pi^2}{\omega^2}\left(1+\frac{x^2}{4k^2\pi^2}+\frac{x}{\omega}\right). \tag{7}$$

In addition we have a factor $a-b = x/\omega$, and if ω is even, a factor $a+b = 2+x/\omega$, which Euler did not in fact consider, as the fraction x/ω is infinitely small.

4. "In view of this", writes Euler, "apart from the factor x, the expression e^x-1 will have infinitely many factors

$$\left(1+\frac{x^2}{4\pi^2}+\frac{x}{\omega}\right)\left(1+\frac{x^2}{16\pi^2}+\frac{x}{\omega}\right)\left(1+\frac{x^2}{36\pi^2}+\frac{x}{\omega}\right)\ldots, \text{ and so on.} \tag{8}$$

These all include the infinitely small term x/ω. As this term is in all the factors, and multiplying through by $\omega/2$ produces a term $x/2$, it cannot be omitted".

5. "To avoid this difficulty" in § 156 Euler proceeds to look at the function

$$2\sinh x = e^x - e^{-x} = \left(1+\frac{x}{\omega}\right)^\omega - \left(1-\frac{x}{\omega}\right)^\omega, \tag{3a}$$

where again ω is an infinitely large natural number.

6. For this function we take $a = 1+\frac{x}{\omega}$, $b = 1-\frac{x}{\omega}$, and then

$$t_k = \left(1+\frac{x}{\omega}\right)^2 + \left(1-\frac{x}{\omega}\right)^2 - 2\left(1-\frac{x^2}{\omega^2}\right)\cos\frac{2k\pi}{\omega}. \tag{5a}$$

After the transformation (6) these factors take the form

$$t_k = \frac{4k^2\pi^2}{\omega^2}\left(1+\frac{x^2}{k^2\pi^2}-\frac{x^2}{\omega^2}\right). \tag{7a}$$

7. "Thus, the function e^x-e^{-x} can be divided by

$$1+\frac{x^2}{k^2\pi^2}-\frac{x^2}{\omega^2}, \tag{8a}$$

where the term x^2/ω^2 can be omitted without danger, since even after multiplying by ω it remains infinitely small." Moreover $a-b = 2x/\omega$, that is, Euler continues, "the first factor is x. Hence, if we put these factors in order, we have

$$\frac{e^x - e^{-x}}{2} = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\ldots\text{''}. \tag{9a}$$

8. "By multiplying through by an appropriate constant, I bring the separate factors into a form so that under the actual multiplication we obtain the first term x." Euler appeals to the Maclaurin series for the hyperbolic sine function, obtaining it from the series for the exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

9. Thus we have completed the factorization of sinh. Next, in § 157 of [1], by considering the factors of the expression $a^\omega + b^\omega$ in a suitable way a factorization of the hyperbolic cosine is obtained. Then in § 158, substituting the "imaginary quantity" ix for x, the factorizations for sine (formula (2)) and cosine are obtained.

The rest of Chapter IX (§ § 159–164) in [1] contains factorizations for more difficult combinations of exponentials and trigonometric functions. Euler used these results in the later chapters of [1] and also in "Differential Calculus" [2] and other works, to sum numerical series, and for other purposes.

But is the approach we have outlined correct? This question appears rather naive, because almost every point raises problems of rigour and justification. At the same time, a modern mathematician should at least ask the author of [1] the following questions:

(A) What is an infinitely large number and in what sense should we understand (3) and (3a)?

(B) If we agree to understand an infinitely large number as some special numerical entity, larger than any ordinary "finite" number, then how can we apply the result in item 2 on factorization to such a number?

(C) Further, how can we verify that (6) is valid, and also the expressions for t_k obtained from it (items 3 and 6) if we have in view only finite values of k. In this case the arc $2k\pi/\omega$ is in fact infinitely small, and the factors (5) and (7) and also (5a) and (7a) are of the second order in ω^{-1} and differ only by terms of the fourth and higher orders; in fact, in each pair they can be considered interchangeably (at least from our point of view). But how can k be infinitely large with the same order of magnitude as ω, when the ratio of k to ω is no longer infinitely small? This is a very controversial point, and Euler's proof would appear to be consistent with neither rigour nor informal intuition.

(D) How can we make rigorous the statement that it is not possible to omit "the infinitely small term x/ω" from the factors in the product (6), and that it is possible to omit the corresponding term x^2/ω^2 in (6a)? Euler's explanation only works on an intuitive level.

(E) Can we correctly determine the magnitude of the numerical factor in (9a) by comparing the coefficients of x in the product (9a) and the Maclaurin series for $\sinh x$, or in other words, can the known properties of polynomials be extended to series and infinite products if they are considered as "polynomials of infinitely large degree"?

(F) Where do the constant coefficients $4k^2\pi^2/\omega^2$ (in (7a) for t_k) and $2/\omega$ for x in the final term in (9a) disappear to? Their product, according to the calculated magnitude of the coefficients, should equal 2 (or 4 if we also have the factor $a+b=2$), but a simple calculation using Stirling's formula (see §5) shows that this product is infinite. How do we remove this discrepancy?

(G) Is it possible to bring Euler's incomplete calculation for e^x-1 to a conclusion (items 3 and 4)? In principle this question does not have a direct bearing on the analysis of the deduction of (1) and (2), but an answer is necessary if we want to fully understand Euler's method.

(H) In writing down (9a) we implied a product of factors $1+x^2/k^2\pi^2$ with finite natural numbers k. Euler's point of view is not entirely clear, but in any case we need to show what the situation is for factors t_k corresponding to infinitely large indices k in the composition of the final formula (9a).

Thus, the aim of this paper is to obtain an exhaustive explanation of all these controversial points, and carry out carefully and completely rigorously all the constructions for the factorization of the hyperbolic and circular sine functions into infinite products obtained by Euler. In our view this will be both sufficient and necessary in order to draw any conclusions on the correctness of Euler's method.

§2. Non-standard analysis

The starting point for the construction presented in §1 is the use of an infinitely large natural number, and this is generally a key aspect for choosing a path to justify all the arguments.

It is well known that the infinitely large and infinitely small constants (or, taking the terminology from logic and philosophy, "actuals" that is, given in complete form) were widely, though not without hesitation, used by Euler and his contemporaries, but were banished from mathematics in the course of the reconstruction of the foundations of analysis carried out in the 19th century. To be precise, these concepts had a quite different meaning, and were interpreted as an indication of the convergence of a variable quantity to infinity, or to zero.

In principle, with this present-day understanding of infinity it is possible to give a correct and rigorous interpretation to Euler's proof of (1) and (2).

However, to do this we would need such a fundamental reconstruction of all the calculations that it is much more attractive, and more in the spirit and letter of Euler's work, to give another interpretation of mathematical infinity, known as non-standard analysis.

The system with this name was developed in the early 60's by Robinson [5]. A detailed account of the foundations of non-standard analysis can be found in several works, of which a number are in Russian [5]-[8] (with an extensive bibliography in [8]). Here we only go into those aspects of non-standard analysis that are necessary to establish our results, and we present them in the simplest possible way, without pretending to strictly adhere to this or that version of the rigorous axiomatics of non-standard analysis ([10], [11]) as far as the formal details are concerned.

In the framework of the system we are discussing, ordinary real and complex numbers are called *standard*. However, it is postulated that non-standard numbers also exist, and that these are divided into three categories:

1) *Infinite* numbers. The number ω is infinite if $|\omega| > c$ for any standard real number c.

2) *Infinitesimal* numbers. The number α is infinitesimal if $|\alpha| < \varepsilon$ for any standard real $\varepsilon > 0$. According to this definition, 0 is an infinitesimal. Indeed, it is the only standard number that at the same time is infinitesimal.

3) Numbers of the form (standard) + (infinitesimal but non-zero), which are appropriately called *near-standard*. It is clear that all the infinitesimals apart from zero are in this category.

The near-standard numbers are combined with the standard numbers under the general title of *finite* numbers, by contrast with infinite numbers.

It is very important that the extended continuum $^*R$ of non-standard analysis, which includes the non-standard numbers together with the standard ones, has all the intrinsic properties of the standard continuum. (We consider as "intrinsic" any mathematical property that can be formulated without appealing to standardness or non-standardness. Without this proviso it is possible to obtain a contradiction; see below.) This fundamental statement—the transfer principle—enables us to deduce the existence of the infinite natural number ω. Indeed, in the standard continuum any real number s is dominated by the natural number ω. According to the transfer principle, this is also true in the extended continuum. We apply it to any infinite real number $s > 0$ and we obtain the infinite natural number $\omega \geqslant s$.

For natural numbers the concepts "standard" and "finite" are equivalent, so that a non-standard sequence of natural numbers begins with standard natural numbers, which form a proper initial segment, and is then continued by the infinite natural numbers. It is clear that there is no maximum in the set of finite natural numbers, and no minimum in the set of infinite numbers (otherwise what do we get by adding or subtracting one?).

There is no contradiction to the transfer principle in this violation of the principle of induction, as the property of "being finite" is not intrinsic.

For those sets that have been extracted from the sequence of non-standard natural numbers because of their intrinsic (that is, ordinary mathematical) properties, the principle of induction follows as a corollary of the transfer principle.

In general the transfer principle implies that we can manipulate non-standard numbers in exactly the same way as the elements of the ordinary "standard" continuum, at least when we are considering ordinary mathematical sets and relations.

The system of non-standard analysis enabled Robinson and his followers to make the foundations of the differential and integral calculus of Leibniz and Euler completely mathematically rigorous, by using infinite numbers and infinitesimals. In non-standard analysis some important mathematical concepts take on an unusual form. In particular, the non-standard definition of a limit becomes:

$\lim_{n\to\infty} a_n = A$ when $a_\omega \simeq A$ for any infinite natural index ω.

The sign $\simeq$ denotes infinitesimal closeness, that is,

$u \simeq v$ when the difference $u - v$ is infinitesimal (in particular, it is possible that it is exactly zero).

This definition of limit is equivalent to the standard one. Now the start of Euler's calculations becomes clear. For in formulae (3) and (3a) we replace the equals sign by $\simeq$. Then

$$e^x \simeq \left(1+\frac{x}{\omega}\right)^\omega \text{ and } e^x - e^{-x} \simeq 2\sinh_\omega x, \tag{10}$$

where

$$\sinh_\omega x = \frac{1}{2}\left[\left(1+\frac{x}{\omega}\right)^\omega - \left(1-\frac{x}{\omega}\right)^\omega\right],$$

for any infinite natural number ω and any standard real or complex x. It is clear that our aim is to show that

$$\sinh x \simeq x \prod_{k=1}^{\nu} \left(1+\frac{x^2}{k^2\pi^2}\right) \tag{11}$$

for any infinite natural number ν and any standard real or complex x. Formula (11) is the non-standard analogue and equivalent of (1).

We now present the proof of what we were aiming for. This proof was obtained by Luxemburg [9], and differs significantly from Euler's arguments. It does not clear up all the questions we raised earlier, but all the same it is simple enough and serves as a good introduction to the analysis of Euler's calculations that follows.

§3. The derivation of Euler's factorizations using non-standard analysis

We fix a standard complex number x, where we assume that $x \neq k\pi i$ for any integer k. This is without loss of generality, since (1) is obvious if $x = k\pi i$.

We also fix an infinite natural number ν, for which (11) is to be proved. We set $\omega = 2\nu + 1$, and we try to factorize the polynomial $\sinh_\omega x$, which by (10) approximates the function $\sinh x$ with an infinitesimal error in the standard x. The transfer principle allows us to apply the statement on factorization given in item 2 of §1 of this polynomial of infinitely large degree. Accordingly, the factors (5a) can be reduced to the form

$$t_k = 2 + 2\frac{x^2}{\omega^2} - 2\left(1 - \frac{x^2}{\omega^2}\right)\left(1 - 2\sin^2\frac{k\pi}{\omega}\right) =$$
$$= 4\sin^2\frac{k\pi}{\omega}\left(1 + \frac{x^2}{\omega^2\tan^2\frac{k\pi}{\omega}}\right) = 4\sin^2\frac{k\pi}{\omega}\left(1 + \frac{x^2}{k^2\pi^2}h_k\right),$$

where

$$h_k = \frac{k^2\pi^2}{\omega^2\tan^2\frac{k\pi}{\omega}}.$$

In addition to these factors (for which $1 \leqslant k \leqslant \nu$) we also have the factor $a - b = 2x/\omega$. Thus

$$\sinh_\omega x = \frac{1}{2}\left[2^\omega \prod_{k=1}^{\nu} \sin^2\frac{k\pi}{\omega}\right] x \prod_{k=1}^{\nu}\left(1 + \frac{x^2}{k^2\pi^2}h_k\right).$$

The resulting equality represents an identity transformation of polynomials in x, and then by comparing the coefficient of x on the left-hand side (where it equals 1) and right-hand side it follows that the expression in the square brackets equals 2, and so

$$\sinh_\omega x = x \prod_{k=1}^{\nu}\left(1 + \frac{x^2}{k^2\pi^2}h_k\right). \tag{12}$$

The next theorem guarantees that we can remove the term h_k from this expression. Although it does not appear in [9], it is clear that in general it parallels the arguments introduced there. We recall that ν is a fixed infinite natural number, and for simplicity we assume that the symbols $\sum$ and $\prod$ without indication of the limits denote the operations $\sum_{k=1}^{\nu}$ and $\prod_{k=1}^{\nu}$.

Theorem. *Let each natural number k, $1 \leqslant k \leqslant \nu$, be put in correspondence with the complex (possibly non-standard) numbers p_k and u_k. Firstly, if $\sum |p_k| \simeq 0$ and the following condition is satisfied:*

(a) *there is a real standard number $\varepsilon > 0$ such that $|u_k| > \varepsilon$ for all k with $1 \leqslant k \leqslant \nu$, then we have:* $[\prod (u_k + p_k)/\prod u_k] \simeq 1$.

Secondly, for the sum $\sum |p_k|$ *to be infinitesimal it is sufficient that the following two conditions hold*:

(b) $p_k \simeq 0$ *for each finite* k, *and*

(c) *there is a real standard number* $\gamma > 0$ *such that* $|p_k| \leqslant \gamma k^{-2}$ *for all infinite* $k \leqslant \nu$.

(*Remark.* The correspondence mentioned in the hypotheses of the theorem is assumed to be intrinsic, in the sense that this concept was explained in §1. In other words, the rule that defines p_k and u_k with respect to the index k must not be dependent on the concept of standardness. They are of the same type as, for example, in (13) below and all the other applications of the theorem in this paper. However, if we define $p_k = 1$ or 0 depending on whether k is finite or infinite, then the correspondence will not be intrinsic.

In the proof of the theorem that we give here, the requirement that the correspondence be intrinsic is only used in order to guarantee that operations such as $\sum |p_k|$ are carried out in the system of non-standard analysis, for which, strictly speaking, we need to use the transfer principle. For the "exterior" collection of numbers p_k introduced a few lines earlier the operation $\sum |p_k|$ is not defined in the non-standard domain $^*R$.)

Proof of the theorem. We simplify matters by setting $z_k = p_k/u_k$. Then

$$\sum |z_k| \leqslant \frac{1}{\varepsilon} \sum |p_k|, \text{ that is, } \sum |z_k| \simeq 0.$$

However, multiplying out the brackets on the right-hand side of

$$D = [\prod (u_k + p_k)/\prod u_k] - 1 = \prod (1 + z_k) - 1,$$

we obtain the sum of all possible products of the numbers z_k with each other. It is clear that the absolute value of this sum will not decrease if we replace each z_k by $|z_k|$. Thus, also taking into account that $1 + r \leqslant \exp(r)$ for any real r, we obtain

$$|D| \leqslant \prod (1 + |z_k|) - 1 \leqslant \exp\left(\sum |z_k|\right) - 1,$$

that is, $D \simeq 0$, since, as we saw above, $\sum |z_k|$ is infinitesimal.

In the proof of the second part of the theorem (that (b) and (c) are sufficient for $\sum |p_k|$ to be infinitesimal) let us agree to denote the expressions $\sum\limits_{k=n+1}^{\nu}$ and $\sum\limits_{k=1}^{n}$ by $\sum\limits_{n+1}$ and $\sum\limits^{n}$ respectively.

Step 1. The number ν is a natural number in the system of non-standard analysis, and thus any "intrinsic" set of ν real numbers contains a largest member (by the transfer principle). The number $p = \max\limits_{1 \leqslant k \leqslant \nu} |p_k|$ is infinitesimal, since each p_k is infinitesimal because of (b) and (c). We let n denote the integer part of the (infinite) number $p^{-1/2}$ (if $p = 0$, we take

$n = \nu$). Then

$$\sum^{n} |p_k| \leqslant np \leqslant p^{-\frac{1}{2}} p = p^{\frac{1}{2}} \simeq 0.$$

Step 2. We now show that the complementary sum $\sum_{n+1} |p_k|$ is also infinitesimal. We use the following estimate:

$$\sum_{n+1} |p_k| \leqslant \gamma \sum_{n+1} k^{-2} \leqslant \gamma \int_n^{\nu} \xi^{-2}\, d\xi \leqslant \gamma n^{-1}.$$

Thus, $\sum_{n+1} |p_k| \simeq 0$ because n is infinite, and since in addition $\sum^{n} |p_k| \simeq 0$, the whole sum $\sum |p_h| = \sum^{n} |p_k| + \sum_{n+1} |p_k|$ is also infinitesimal. This completes the proof of the theorem.

In order to apply the theorem to the number ν fixed at the start of §3 and the numbers

$$u_k = 1 + \frac{x^2}{k^2\pi^2}, \quad p_k = \frac{x^2}{k^2\pi^2}(h_k - 1), \tag{13}$$

we must show that the corresponding conditions are satisfied.

(a) For numbers k such that $|k| > |x|$ we have $|u_k| \geqslant 1/2$. Thus for ε we can take the smaller of the numbers $1/2$ and

$$\min\{|u_k|:\ -|x| \leqslant k \leqslant |x|\}.$$

Under this definition $\varepsilon > 0$, since at least one of the u_k is non-zero (by hypothesis x does not have the form $k\pi i$).

(b) If k is finite, then $k\pi/\omega \simeq 0$, since ω is infinite. But $(\tan\alpha)/\alpha \simeq 1$ for any infinitesimal α. Therefore $h_k \simeq 1$ and $p_k \simeq 0$ for finite k.

(c) In the quadrant $0 < \alpha < \pi/2$ the inequality $0 < \alpha < \tan\alpha$ holds, and so taking $\alpha = k\pi/\omega$ we obtain $0 < h_k < 1$ for $1 \leqslant k \leqslant \nu$. Consequently, $|p_k| < \gamma k^{-2}$, where $\gamma = |x|^2/\pi^2$.

Thus, we can in fact apply the theorem to the set of numbers in (13), and this means that (11) is a consequence of (12) and the relation (10). This completes the derivation of (11).

It is obvious that the proof we have given clarifies some elements of Euler's calculations given in §1. In particular, the answer to question (A), given by the concepts of non-standard analysis and the relation (10), even if it does not correspond completely to Euler's point of view (given in detail in "Differential calculus" [2]), is in any case sufficiently in accordance with his method of reasoning about infinity. Question (B) is completely answered on the basis of the transfer principle. But the later calculations of Luxemburg are essentially removed from the construction in [1], thus avoiding all the remaining questions.

§4. Euler's construction in the system of non-standard analysis

Now, keeping in mind the main problem of the paper, we look carefully at Euler's construction, in the form given in §1, and give a justification in the system of non-standard analysis of each of the separate key items in §1. On the way we try to answer questions (A)–(E). (The remaining three questions will be dealt with later.)

First, we take x to be a standard real or complex number, with $x \neq k\pi i$ for any integer k. Euler did not make this proviso, but it is unimportant, as when $x = k\pi i$ the result we are interested in is obvious.

Items 1 *and* 5. The concept of an infinite number and the sense of equalities (3) and (3a) have already been explained. In particular, (3a) can be understood as the second relation in (10), where ω is an infinite natural number. This answers (A), and is also the starting point for all the constructions that are aimed at deriving (1) in the form of (11), where an appropriate choice for ν is the integer part of $(\omega - 1)/2$, so that $\omega = 2\nu + 1$ or $\omega = 2\nu + 2$. We shall see how this is done.

Item 2. We have already discussed the answer to (B) at the end of §3. Thus, the approximating polynomial

$$2\sinh_\omega x = \left(1 + \frac{x}{\omega}\right)^\omega - \left(1 - \frac{x}{\omega}\right)^\omega$$

is a product of factors t_k of the form (5a) $(1 \leqslant k \leqslant \nu)$ together with the factor $2x/\omega$, and if $\omega = 2\nu + 2$ is even, a constant factor 2.

Items 3 *and* 4, which do not touch on the factorization of the sines directly, will be left until §6.

Item 6. We shall now look at some peculiarities of the transformation (6) when it is applied to the factors (5a). Replacing $\cos 2k\pi/\omega$ by $1 - 2k^2\pi^2/\omega^2$, Euler wrote down the sum of all the remaining terms of the Maclaurin series for $\cos\alpha$ (well known at that time, and derived in [1], §134) for $a = 2k\pi/\omega$. In what follows we denote this sum by $2k^2\pi^2 a_k/\omega^2$, where, consequently,

$$(14) \qquad a_k = \frac{\omega^2}{2k^2\pi^2}\left(\cos\frac{2k\pi}{\omega} - 1 + \frac{2k^2\pi^2}{\omega^2}\right) = 1 - \left(\frac{\omega \sin\frac{k\pi}{\omega}}{k\pi}\right)^2.$$

In this notation the correct form of the factor (5a) will be

$$(15) \qquad t_k = 2 + \frac{2x^2}{\omega^2} - 2\left(1 - \frac{x^2}{\omega^2}\right)\left(1 - \frac{2k^2\pi^2}{\omega^2} + \frac{2k^2\pi^2}{\omega^2} a_k\right) =$$
$$= 4\frac{k^2\pi^2}{\omega^2}(1 - a_k)\left(1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{\omega^2} + \frac{a_k}{1 - a_k}\frac{x^2}{k^2\pi^2}\right).$$

Thus,

$$(16) \qquad \sinh_\omega x = \mathrm{Const}\cdot x \prod\left(1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{\omega^2} + \frac{a_k}{1 - a_k}\frac{x^2}{k^2\pi^2}\right),$$

where the constant is independent of x and $\prod$ denotes $\prod\limits_{k=1}^{\nu}$ just as in §3.

But Euler, applying (6), obtained a different form:

$$\sinh_\omega x \simeq \mathrm{Const}\cdot x \prod \left(1+\frac{x^2}{k^2\pi^2}-\frac{x^2}{\omega^2}\right), \tag{17}$$

where again the constant is independent of x. This step is justified by applying the theorem in the last section to the collection of numbers

$$u_k = 1+\frac{x^2}{k^2\pi^2}-\frac{x^2}{\omega^2}, \quad p_k = \frac{a_k}{1-a_k}\,\frac{x^2}{k^2\pi^2}\,. \tag{18}$$

Let us check that the hypotheses of the theorem are satisfied.

(a) The term x^2/ω^2 in u_k is infinitesimal for standard x and infinite ω, and hence the arguments given in §3 to show that (a) holds for numbers of the form (14) are also valid here.

(b) If k is finite, then $k\pi/\omega \simeq 0$ and $a_k \simeq 0$, since $(\sin\alpha)/\alpha \simeq 1$ for infinitesimal α. Thus, in this case $p_k \simeq 0$.

(c) In the sector $0 < \alpha < \pi/2$ we have $2/\pi < (\sin\alpha)/\alpha < 1$. We set $\alpha = k\pi/\omega$. Then, by definition (16),

$$\frac{a_k}{1-a_k} = \left(\frac{\alpha}{\sin\alpha}\right)^2 - 1, \text{ that is, } 0 < \frac{a_k}{1-a_k} < \frac{\pi^2}{4} - 1 < 2, \tag{19}$$

and we see that we can choose $\gamma = 2|x|^2/\pi^2$ to estimate p_k.

Thus, the theorem in §3 can be applied, and so (17) follows from (16) (with the same constant).

Remark 1. It is clear that, as far as Euler was concerned, making the transformation (6) in the factors (5a) reduced to multiplying the true product of the factors (15) by a constant (whilst rejecting the coefficients $1-a_k$ that are independent of x) and to replacing the quadratic factors by simpler ones; this amounts to multiplying by a coefficient which, although it depends on x, is infinitely close to 1 for any x of the form under consideration. In the given case these changes do not affect the final result, since the constants can all be chosen to be equal from other considerations. This also answers question (C): the transformation (6) remains infinitely close not in the absolute sense, but to within a constant factor.

Remark 2. Some of the deficiencies in our justification of the transition from (16) to (17) can be overcome by using (19), which is simple but not so obvious and, which is the main thing, completely missing from Euler's computations. This estimate is derived here on the basis of the concrete properties of the cosine function, and it may seem that Euler was rather lucky in that $2/\pi < (\sin\alpha)/\alpha < 1$ in the quadrant $0 < \alpha < \pi/2$, which is the starting point for (19). But actually the fact that the fraction $a_k/(1-a_k)$ can be bounded above by a standard number may be shown by completely different arguments.

For suppose on the contrary that $a_k \simeq 1$ for some k. Then $\cos(2k\pi/\omega) \simeq 1$, by the definition of a_k, that is, $2k\pi/\omega \simeq 0$. However, by definition, a_k must be infinitesimal for infinitesimal $\alpha = 2k\pi/\omega$ (it has a factor α^2 in it), and this contradicts the original hypothesis that $a_k \simeq 1$. In this method of reasoning we only use quite general facts about Maclaurin series, and also the remark that the equation $\cos\alpha = 1$ has the unique solution $\alpha = 0$ in the quadrant $0 \leqslant \alpha \leqslant \pi/2$.

Item 7. Here we give another simplification of the factors, and (17) is transformed to

$$\sinh_\omega x \simeq \mathrm{Const}\cdot x \prod_{k=1}^{\nu}\left(1+\frac{x^2}{k^2\pi^2}\right). \tag{20}$$

This change is quite easily justified by the theorem in §3 with

$$u_k = 1 + x^2/k^2\pi^2,\ p_k = -x^2/\omega^2,$$

where in this case

$$\sum |p_k| = \nu x^2/\omega^2 \simeq 0,$$

that is, essentially, just as in Euler's work, everything reduces to the fact that the term x^2/ω^2 remains infinitesimal even after being multiplied by ν (but $\nu/\omega \simeq 1/2$).

Thus, Euler's arguments that we mentioned in question (D) can be completely justified insofar as they concern the factors in (6a).

Item 8. Euler drew the conclusion that the constant in (20) was equal to 1, based on a comparison between the infinite product and the Maclaurin series for $\sinh x$. In principle, the size of the constant can be determined without this. In fact, (16) is obtained by an identity transformation of the polynomial, so that the constant in (16) is precisely equal to 1. Now the transition to (17) and then to (20) is carried out by using the theorem of §3, and so the size of the constant is unchanged. Nonetheless, since we want to justify Euler's calculations per se, we need to explain what the basis for comparing the coefficients is.

For convenience we introduce the following notation:

$$P_n(x) = \sum_{k=0}^{n}\frac{x^{2k}}{(2k+1)!},\quad Q_n(x) = \prod_{k=1}^{n}\left(1+\frac{x^2}{k^2\pi^2}\right). \tag{21}$$

Then for infinite n and standard $x \neq k\pi i$ we have

$$xP_n(x) \simeq \sinh x \simeq CxQ_n(x), \tag{22}$$

where C is the constant in (20). In fact, the left-hand relation is the nonstandard analogue and equivalent of the decomposition

$$\sinh x = x\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)$$

(see the end of §2), and the right-hand one is the derivative of (20) (for $n = \nu$) and (10). If in (22) we divide through by x, and again take $n = \nu$, we obtain

$$P_\nu(x) \simeq CQ_\nu(x) \quad \text{for standard} \quad x \neq k\pi i. \tag{23}$$

Formally we cannot put $x = 0$ here, but in fact we would like to, since $P_\nu(0) = Q_\nu(0) = 1$, and we would then find that $C = 1$. If P_ν and Q_ν were standard polynomials, and (23) were satisfied with equality, then the required result would be given by the following elementary argument, which it is completely possible that Euler used to obtain the comparison of the coefficients:

"Choosing $x \neq 0$ sufficiently small, we can arrange that the values of $P_\nu(x)$ and $Q_\nu(x)$ are arbitrarily close to $P_\nu(0)$ and $Q_\nu(0)$, respectively. But then $P_\nu(x) = CQ_\nu(x)$, that is, we find that $P_\nu(0)$ and $CQ_\nu(0)$ are arbitrarily close to each other, that is, their values are equal, which is what we required."

We shall try to apply this line of argument to our problem.

First we must choose a "sufficiently small" value of x. In particular, (23) will be satisfied if $x = m^{-1}$, where m is a finite natural number, that is, for such m

$$|P_\nu(m^{-1}) - CQ_\nu(m^{-1})| < m^{-1}.$$

A simple argument from the proof of the theorem in §3 (the first step) shows that this inequality remains true for some infinite m, and then setting $x = m^{-1}$ we find that $P_\nu(x) \simeq CQ_\nu(x)$ where x is infinitesimal.

Now, in order to show that $C \simeq 1$, it is enough to use the following fact:

$$P_\nu(x) \simeq 1 \simeq CQ_\nu(x). \tag{24}$$

The proof of (24) reduces to simple estimates:

1) $$|P_\nu(x) - 1| \leqslant \sum_{k=1}^{\nu} |x|^{2k} \leqslant |x|^2/(1-|x|^2) \leqslant 2\,|x|^2 \simeq 0.$$

2) $$|Q_\nu(x) - 1| \leqslant \exp\left(\sum_{k=1}^{\nu} (|x|^2/k^2\pi^2)\right) - 1 \leqslant$$
$$\leqslant \exp(|x|^2 \sum (1/k^2\pi^2)) - 1 \leqslant \exp(|x|^2) - 1 \simeq 0.$$

(*Remark.* It can be shown that whether the first estimate holds depends essentially on the actual form of the coefficients $a_k = 1/(2k+1)!$ in the polynomial P_ν, since we have used the fact that they are bounded. However, this can be obtained directly from (22). Indeed, if $k \leqslant \nu$ is infinite, then

$$a_k = P_k(1) - P_{k-1}(1) \simeq \sinh 1 - \sinh 1 = 0,$$

and so $a_k \simeq 0$ for infinite k. Thus, the largest natural number $k \leqslant \nu$ such that $|a_k| \geqslant 1$ must be finite. We now take as the bounding number M the largest of the numbers 1, $|a_0|$, $|a_1|$, ..., $|a_k|$. The number M is finite, and $|a_k| \leqslant M$ for any $k \leqslant \nu$.)

Thus, we have shown that $C \simeq 1$, and so we can set the constant C in (20) simply equal to 1. If we now recall (10), we obtain our final result:

$$\sinh x \simeq x \prod_{k=1}^{\nu} \left(1 + \frac{x^2}{k^2\pi^2}\right). \tag{25}$$

This completes our analysis of item 8 of Euler's arguments, and on the way we have also cleared up the points discussed in question (E).

Item 9 is of no interest, and we do not need to consider it.

Thus we have shown that Euler's construction for the factorization of the hyperbolic and circular sine functions into an infinite product is correct, or, more precisely, can be put on a completely rigorous and correct footing on the basis of:

1) some elementary facts from mathematical analysis, such as the convergence of the series $\sum k^{-2}$ or the fact that the ratio of $\sin \alpha$ to α tends to 1 as $\alpha \to 0$;

2) some simple estimates based on these facts;

3) the setting of non-standard analysis, which more or less adequately interprets the methods characteristic to Euler of dealing with infinity in mathematical analysis.

This also gives us the courage to regard the main thesis of this paper, namely that Euler's methods were right, as well-founded.

Having thus dealt with the basic problem we now dwell in more detail on some of the questions we considered above only to the extent that they directly concerned Euler's constructions. However, as we hope to show, they deserve more attention. We will look at (F), (G), (H) together.

§5. Factors and coefficients in Euler's construction

In this section it is a question of the ambiguity in question (F) from § 1. First we consider constant coefficients for factors of the form (7a) and present the calculation that gives an infinitely large quantity for their product.

For definiteness, suppose that ω is odd: $\omega = 2\nu + 1$. We apply Stirling's formula $\nu! \sim \nu^\nu e^{-\nu}\sqrt{2\pi\nu}$, where the notation $r \sim q$ means that $r/q \simeq 1$. Thus,

$$\frac{2}{\omega}\prod_{k=1}^{\nu}\frac{4k^2\pi^2}{\omega^2} = \frac{2^\omega \pi^{\omega-1}(\nu!)^2}{\omega^\omega} \sim \frac{2^\omega\pi^{\omega-1}\nu^{2\omega}\cdot 2\pi\nu}{e^{2\nu}\omega^\omega} =$$

$$= \frac{(2\nu)^{2\nu+1}\cdot 2\pi^\omega}{e^{2\nu}(2\nu+1)^{2\nu+1}} = \frac{2\pi^\omega}{e^{\omega-1}\left(1+\frac{1}{2\nu}\right)^{2\nu+1}} \sim 2\,\frac{\pi^\omega}{e^\omega},$$

that is, it is an infinite quantity, and in no way has the value 2 which follows from comparing indices.

This incongruity has a simple explanation: it is enough to look at the correct form of (15) for the factors t_k, from which it is clear that in the structure of the constant coefficient, besides the term $4k^2\pi^2/\omega^2$, we also have a term $1-a_k$, which Euler did not take into account. These numbers $1-a_k$ multiplied together ($1 \leqslant k \leqslant \nu$) give an infinitesimal quantity which exactly corresponds to the calculated value $2\pi^\omega/e^\omega$ in the sense that the generalized product will equal 2. This comes from the fact that

$$\frac{4k^2\pi^2}{\omega^2}(1-a_k) = 4\sin^2\frac{k\pi}{\omega},$$

and the product of these factors (with the additional factor $2/\omega$), calculated in §3, is exactly equal to 2.

§6. A factorization that Euler did not want to complete

We now consider the invalid argument used by Euler in order to factorize $e^x - 1$ (items 3 and 4 of §1). It is not impossible that the famous author of "Introduction to infinitesimal analysis" assumed this calculation to be a consequence of the basic result—the factorization of the hyperbolic sine function—as an additional and intuitive argument to justify the possibility of omitting the term x^2/ω^2 from the factors of (8a). Nonetheless, Euler's arguments can be completed to give the desired result, as we shall now show.

First we introduce the correct form of the factorization (in standard notation):

$$e^x - 1 = e^{x/2}\sinh\frac{x}{2} = e^{x/2}\, x \prod_{k=1}^{\infty}\left(1+\frac{x^2}{4k^2\pi^2}\right). \tag{26}$$

After a comparison with (8) we can observe that if, notwithstanding Euler's assertion, we omit the term x/ω from the factors in (8), then we obtain an incorrect result, that differs from (26) by a factor $e^{x/2}$. Further, the same factor can be considered as $\left(1+\frac{x}{\omega}\right)^{\omega/2}$ in correspondence with (3), where $\omega/2$ is the total number of factors of (8) (to be precise, it is the integer part of $(\omega-1)/2$). Thus, the way to obtain the factorization (26) may be to change each factor of (8) into a product of two factors:

$$1+\frac{x}{4k^2\pi^2}+\frac{x}{\omega} \text{ into } \left(1+\frac{x}{\omega}\right)\left(1+\frac{x^2}{4k^2\pi^2}\right).$$

We can try to imagine how Euler would have carried through his argument if he had asked himself such a question.

(A continuation of the discussion of item 4.) "In order to overcome the difficulties that have arisen, we represent each of the resulting factors as a product of two cofactors, the first of which has, for all the factors, the common form

$$1+\frac{x}{\omega},$$

and which for $\omega/2$ such cofactors gives a value

$$\left(1+\frac{x}{\omega}\right)^{\frac{\omega}{2}}=e^{\frac{x}{2}},$$

and the second cofactors form the product

$$\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x}{16\pi^2}\right)\left(1+\frac{x^2}{36\pi^2}\right)\cdots$$

The additional term formed from this representation

$$\frac{x^2}{4k^2\pi^2}\cdot\frac{x}{\omega}=\frac{x^3}{4k^2\pi^2\omega}$$

can be omitted without risk, because, as ω is infinite, it is infinitesimal relative to the remainder term $x^2/4k^2\pi^2$:

$$\frac{x^3}{4k^2\pi^2\omega}\Big/\frac{x^2}{4k^2\pi^2}=\frac{x}{\omega}\simeq 0\text{''}.$$

(All of this argument is just a poor reflection of the depth of Euler's work and the refined style of his translators. The only apology we make for our attempt is that we consider it a natural desire to have the role, unusual in present-day mathematics, of the direct successor to, and interpreter of, the ideas of the great man Euler, albeit in connection with questions which have little current importance, and are in general not very significant.)

We now show how the relevant derivation of (26), which consists of Euler's arguments (items 3 and 4 of §1) and their possible conclusion that we have suggested, becomes rigorous from the viewpoint of non-standard analysis.

We suppose, just as before, that ω is an infinite natural number, and ν is the integer part of $(\omega-1)/2$. We fix a standard $x\neq k\pi i$. The factor (5) takes the form

$$t_k=\frac{4k^2\pi^2}{\omega^2}(1-a_k)\left(1+\frac{x^2}{4k^2\pi^2}+\frac{x}{\omega}+\frac{a_k}{1-a_k}\frac{x^2}{4k^2\pi^2}\right),$$

where a_k is given by (14). The fourth term in the right-hand bracket, when we multiply out the factors t_k, can be dropped according to the theorem in §3, in the same way as we did in §4 (in going from (16) to (17)). Besides this, on the same basis, we can add the term we need, $x^3/4k^2\pi^2\omega$ (it satisfies requirements (b) and (c) of the theorem in §3). Hence,

$$e^x-1\simeq\text{Const}\cdot x\prod\left(1+\frac{x^2}{4k^2\pi^2}+\frac{x}{\omega}+\frac{x^3}{4k^2\pi^2\omega}\right)=$$
$$=\text{Const}\cdot x\left(1+\frac{x}{\omega}\right)^{\nu}\prod\left(1+\frac{x^2}{4k^2\pi^2}\right)\simeq\text{Const}\cdot e^{x/2}x\prod\left(1+\frac{x^2}{4k^2\pi^2}\right),$$

where the product $\prod$ is taken to mean $\prod\limits_{k=1}^{\nu}$ in the non-standard version and $\prod\limits_{k=1}^{\infty}$ in the standard one. Finally, by analogy with the corresponding point in §4, we can see that the constant is infinitely close to one, and so, omitting the constant, we arrive at the required factorization (26).

This shows the correctness of the deduction of the factorization (26) for the function $e^x - 1$ that Euler outlined but did not carry through to its conclusion. It answers question (G) of §1.

§7. The summation of infinitesimals

In this section we analyse in more detail than we did in §3 the part of the theorem there that asserts that the sum

$$\sum |p_k| = \sum_{k=1}^{\nu} |p_k|$$

is infinitesimal under the hypotheses

(b) the term p_k is infinitesimal for any finite index k;

(c) there is a standard real number $\gamma > 0$ such that for any infinite k we have $|p_k| \leqslant \gamma k^{-2}$.

The method of proof of this proposition consists firstly in the fact that that the sum

$$\sum^{n} |p_k| = \sum_{k=1}^{n} |p_k|$$

is infinitesimal for suitable infinite $n < \nu$, and secondly that the additional sum

$$\sum_{n+1} |p_k| = \sum_{k=n+1}^{\nu} |p_k|$$

is also infinitesimal, in fact for any infinite $n < \nu$. The first statement is based on hypotheses (b) and (c). The second is proved by means of a very simple estimate, using hypothesis (c), and we can show that the result remains true in the case when, in place of the bound given in (c), we bound the numbers $|p_k|$ above by the terms of any series of positive terms that is convergent in the standard sense.

In this argument we can see some incompatibility, which is expressed in the indeterminate choice of the "critical" value n: any infinite $n' < n$ can be taken in place of n. Here we would like to use the smallest n that will work in the proof, that is, in this case, simply the smallest of the infinite natural numbers n. The trouble with this is that in the system of non-standard analysis there is no such number, and indeed, the assumption that it exists immediately leads to a contradiction (see §2).

We shall assume nonetheless that there is a "number" ∞, whose nature is ambiguous, that lies on the number axis between the finite and the infinite numbers. In this case its inverse "number" ∞^{-1} will separate the infinitesimal numbers from the positive numbers that are not infinitesimal.

We now introduce a line of argument whose aim is to show that the sum $\sum |p_k|$ is infinitesimal under hypotheses (b) and (c).

The summation for $k < \infty$ is carried out separately. To estimate the first sum, we note that for any finite k, since the term p_k is infinitesimal, we have $|p_k| < \infty^{-1}$, and further, $|p_k| < \infty^{-2}$, because $|p_k|^{\frac{1}{2}}$ is infinitesimal together with p_k. Hence the quantity $\sum^{\infty} |p_k|$ is infinitesimal, as the sum of ∞ terms that do not exceed ∞^{-2}.

The estimate of the second sum is an exact repetition of that given in §3:

$$\sum_{\infty+1} |p_k| = \sum_{\infty<k<\nu} |p_k| \leqslant \gamma \int_{\infty}^{\nu} \xi^{-2}\, d\xi \leqslant \gamma \infty^{-1},$$

and hence it is also infinitesimal.

One may suppose that this "proof" would have been more readily acceptable to Euler than the more precise arguments in §3, where ∞ was interpreted as a "sufficiently small" infinite natural number n. What this "ideal" interpretation for ∞ may be we shall see below, but meanwhile we mention two more arguments.

The sum of the series. From what we said at the end of §2 about the limit of a sequence, for the standard series $\sum_{k+1}^{\infty} a_k$ to converge to the standard number A it is necessary and sufficient that $\sum_{k=1}^{n} a_k \simeq A$ for any infinite n, and the criterion for the existence of such an A is the requirement that the sum $\sum_{k=n+1}^{\nu} a_k$ be infinitesimal for any pair of infinite natural numbers $n < \nu$ (the non-standard Cauchy criterion). Taking ∞ in place of n here, we can obtain the following interpretation of the relationship between standard and non-standard convergence:

$$\sum_{k=1}^{\nu} a_k = \sum_{k=1}^{\infty} a_k + \sum_{k>\infty}^{\nu} a_k.$$

where the second sum on the right is infinitesimal, and so the result of summing to infinite ν is determined (to within an infinitesimal summand) simply by the terms with finite indices k.

Infinite products also allow a similar analysis, but in place of infinitesimals we need to consider factors becoming infinitely close to unity. In particular, our definitive formula (25) can be written in the form

$$\sinh x \simeq x \prod_{k=1}^{\infty} \left(1+\frac{x^2}{k^2\pi^2}\right) \prod_{k>\infty}^{\nu} \left(1+\frac{x^2}{k^2\pi^2}\right), \tag{27}$$

where the second product is infinitely close to unity (and this will be rigorous in non-standard analysis if in place of ∞ we put any infinite $n < \nu$).

On the basis of this representation we will try to establish the role of the factors t_k with finite and infinite indices k in Euler's construction.

For definiteness, we suppose that ω is an odd infinite number: $\omega = 2\nu + 1$. By the computations in §4 each factor t_k of the form (5a) can be represented as follows:

$$t_k = \frac{4k^2\pi^2}{\omega^2}(1-a_k)\left(1+\frac{x^2}{k^2\pi^2}\right) = 4\sin^2\frac{k\pi}{\omega}\left(1+\frac{x^2}{k^2\pi^2}\right).$$

(We have omitted for simplicity a factor that depends on x but is infinitely close to unity for any standard x. This factor appears when we apply the theorem in §3 in the transitions (16) → (17) → (20); see Remark 1 in §4.) In addition, the constant in (20) that equals 1 (on going over to (25)) is the product

$$C_\nu = \frac{1}{2}\cdot\frac{2}{\omega}\prod_{k=1}^{\nu}\frac{4k^2\pi^2}{\omega^2}(1-a_k)$$

of the constant coefficients of the factors t_k with the coefficient $2/\omega$ of the factor $2x/\omega$ and also the $1/2$ from the formula for sinh.

Thus, returning to the representation (27) for formula (25), we can see that the functional structure of the hyperbolic sine is determined by the factors t_k with finite indices k (and the separate factor $2x/\omega$). The factors t_k with infinite indices are only involved in forming the constant C_ν that is equal to 1, and their formal dependence on x is expressed in multiplying by quantities that are infinitely close to 1 for any standard x (both separately and as a product). At the same time it is not possible to simply omit those factors t_k with infinite k, since the regular factors $1+x^2/k^2\pi^2$ in the composition of the "unnormalized" t_k are connected with the infinitesimal numerical coefficients (for finite k), and it is a function of the factors t_k with infinite index to "neutralize" them.

We present these arguments as an answer to the last question (H) in §1. It is clear that in the system of non-standard analysis all the concepts have a rigorous meaning provided only that we replace ∞ by an arbitrary infinite natural number $n < \nu$, which corresponds to adjoining to the finite natural indices some "arbitrarily small" part of the infinitely large indices. Now the question is how to find an "ideal" interpretation of the symbol for formal infinity ∞ that reflects precisely the requirement we started with: that it separates the finite numbers from those that are positive infinite.

§8. Formal infinity as a mathematical object

Here we present a "model" of formal infinity that is very simple: it is based on the method of the Dedekind cut, which gives us the definition of the real numbers "due to Dedekind". In fact, we let ∞ denote the cut $I|J$ of the non-standard continuum $^*R$, where the lower class I is the set of all non-standard real numbers that are not positive infinite (that is, finite plus negative infinite) and the upper class is the set of all positive infinite numbers (amongst these are the infinite natural numbers).

Here we should make some general remarks about the cuts of $^*R$. It is known that each cut $A|B$ of the standard continuum R is a Dedekind cut (that is, either the lower class A has a largest element, or the upper class B has a smallest element), and this statement is one of the formulations of the fact that R is dense. By the transfer principle (see §2), and cut of $^*R$ given by the intrinsic property will also be a Dedekind cut. However, among the cuts of $^*R$ there are also "non-intrinsic" ones: an example is given by the cut $\infty = I|J$ just introduced, which is clearly not a Dedekind cut. The definition of this cut is not intrinsic, as the concept of an infinite number appeals to an "external" property of the standard system.

Thus, the Dedekind completion $\mathscr{R}$ of the non-standard continuum $^*R$ is formed by all possible non-Dedekind cuts of $^*R$, to which we append Dedekind cuts of the form

$$\{x \in {}^*R : x \leqslant a\} | \{x \in {}^*R : x > a\},$$

where $a \in {}^*R$, identified with the corresponding standard or non-standard number a.

Among other things, the "supercontinuum" $\mathscr{R}$ contains the "number" ∞, its inverse "number" $\varPhi$, that is, the cut which has as upper class the set of all positive $x \in {}^*R$ that are not infinitesimals, the "numbers" with the opposite sign to these, namely $-\infty$ and $-\varPhi$, and so on. If we define a linear ordering on $\mathscr{R}$ in the usual way (that is, corresponding to the closures of the lower classes) we obtain $x < \infty < \xi$ for any finite $x \in {}^*R$ and any infinite $\xi \in {}^*R$, and hence in $\mathscr{R}$ the "number" ∞ separates the finite numbers from the positive infinite ones, which is what we expected. In exactly the same way, the inverse "number" $\varPhi$ separates the infinitesimals from the positive numbers that are not infinitesimal.

Hence, the symbol ∞ from the arguments in §7 has the precise meaning of a cut in $^*R$ and of a "number" in the Dedekind completion $\mathscr{R}$ of the non-standard continuum $^*R$.

In an attempt to define the arithmetic structure of $\mathscr{R}$ we encounter a phenomenon that has no analogue in the Dedekind completion of the field of rational (standard) numbers to the standard continuum R. This is most easily seen in the example of taking differences, which according to Dedekind has the following definition:

$$(A_1|B_1) - (A_2|B_2) = (A_1 - B_2)|(B_1 - A_2),$$

where the class $A_1 - B_2$ consists of all possible differences $a_1 - b_2$, where $a_1 \in A_1$ and $b_2 \in B_2$, and $B_1 - A_2$ is defined analogously. However, by contrast with the Dedekind completion of the field of rationals, here it can happen that the pair $(A_1 - B_2)|(B_1 - A_2)$ does not form a cut of $^*R$. For instance, if we want to find $\infty - \infty$ from the given definition, then the lower class $I-J$ contains the negative infinite numbers, and the upper class $J-I$ contains all the positive infinite numbers; the finite numbers remain outside both classes. Thus, $\infty-\infty$ is not realizable in $\mathscr{R}$ (in the framework of Dedekind's definitions) and in this we can see a parallel with one of the main indeterminancies in the classical theory of limits. We get exactly the same situation with the other indeterminate operations, such as $0\cdot\infty$, ∞/∞, 1^∞, and so on; in this connection the classical zero corresponds to the "number" $\varnothing \in \mathscr{R}$.

In other cases the operations in $\mathscr{R}$ lead to definite results, that are not without interest. Thus,

$$\infty+2=\infty\cdot2=\infty^2=\infty;\ \varnothing=\frac{1}{\infty};\ \varnothing+\varnothing=\varnothing.$$

These and similar equalities are well known in the theory of limits, where the symbol ∞ is treated in a formal sense, as the symbol of a quantity that, in the given case, tends to plus infinity, and the equalities themselves are treated in the same way. Here they become mathematically rigorous consequences of the given definitions, giving the symbol ∞ the sense of a definite mathematical object.

Now let us return to the questions we considered in §7. The fact that $\varnothing\cdot\infty$ is indeterminate unfortunately does not allow us to give a meaning to the proof we set out in §7 that $\sum |p_k|$ is infinitesimal, simply on the basis of the formal rules of operation with the "numbers" ∞ and $\varnothing=\infty^{-1}$. To overcome this difficulty we must use the following definition.

Definition. Let F be an intrinsic function of a single real variable, and A a standard number. Then the relation $F(\infty)\simeq A$ holds in either of the two cases:

1) (lower limit) $\lim\limits_{x\to\infty} F(x)=A$ in the standard sense, that is, for any positive $\varepsilon\in R$ there is a number $x_\varepsilon\in R$ such that $|F(x)-A|<\varepsilon$ for any $x\in R$, $x>x_\varepsilon$;

2) (upper limit) there is an infinite $X\in {}^*R$ such that $F(x)\simeq A$ for any infinite $x\leqslant X$.

(These definitions are not contradictory, that is, the two cases cannot give two different standard values for $F(\infty)$; this is not hard to show on the basis of the transfer principle.)

In the simplest case of a constant function $F(x)=0$ we find that $F(\infty)\simeq 0$. This is also true when $F(x)=x^{-1}$. However, if we are trying to find the correct value for $F(\infty)$, then we must write $F(\infty)=0$ in the first case, and $F(\infty)=\varnothing$ in the second. This suggests that the given definition may be stated in the form of an equality, rather than a relation of infinite closeness , where as possible values for $F(\infty)$ we could have: a) standard numbers; b) ideal "numbers" in $\mathscr{R}$ of the form (standard) $\pm\varnothing$; and c) the "numbers" $\pm\infty$. We shall not investigate further the interesting question of the construction of the analytic structure of $\mathscr{R}$. There may be many absorbing problems here, and they need a separate investigation.

As for the fact that the sum $\sum |p_k|$ given in §7 is infinitesimal, from the point of view of the definitions we have given and the interpretation of ∞ we have introduced, the computations in §7 have a completely definite meaning. A more curious argument is obtained when we prove that the sum is infinitesimal in the region $k < \infty$, using the definition in the form of an upper limit. In principle, here we could have used the computation from the proof of the theorem in §3, where we showed that the sum $\sum^{n} |p_k|$ is infinitesimal for some infinite $n \leqslant \nu$ (and then the result holds for any $n' \leqslant n$). But this calculation has a deficiency, namely the use of hypothesis (c), since the corresponding argument in §7 (that is, the proof that the sum is infinitesimal in the region $k < \infty$) is not based on (c).

This defect may be overcome as follows. We consider the greatest natural number $\mu \leqslant \nu$ (where ν is the upper bound of the summation in §7) that has the property that $|p_k| < k^{-1}$ for any $k \leqslant \mu$. The number μ is infinite, since for finite k the inequality $|p_k| < k^{-1}$ is satisfied, as the p_k are infinitesimal by hypothesis (b). Moreover, p_k will be infinitesimal for any $k \leqslant \mu$—for infinite k by the construction of μ, and for finite k by hypothesis (b).

Further, taking the μ we have found in place of ν in the computation (which we have discussed) from the proof of the theorem in §3, we find an infinite $n \leqslant \mu$ that satisfies the relation $\sum^{x} |p_k| \simeq 0$ for $x = n$. This relation will clearly also be satisfied for any $x < n$ (in place of x we need to take the integer part of x). Now using the definition we quickly find that $\sum^{\infty} |p_k| \simeq 0$, as required.

We can interpret the other arguments in §7 along the same lines, and also Euler's construction as a whole. However, the problem of constructing another interpretation seems to be of great interest. What we have in mind is to take $\nu = \infty$ (or $\omega = \infty$) in the arguments presented in §§1, 3, 4. There are perhaps no fundamental difficulties here, and the question is now that of finding a proof that does not reduce to a straight application of the given definition. In particular, the following nuance deserves attention: for $\nu = \infty$ the only factors t_k that remain under consideration are those corresponding to finite indices k. The analysis of this case probably reveals a new side to Euler's method.

We will make some further remarks about one problem. The indeterminate nature of the analytic structure of $\mathscr{R}$ can arise not only as a consequence of the Dedekind definitions of the operations. For instance, is there a whole "number" ∞ in the structure of $\mathscr{R}$? We cannot answer this question on the basis of our definitions, as any neighbourhood of ∞ in $\mathscr{R}$ contains both integers and non-standard fractions (both on the left and the right).

We need to find a wider definition which will include a reasonable answer to the question of the "wholeness" of ∞ in $\mathscr{R}$.

More generally, we can give (in a consistent way) a value in $\mathscr{R}$ for each indeterminacy (of any type and origin), taking these indeterminancies one by one (we have in mind a construction that is similar to the construction of an ultrafilter, using the axiom of choice), or in some other way, and how can we give a meaning to all this? To show that this is a non-trivial problem and that there may be interesting arguments in connection with it, we give the following example.

We assert that $\sin\infty = 0$. In fact, since

$$\infty+\pi=\infty \quad \text{and} \quad \sin(x+\pi)=-\sin x,$$

there is no alternative. Similarly, $\cos\infty = 0$. But we have $\sin^2 x+\cos^2 x = 1$ for any standard or non-standard x, that is, here we lose the associativity of composition:

$$(\sin^2 x+\cos^2 x)|_{x=\infty} \neq (\sin\infty)^2+(\cos\infty)^2.$$

§9. Representations of the exponential as a series and as a power

The aim of this section is to examine the arguments Euler used to obtain the formulae

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots, \tag{28}$$

$$e^x=\left(1+\frac{x}{\omega}\right)^{\omega}, \quad \text{where } \omega \text{ is infinite.} \tag{29}$$

We recall that the second formula is the starting point for Euler's arguments in §1, and we have written it in the form in which it was used in "Introduction to infinitesimal analysis". It is clear that instead of the equals sign we should write the symbol for infinite closeness $\simeq$. The first formula is used to determine the size of the constant factor in the product (9a) (see item 8 in our §1 and also §4 where we analyse it).

At the same time, formulae (28) and (29) were obtained by Euler in [1] in a manner entirely different from the way in which this material is presented in modern books on the foundations of mathematical analysis. Euler's arguments in this respect are very close in spirit and approach to infinity to the factorization of sine, and we consider them in consequence of this closeness, and also to make our survey of the whole of Euler's construction more complete.

Having fixed a finite number a, in §114 of Chapter VII of [1] Euler writes the equality $a^\varepsilon = 1+\psi$, where ε and also ψ are infinitesimals, and he connects these numbers by the relation $\psi = s\varepsilon$, where s is a finite number that depends on a (in the important case when $a = 10$ we have $s = 2.30258...$). Hence $a^\varepsilon = 1+s\varepsilon$.

If we now take $x = \omega\varepsilon$, where ω is an infinite natural number, then $\varepsilon s = sx/\omega$ and (§ 115)

$$a^x = (1+s\varepsilon)^\omega = \left(1+\frac{sx}{\omega}\right)^\omega = \tag{30}$$

$$= 1+sx+\frac{\omega-1}{\omega}\frac{s^2x^2}{2!}+\frac{(\omega-1)(\omega-2)}{\omega^2}\frac{s^3x^3}{3!}+\cdots = \tag{31}$$

$$= 1+sx+\frac{s^2x^2}{2!}+\frac{s^3x^3}{3!}+\cdots, \tag{32}$$

since if ω is infinite we have

$$\frac{\omega-1}{\omega}=1,\quad \frac{\omega-2}{\omega}=1,\quad \frac{\omega-3}{\omega}=1\text{, and so on,}$$

because "the larger the number we substitute for ω, the closer the fraction approaches to unity; if ω becomes larger than any given number, then the fraction becomes equal to unity" (§ 116).

Afterwards, Euler goes over for a time to logarithms, and then returns to the exponential, and in § 122 he consider the particular case when $s = 1$. He denotes the corresponding value of a according to (32) with $x = 1$ by e:

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots;$$

in passing he obtains an approximation to the value of e to 23 decimal places.

Next, to obtain formulae (28) and (29) for $s = 1$ and $a = e$ we use (32) and (30) respectively. These are introduced by Euler in § § 123 and 125, where in the first edition of the Russian translation of [1] used by the author of this article there is a misprint (on p. 122, in the 7th line from the bottom, the exponent is omitted).

(We should mention that we have somewhat changed the system of letters that Euler used. For instance, the infinite number is denoted in [1] by i and not by ω, and in place of s he used k.)

It is impossible not to remark on the simplicity and elegance of Euler's arguments by comparison with the modern way of setting out the material under consideration. As for the question of rigour, non-standard analysis enables us to obtain satisfactory and well-defined interpretations of all the elements of his construction.

Thus, let a and x be standard numbers, with $a > 0$ and ε infinitesimal. Then the difference $\psi = a^x - 1$ can be put in the form $\psi = s\varepsilon + \theta\varepsilon$, where s is the value of $(a^x)'$ at $x = 0$ and θ is an infinitesimal that depends on a and ε. (Here we are using the differentiability of the exponential function, that is, in principle there is an element of a circular argument, but we shall not pay attention to it.) Then

$$a^\varepsilon = 1+s\varepsilon+\theta\varepsilon = 1+\sigma\varepsilon,$$

where $\sigma = s + \theta$.

Now if $x = \omega\varepsilon$, where ω is an infinite natural number, then using the binomial theorem we can write

$$a^x = a^{\varepsilon\omega} = \left(1 + \frac{\sigma x}{\omega}\right)^\omega = \tag{33}$$

$$= \sum_{k=0}^{\omega} 1\left(1 - \frac{1}{\omega}\right)\cdots\left(1 - \frac{k-1}{\omega}\right)\frac{\sigma^k x^k}{k!}. \tag{34}$$

However, we also need to deduce another formula

$$a^x \simeq \sum_{k=0}^{\omega} \frac{s^k x^k}{k!}, \tag{35}$$

which represents the non-standard equivalent of (32). To do this we denote the terms of the sums (34) and (35) by v_k and w_k, respectively. As ω is infinite and θ is infinitesimal, $v_k \simeq w_k$ for any finite k. If k is infinite, then $|v_k| < k^{-2}$ and $|w_k| < k^{-2}$, since the factorial grows faster than any product of powers with finite exponents and exponentials with finite bases (of course, we can obtain concrete estimates here). Hence, we are in the domain of action of the theorem in §3, more precisely, a statement about the infinitesimal nature of the sum $\sum |p_k|$ for $p_k = v_k - w_k$ (and for $\nu = \omega$, since ν is an upper bound for the sums and products in §3). This implies that the sums (35) and (34) are infinitely close, that is, (35) follows from (34), which we wanted to prove.

Omitting some of the subtleties of the proof that to each standard s in our argument there corresponds a definite standard a (clearly, $a = e^s$), we denote by e the value of a that corresponds to $s = 1$. Then

$$e \simeq \sum_{k=0}^{\omega} \frac{1}{k!} \text{ and } e^x \simeq \sum_{k=0}^{\omega} \frac{x^k}{k!},$$

and the latter relation is the analogue and non-standard equivalent of (28).

Finally, in order to derive (29) (with $\simeq$ in place of the equals sign), it is enough to use (33) with $s = 1$ and $a = e$, showing first that

$$\left(1 + \frac{x}{\omega}\right)^\omega \simeq \left(1 + \frac{x}{\omega} + \frac{\theta x}{\omega}\right)^\omega. \tag{36}$$

The theorem in §3 helps us again here. In fact, hypothesis (a) is satisfied for $u_k = 1 + x/\omega$ (which is independent of k), since x is finite and ω is infinite, and if we set $p_k = \theta x/\omega$, then the sum

$$\sum |p_k| = \omega\frac{|\theta x|}{\omega} = |\theta x|$$

will be infinitesimal, as θ is. Thus, the theorem in §3 is applicable, and we arrive at (36).

To close this section we make some comments. The editor of the translation of the first edition of "Introduction to infinitesimal analysis"

([1], 328) showed that in the particular case $x = 1$ the formula connecting the right-hand sides of (28) and (29) was obtained by D. Bernoulli in 1728. However, the method of binomial expansion with an infinite power and the consequent representation of the coefficients with the omission of infinitesimal "distortions" leading to a series was found much earlier. In any case, such a representation had already been used by Taylor in 1714 to construct the series named after him. Taylor's arguments are given, for example, in Euler's "Differential calculus" ([2], Ch. III of the second part) and analysed in the "Introductory remarks" in [2] and, from the standpoint of non-standard analysis, by the author of this article in [13].

Another construction that includes the binomial expansion for infinitely large powers was used by Euler in [1], §§133–134, to obtain power series for the circular sine and cosine functions, where Euler, not satisfied by the simpler method connected with the expression of the circular functions by exponentials, gave a much more subtle argument, starting from de Moivre's theorem.

§10. Comparing the coefficients of non-standard polynomials

Here we return to a question for which the arguments we gave in §4 in order to compare the coefficients obtained by Euler do not give an answer in the general case. Let P and Q be two polynomials, possibly having infinitely large degree, where for any standard $x \neq 0$ we have $P(x) \simeq Q(x)$. Is it then the case that the constant terms of P and Q are equal, or perhaps infinitely close to each other? The problem obviously reduces to the following: if $P(x) \simeq 0$ for any standard $x \neq 0$, then is it true that the constant term of P is infinitesimal or exactly equal to 0? We will also consider this problem, and we start with a negative result.

Lemma. *There is a polynomial P with constant term* 1 *such that $P(x) \simeq 0$ for any standard $x \neq 0$.*

Proof. We fix an infinite natural number ω. By the Stone-Weierstrass theorem ([12], 251) there is a polynomial $Q(x)$ such that

$$\left|Q(x) + \frac{1}{x}\right| < \omega^{-1} \quad \text{for} \quad \omega^{-1} \leqslant |x| \leqslant \omega.$$

But any standard $x \neq 0$ lies in the annular region

$$K_\omega = \{x : \omega^{-1} \leqslant |x| \leqslant \omega\},$$

and so, if we multiply by x, we obtain

$$|1 + xQ(x)| < |x|\, \omega^{-1}$$

for any standard $x \neq 0$. Thus, the polynomial $P(x) = 1 + xQ(x)$ has the desired property.

Remark. If we complicate the proof somewhat we can obtain the desired polynomial in a form such that its coefficients are given by means of a standard function of the index of the corresponding power of x. In particular, the coefficients of finite powers of x can be expressed as standard numbers. Here is a sketch of this more complicated construction.

Let $P_0 = 1$. If we have constructed the polynomial P_n of degree m, then as before there is a polynomial Q_n with the property that

$$|P_n(x)+x^{m+1}Q_n(x)|<n^{-1} \text{ for any } x \text{ in the annulus } K_n,$$

and so this polynomial can be defined by P_n in a single-valued fixed way.

(In fact, we first pick out those polynomials Q_n that have the stated property and the smallest possible degree. Next we select from them only those that have rational coefficients, and finally we minimize the numerators and denominators of the coefficients represented as irreducible fractions from the lowest to the highest.)

With a view to the single-valued definition of Q_n from P_n, we take as P_{n+1} the polynomial $P_n(x) + x^{m+1}Q_n(x)$. It is clear that P_{n+1} contains all the terms of P_n and their coefficients, and in addition has only terms that have degree higher than the degree of P_n.

The given construction is an intrinsic construction in analysis, and in the system of non-standard analysis it can be extended to any infinite natural number ω. The corresponding polynomial $P = P_\omega$ "inherits" the constant term 1 from P_0 and at the same time satisfies the inequality $|P(x)| < \omega^{-1}$ on the annular region K_ω, which clearly contains all standard $x \neq 0$.

One of the possibilities for finding a positive answer to the question we are discussing consists in bounding the coefficients. If we assume that all the coefficients of the polynomial P are bounded in absolute value by a standard number M, then

$$|P(x)-P(0)| \leqslant \frac{M|x|}{1-|x|}$$

for any x, and hence taking a standard x that is small enough, we can arrange things so that $|P(x)-P(0)|$ remains smaller than any preassigned standard $\varepsilon > 0$. Thus, if in this case $P(x) \simeq 0$ for all standard $x \neq 0$, then $P(0) \simeq 0$. Essentially, this is roughly how we have argued in §4.

Finally, the method of ensuring that the coefficients of the polynomial are bounded, which was given at the end of §4 (the remark in brackets), can also be carried out in a sufficiently general setting, but we will not dwell on this.

Conclusion

The works of Leonhard Euler are probably among the mathematical works of the 18th century most strongly characterized by their systematic and exceptionally fruitful application of the infinite. However, the arguments that this article has dealt with still essentially only occur in the work of experts in non-standard analysis. Euler made other experiments with infinity which have still not been the subject of detailed study from the position of non-standard analysis, among them beautiful and complicated constructions such as the Euler-Maclaurin series, the interpolation of functions, and others (see "Differential calculus" ([2], the second part of the book), the analysis of which can lead to posing new problems both in the foundations of mathematics and possibly also in more concrete areas.

We have already mentioned one such problem—the determination of the analytic structure of the "supercontinuum" $\mathscr{R}$—at the end of §8. Another possible problem is the determination of the exact meaning and the bounds of applicability of the following statement of Euler's:

"I assume that each series must attain a definite value. However, in order to handle all the difficulties that arise here, this value should not be called the sum, since we usually associate a meaning with this word as if the sum can in fact be obtained as a result of an actual summation, and this idea has no place in divergent series...". (Quoted from [14], 29.)

Thus, Euler assumes that any series has a numerical characteristic, which could properly be called the sum, had this concept not been needed in connection with the known method of defining this sum for convergent series. But how do we define the "sum" of a divergent series? Deep ideas and concrete methods of summation, developed by Euler in "Introduction to infinitesimal analysis" [1] and particularly in "Differential calculus" [2], make a significant contribution to the foundations of the theory of the summation of divergent series. This area of the theory of functions has been successfully developed (thus, in the survey [15], which is especially devoted to different questions on the summation of divergent sequences and series, a bibliography of 870 names is given, and the author of the survey mentions that some directions have yet to be investigated). Many and various methods of summation have been developed, that is, ways of defining (or "finding") the sum of a divergent series, each of which has its own characteristics and range of applications. However, such a multiplicity of approaches is indicative of the absence of a fundamental answer to the most important question from the point of view of the foundations of mathematics: is it true that any series has a unique "natural" value of the sum, as Euler believed, and if so, how do we find that value? It would be of great interest to find an approach to this question from the standpoint of non-standard analysis.

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