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Kolmogorov's ideas in the theory of operations on sets

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CONTENTS

Foreword

The theory of operations on sets is one area of mathematics in which the role of Kolmogorov is generally recognized, as the originator of the subject and contributor of the principal concepts and the first fundamental results. It is quite true, as asserted by Novikov and Lyapunov in the survey article "Descriptive set theory" [26], that the whole theory of operations on sets originates from work carried out in the 20's by Kolmogorov. The first part of his study, dated January 1922, appeared in the paper "Operations on sets" [1], where the foundations of a general theory of operations are laid down:

— the definition of a δs-operation;

— the definition of the complement of a given operation;

— the theorem on complements, according to which every δs-operation can be used to produce from closed sets a set whose complement cannot be so obtained;

— a general method for constructing a hierarchy of classes of point sets, indexed by the finite and countable ordinals: this is based on the joint application of a given δs-operation and its complement;

— the theorem on non-vacuousness of the classes: under wide conditions on the initial operation this guarantees that the resulting hierarchy does not degenerate at a countable level.

The second part was published under the same title [2] in the third volume of Kolmogorov's selected works [6]. In it he introduced the R-transform into set theory—an exceptionally important concept, associated with the most productive line of investigation in the classical theory of operations and of very significant interest to current experts. This second part, which remained unpublished for more than half a century (it is dated February 1922) was, however, as stated in the introductory words 'From the author' in [2], made available to a number of experts in descriptive set theory.

The memoir "Operations on sets" (both parts) was a rich source of ideas for later research by those following Kolmogorov in this domain. Kantorovich and Livenson in the extensive two-part "Memoir on the analytical operations and projective sets" [10], [11] developed a general theory of analytic operations, which also covers the δs-operations, characterized within this wider class by the property of 'positivity'. The second part of their memoir includes, with an acknowledgement to Kolmogorov's "kind permission", certain material from the manuscript [2], including the definition of the R-transform.

Using the idea, included in [2], of the interchange of the R-transform with passage to the complementary operation, Lyapunov constructed ([19], [20]) what is known as the normal series of R-operations, and introduced the class of R-sets, obtained from closed sets by applying the normal series. To a large extent, Lyapunov based the proofs of the basic theorems about R-sets on an analysis and reworking of what had been done by Kolmogorov. Such results, in Lyapunov's most significant contribution "R-sets" [20], as the construction of the classes and subclasses of the hierarchy of R-sets and the proof that these are non-empty, the proof of measurability for the R-sets, the partitioning of R-sets into $\aleph_1$ Borel sets, the development of an apparatus of indices for the R-transform—all these were direct extensions of the ideas and constructions in [2].

And this important piece of research on "Operations on sets", which opened up a new area of set theory, was carried out by an 18-year-old student who had only just recently been invited by N.N. Luzin to join his group. Reminiscing about the years of his youth in the commentary "Papers in the theory of functions and set theory" [5], pp. 363–364, Kolmogorov rather delicately indicates that his plan for research in set theory turned "in a direction not envisaged by Luzin". We shall probably never know the exact form in which disapproval from the then leader of the Moscow mathematicians actually manifested itself in this situation, nor the extent to which this may have influenced a very young man taking his first steps in research. However, it is undoubtedly true that in his published work on set theory Luzin never referred to Kolmogorov's contributions, while Kolmogorov himself essentially never returned subsequently to this area in his published papers (if we except the article [3] written in honour of P.S. Aleksandrov) even though, as is stated in the commentary referred to above, he had an extensive plan for further research at the time. One can but regret the loss to set theory as a whole that was brought about by Kolmogorov's departure from the field.

With Lyapunov's work on R-sets the classical theory of operations on sets took on a definitive form, on the whole. The concepts and methods introduced by Kolmogorov had proved to be conclusive in the sphere of the traditional problems of descriptive set theory and the classical descriptive problem area(1). Later research (beginning in the 50's) had as its principal purpose the extension of classical results to the case of uncountable indices and arbitrary topological spaces, with a study of the resulting effects. Thus, the theory of operations on sets gradually diverged from the clear descriptive direction initially inherent in it and emphasized by Kolmogorov in his commentary referred to above. It is possible that this fact, and also the exceptionally difficult notation and techniques in the theory of operations in the version arising from Lyapunov's work (in sharp contrast with the brilliant style of the papers [1], [2], and [3]), was responsible for the subsequent absence of real progress in this field and its separation from the mainstream of the development of set theory.

However, the last 20–25 years have been a period of exceptionally rapid development in set-theoretical investigations, based on fundamental discoveries such as the apparatus of constructive sets, the method of forcing, the general notion of recursion, infinite games, and determinacy. The new methods have led to the complete solution (in most cases, in the form of a proof of undecidability) of many fundamental problems of set theory, such as, for example, the continuum problem, the problem of measurability of projective sets, problems of separation and uniformization for the projective classes, and problems on the constituents. Neither has this development left the theory of operations on sets to one side. Thus, at the beginning of the 80's a "game-theoretical" representation of R-sets was found, together with a simpler construction of C-sets, from which it became clear that the Borel sets, the C-sets, and the R-sets constitute the three initial levels of a certain natural hierarchy. It became possible definitively to clarify the relationship between the R-sets and the sets of the second projective level. In their turn, the concepts of the theory of operations lie at the heart of certain fundamental definitions of the general theory of recursion, and Kolmogorov's construction of the R-transform is now regarded as the first example of an infinite game with perfect information.

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(1)For the development of descriptive set theory and the main ideas and results, see the surveys [26], [13], [33], and [9].

The present author has for some time had in mind the preparation of a survey article describing the evolution of the ideas which were introduced by Kolmogorov as a basis for the theory of operations on sets, and the form that these ideas have taken or are taking in modern set-theoretical research, and also clarifying some points of terminology and priority that have not always been completely or accurately covered in the literature. It is a sad circumstance, tragic for world mathematics, that has given a reason and an opportunity for the accomplishment of this purpose.

Introduction

The present paper is not, strictly speaking, a survey article on the theory of operations on sets, and does not claim to present a complete analysis of all lines of research in this area. Rather, as can be inferred from the title, it is concerned only with research carried out under the influence of Kolmogorov's ideas, or exhibiting them from a new and more modern point of view[(1)]. Naturally, a certain amount of space (§§1 and 2) is given up to an exposition of the foundations of the theory of operations as created by Kolmogorov.

Most attention is paid to Kolmogorov's R-transform—one of the most significant discoveries in the whole of classical descriptive set theory. In §3 the initial definition of the R-transform is given, together with the modern reformulation in the language of infinite games and generalized quantifiers, and certain methods of proof in connection with these transforms. Then §4 is devoted to the method of indices (inductive analysis)—the principal technique of both the classical and, along with 'game-theoretical' analysis, also the modern theory of operations on sets. Kolmogorov introduced this method in [2] in a general form, aware of the earlier construction of indices in a special case (the A-operation) carried out by Luzin and Sierpiński. Within the framework of a 'game-theoretical' approach, the method of indices takes the form of the definition of winning positions.

In §5 we give Lyapunov's definition of R-sets by means of his construction of the operations of the normal series. In §6, modern ideas of the nature of R-sets are considered, obtained by a deep re-working and far-reaching generalizations of Kolmogorov's ideas and constructions from [2]. Then §7 gives a short survey of research on R-sets.

The concluding §8 is concerned with the Kolmogorov problem (published [4] in 1935) on the length of the Borel hierarchy, which has attracted a great deal of interest and recently received a complete solution.

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(1)For other directions of research in the theory see [13], [23]-[26], [29], [30], and [36].

§1. Introduction to the theory of operations on sets

Keeping to chronological order, we first consider the definition of a δs-operation given by Kolmogorov in [1], and examples of such operations, and we then pass to the more general concept of an analytical operation due to Kantorovich and Livenson.

The concept of a δs-operation.
Let us fix some non-empty set I—the *index set*. Subsets of this, that is, sets $U \subseteq I$, are called *chains*, and sets of chains *bases*; thus a basis is an arbitrary set $Q \subseteq \mathcal{P}(I)$, where $\mathcal{P}(I)$ denotes the set of all $U \subseteq I$.

To avoid trivialities, it is usually assumed that a basis is non-empty and does not contain the empty chain: $Q \neq \emptyset$ and $\emptyset \notin Q$. The two operations excluded by this stipulation will be referred to later.

Every basis $Q \subseteq \mathcal{P}(I)$ gives rise to an operation Φ_Q, acting on an I-indexed family $\{X_i: i \in I\} = \{X_i\}$ of sets X_i, according to the following definition [1]:

$$\Phi_Q\{X_i\} = \Phi_Q\{X_i: i \in I\} = \bigcup_{U \in Q} \bigcap_{i \in U} X_i.$$

The name of δs-*operations* has become attached to operations of the form Φ_Q; the terminology is due to Hausdorff, who arrived independently at the same definition in [34]; Kolmogorov does not give them any particular name in [1].

One and the same δs-operation can be determined, in general, by various different bases; a necessary and sufficient condition for $\Phi_Q = \Phi_P$ is that

$$(1) \qquad \forall U \in Q \exists V \in P(V \subseteq U) \wedge \forall V \in P \exists U \in Q(U \subseteq V).$$

Nevertheless, among all the bases yielding a given δs-operation Φ, we can single out the unique basis

$$(2) \qquad Q(\Phi) = \Phi\{E_i: i \in I\}, \text{ where } E_i = \{U \subseteq I: i \in U\},$$

that satisfies the condition of *completeness*, that is, together with any chain $U \in Q$ it also contains every chain $V \subseteq I$ such that $U \subseteq V$. This unique complete basis for Φ may also be obtained from any other basis Q for Φ by means of the formula

$$(3) \qquad Q(\Phi) = \{V \subseteq I: \exists U \in Q(U \subseteq V)\}.$$

Examples (Kolmogorov [1]). The class of δs-operations includes, in particular, union and intersection:

$$\bigcup_I \{X_i\} = \bigcup_{i \in I} X_i; \quad \bigcap_I \{X_i\} = \bigcap_{i \in I} X_i$$

for an I-indexed family of sets. Complete bases for the operations $\bigcup_I$ and $\bigcap_I$ are as follows:

$$Q(\bigcup_I)=\{U\subseteq I\colon\ U\neq\varnothing\};\quad Q(\bigcap_I)=\{I\}.$$

The basis is the only one possible for intersection, but $\bigcup_I$ can be given in the form Φ_Q for any basis Q containing all one-element chains $U = \{i\}, i\in I$ (but not containing the empty chain).

Another example is Aleksandrov's A-operation, which he used in [7] to solve the problem of the cardinality of Borel sets. The index set of the A-operation is the collection

$$S = \{\langle a_0, a_1, \ldots, a_m\rangle : a_0, a_1, a_2, \ldots, a_m \in \omega\}^{(1)}$$

of all strings of natural numbers a_k of arbitrary finite length $m+1$. The result of the A-operation on an S-indexed family $\{X_s\colon s\in S\}$ is defined as follows:

$$A\{X_{a_0\ldots a_m}\}=\bigcup_{\alpha}\bigcap_{m\in\omega}X_{a_0\ldots a_m},$$

where the union is taken over all infinite sequences $\alpha = \langle a_0, a_1, a_2, \ldots\rangle$ of natural numbers a_k. Thus the A-operation is the δs-operation with basis consisting of all possible chains of the form

$$U = \{\{a_0\}, \{a_0, a_1\}, \{a_0, a_1, a_2\}, \ldots\}$$

(of course this basis is not complete).

The above operations may be described as the simplest ones. The most useful way of constructing more complicated operations is by using various transformations through which new δs-operations may be constructed from existing ones. Three classical transformations of δs-operations are well known: passage to the complementary operation, superposition, and the R-transform; they were all introduced by Kolmogorov, in [1] and [2]. Leaving aside the R-transform until §3, we now consider the simpler first two.

The complementary operation.

Assuming that all the sets in question are situated within some fixed set $\mathscr{X}$, for $X\subseteq\mathscr{X}$ we denote by $\breve{X}$ the complement of X, that is, $\breve{X} = \mathscr{X} - X$. The operation $\breve{\Phi}$ *complementary* to a given δs-operation Φ is defined by

$$\breve{\Phi}\{\breve{X}_i\} = \breve{X} \leftrightarrow \Phi\{X_i\} = X.$$

The operation $\breve{\Phi}$ is a δs-operation along with Φ, and a complete basis $\breve{Q}$ for it may be derived from any basis Q of Φ:

$$\breve{Q} = \{V\subseteq I\colon\ \forall U\in Q(U\cap V\neq\varnothing)\},\tag{4}$$

(1) $\omega = \{0, 1, 2, \ldots\}$ is the set of all natural numbers.

where if Q is complete the formula can be simplified to

$$(4^*) \qquad \breve{Q} = \{V \subseteq I: I - V \notin Q\}.$$

It is easy to see that the operations $\bigcup_I$ and $\bigcap_I$ are mutually complementary. The Γ-operation, complementary to the A-operation, was constructed by Aleksandrov in [8].

Superposition.
The sets obtained as the result of applying some δs-operation may themselves serve as arguments for other operations. However, such a double application can be replaced by a single one by means of Kolmogorov's superposition scheme.

Let a δs-operation Φ be given with index set I, and let a δs-operation Φ_i, with index set I_i, be associated with each $i \in I$. Then we can define a new δs-operation $\Theta = \Phi \mid \{\Phi_i: i \in I\}$ with index set

$$J = I \mid \{I_i\} = \{\langle i, j\rangle: i \in I \wedge j \in I_i\},$$

which acts on a J-indexed family:

$$\Theta\{X_{ij}: \langle i, j\rangle \in J\} = \Phi\{\Phi_i\{X_{ij}: j \in I_i\}: i \in I\}.$$

In other words, we first construct the sets

$$X_i = \Phi_i\{X_{ij}: j \in I_i\},\ i \in I,$$

and then apply the operation Φ to these in order to obtain the final result.

In order to define a basis for Θ, we fix a basis Q of Φ and for each $i \in I$ a basis Q_i of Φ_i. The required basis for Θ can consist of all chians of the form

$$V = \{\langle i, j\rangle: i \in U \wedge j \in U_i\},$$

where $U \in Q$ and $U_i \in Q_i$ for all i.

In the important special case when each operation Φ_i is identical with some fixed δs-operation Φ', we agree to denote the superposition $\Phi \mid \{\Phi_i\}$ by $\Phi\Phi'$. For example, the operation $\bigcup_I \bigcap_J$ acts on an $I \times J$-indexed family:

$$\bigcup_I \bigcap_J \{X_{ij}\} = \bigcup_{i\in I} \bigcap_{j\in J} X_{ij}.$$

Comparison of operations.
Intuitively, an operation Ψ is stronger than an operation Φ if the action of the second can be replaced by the action of the first. This intuitive idea can be made precise in two ways, as indicated by Kolmogorov in [1].

First, Ψ is stronger than Φ (in the non-strict sense)—in symbols $\Phi \leqslant \Psi$—when there exists a mapping f of the index set J of Ψ into the index set I of Φ such that for every I-indexed family of sets $\{X_i: i \in I\}$ we have

$$(5) \qquad \Phi\{X_i: i \in I\} = \Psi\{X_{f(j)}: j \in J\}.$$

Second, denote by $\Theta(K)$ the collection of all sets which may be obtained by a single application of the δs-operation Θ to a family of sets from the class K: for example, $\bigcup_\omega (K)$ is the collection of all countable unions of sets from K. Now the second definition is that $\Phi \leqslant \Psi$ if $\Phi(K) \subseteq \Psi(K)$ for every class K.

Fortunately, the two definitions are equivalent (this is referred to as a fact in [1] and proved rigorously in [10]). Indeed, let two δs-operations Φ, Ψ satisfy $\Phi \leqslant \Psi$ in the second sense. For a family of sets E_i of the form (2) we obtain

$$\Phi\{E_i : i \in I\} = \Psi\{E_{f(j)} : j \in J\},$$

where f is a suitable mapping of J into I. It is easy to see that this mapping f also satisfies (5) for any other family $\{X_i\}$, which proves that $\Phi \leqslant \Psi$ in the first sense.

The implication in the opposite direction is trivial.

As usual, we define $\Phi \approx \Psi$ to mean that both $\Phi \leqslant \Psi$ and $\Psi \leqslant \Phi$; such operations are called equivalent. Finally, $\Phi < \Psi$ if $\Phi \leqslant \Psi$ holds but not $\Psi \leqslant \Phi$.

Example: $\bigcup_\omega \leqslant A$, where the corresponding function $f : S \to \omega$ satisfying (5) may be defined as follows:

$$f(\langle a_0, \ldots, a_m \rangle) = a_m.$$

Moreover $\bigcup_\omega < A$ in the strict sense, since it is known that the class $A(\mathrm{Int})$ of all sets obtained by the A-operation from open intervals on the real line (that is, the class of Suslin or A-sets, constructed by Suslin in [32]) is much wider than the class $\bigcup_\omega(\mathrm{Int})$ of all open sets on the line.

One equally easily sees that $\bigcap_\omega < A$.

Analytic operations.

Thus certain important operations on sets are included among the δs-operations. On the other hand, for example, the symmetric difference $X \Delta Y = (X - Y) \cup (Y - X)$, the taking of the complement of a set, and projection are not δs-operations. What is the property characterizing the δs-operations among all set-theoretical constructions? This question was considered by Kantorovich and Livenson in [10]. They call an operation Ψ acting on an I-indexed family of sets *analytic* if the question of whether a point x belongs to the set $X = \Psi\{X_i\}$ is completely determined by the set of indices i for which $x \in X_i$. More precisely, it is required that for any pair of points x, y and any two families $\{X_i\}$, $\{Y_i\}$ we have

$$\forall i (x \in X_i \leftrightarrow y \in Y_i) \to (x \in \Psi\{X_i\} \leftrightarrow y \in \Psi\{Y_i\}).$$

An analytic operation is called *positive* in [10] if from $X_i \subseteq Y_i$ for all i it follows that $\Psi\{X_i\} \subseteq \Psi\{Y_i\}$.

Each δs-operation is analytic and positive, clearly. But the converse is also true, and is proved in [10]: every positive analytic operation Φ is a δs-operation, with a complete basis obtainable from (2).

Kantorovich and Livenson also showed that an analytic operation (not necessarily positive) admits a representation

$$\Psi\{X_i\} = \Psi_Q\{X_i\} = \bigcup_{U \in Q} [(\bigcap_{i \in U} X_i) \cap (\bigcap_{i \notin U} \breve{X}_i)], \tag{6}$$

where $Q \subseteq \mathscr{P}(I)$, and in terms of this representation the positive (that is, δs-) operations are characterized among all the analytic operations by the condition of completeness (or extensiveness) of the basis Q:

$$U \in Q \wedge U \subseteq V \subseteq I \to V \in Q,$$

under which $\Phi_Q = \Psi_Q$ is a δs-operation.

Analytic operations represented in the form (6) are called *set-theoretical operations* by Lyapunov in [20].

One example of a non-positive analytic operation is the operation C of taking the complement of a set: $CX = \breve{X}$. This is a unary operation, whose index set contains just one element—for example, $I = \{1\}$. Taking the basis $Q = \{\varnothing\}$ with the single empty chain Φ, we obtain $\Psi_Q\ \{X_1\} = \breve{X}_1$; that is, $C = \Psi_{\{\Phi\}}$.

The degenerate δs-operations.

These are the "null" operation $\mathbb{0}$ with the empty basis $Q_0 = \Phi$, and the "unit" operation $\mathbb{1}$ with the basis $Q_1 = \mathscr{P}(I)$ consisting of all chains, including the empty chain Φ. The actions of these operations

$$\mathbb{0} = \Phi_{Q_0} = \Psi_{Q_0}; \quad \mathbb{1} = \Phi_{Q_1} = \Psi_{Q_1}$$

is very simple: $\mathbb{0}\{X_i\} = \varnothing$ and $\mathbb{1}\{X_i\} = \mathscr{X}$, where $\mathscr{X}$ is the underlying space in which all sets under consideration are situated (we take the intersection $\bigcap_{i \in \Phi} X_i$ over the empty chain to be equal to $\mathscr{X}$ independently of the choice of sets $X_i \subseteq \mathscr{X}$).

There exists, however, a property that distinguishes $\mathbb{0}$ and $\mathbb{1}$ from the δs-operations: for any operation $\Phi = \Phi_Q$ apart from these two, if we put $X_i = X$ for all i in the index set of Φ then we obtain $\Phi\{X_i\} = X$. Of course this is not true for $\mathbb{0}$ and $\mathbb{1}$.

§2. The Kolmogorov hierarchy

Having given the definition of a δs-operation, Kolmogorov considers in [1] the structure of the smallest class $\nabla(\Phi)$ of sets in a given space, containing all closed sets and closed under the application of a fixed operation Φ and the operation of complementation. There is a large set of combinations of Φ and complementation, but Kolmogorov finds it possible to assign the sets of $\nabla(\Phi)$ to classes of increasing complexity, indexed by finite and infinite ordinals.

Taking account of the modern system of definitions and notation in descriptive set theory, it is most convenient to use the notations $\mathbf{\Sigma}_\alpha^\Phi$, $\mathbf{\Pi}_\alpha^\Phi$, $\mathbf{\Delta}_\alpha^\Phi$ for the classes in the Kolmogorov hierarchy, where Φ is a given δs-operation and α is an ordinal number, with respect to which the construction of the classes proceeds by transfinite induction.

The initial level is formed of the classes

$$\mathbf{\Sigma}_0^\Phi = \mathbf{G}, \quad \mathbf{\Pi}_0^\Phi = \mathbf{F}, \quad \mathbf{\Delta}_0^\Phi = \mathbf{G} \cap \mathbf{F}$$

for all open sets, all closed sets, and all open-and-closed sets respectively. For $\alpha \geqslant 1$ we put

$$\mathbf{\Sigma}_\alpha^\Phi = \Phi\Big(\bigcup_{\gamma<\alpha} \mathbf{\Pi}_\gamma^\Phi\Big), \quad \mathbf{\Pi}_\alpha^\Phi = \breve{\Phi}\Big(\bigcup_{\gamma<\alpha} \mathbf{\Sigma}_\gamma^\Phi\Big)$$

and, finally, $\mathbf{\Delta}_\alpha^\Phi = \mathbf{\Sigma}_\alpha^\Phi \cap \mathbf{\Pi}_\alpha^\Phi$. Thus the class $\mathbf{\Sigma}_\alpha^\Phi$ is formed by the action of Φ on all families of sets of the classes $\mathbf{\Pi}_\gamma^\Phi$, $\gamma < \alpha$, and the class $\mathbf{\Pi}_\alpha^\Phi$ in a similar way using the complementary operation $\breve{\Phi}$ on the sets of the classes $\mathbf{\Sigma}_\gamma^\Phi$, $\gamma < \alpha$. It is clear that $\mathbf{\Sigma}_\alpha^\Phi$ consists, for every α, of the complements of the $\mathbf{\Pi}_\alpha^\Phi$ sets (and only these).

The construction of the classes stabilizes at the first ordinal of power strictly greater than that of the index set I of Φ (or at the first infinite ordinal ω, if I is finite). In particular, if I is at most countable, then stabilization occurs no later than at the first uncountable ordinal ω_1 (classically Ω), and we have

$$\nabla(\Phi) = \mathbf{\Sigma}_{\omega_1}^\Phi = \mathbf{\Pi}_{\omega_1}^\Phi = \mathbf{\Delta}_{\omega_1}^\Phi = \mathbf{\Sigma}_\beta^\Phi = \mathbf{\Pi}_\beta^\Phi = \mathbf{\Delta}_\beta^\Phi \quad \text{for } \beta \geqslant \omega_1.$$

Examples. 1. If $\Phi = \bigcup_\omega$ is countable union, then $\nabla(\Phi)$ coincides with the class B of all Borel sets in the given space, and $\mathbf{\Sigma}_\alpha^\Phi$, $\mathbf{\Pi}_\alpha^\Phi$, $\mathbf{\Delta}_\alpha^\Phi$ are the classes of the Borel hierarchy. More precisely, if we use the definitions $\mathbf{\Sigma}_\alpha^0$, $\mathbf{\Pi}_\alpha^0$, $\mathbf{\Delta}_\alpha^0$ of the latter from [41], §2, then

$$\mathbf{\Sigma}_\alpha^\Phi = \mathbf{\Sigma}_{1+\alpha}^0, \quad \mathbf{\Pi}_\alpha^\Phi = \mathbf{\Pi}_{1+\alpha}^0, \quad \mathbf{\Delta}_\alpha^\Phi = \mathbf{\Delta}_{1+\alpha}^0.$$

2. Let Φ be the A-operation. Then $\nabla(A) = C$ is "Luzin's C-domain" as it is phrased in [1], or the class of all C-sets, first constructed in Selivanovskii's paper [31]; the classes $\mathbf{\Sigma}_\alpha^A$, $\mathbf{\Pi}_\alpha^A$, $\mathbf{\Delta}_\alpha^A$ are identical with those in the hierarchy of C-sets.

Remark. In [1], Kolmogorov carried out the construction of the above hierarchy (of course with a different notation for the classes) with sets of the real line $\mathbf{R}$ in view. In modern research on descriptive set theory the underlying space is usually taken to be the more convenient Baire space $\mathcal{N} = \omega^\omega$, consisting of all infinite sequences $a = \langle a_0, a_1, a_2, \dots\rangle$ of natural numbers a_k, endowed with the Tikhonov product topology (with the discrete topology on ω). Both the spaces $\mathbf{R}$ and $\mathcal{N}$ are examples of what are (now) called Polish spaces, that is, separable, metrizable with a complete metric, and (to avoid trivialities) without isolated points. All these spaces are very much alike, if not identical, with respect to the operations considered in the theory of operations on sets, and below we shall assume that the underlying space is (perfect and) Polish.

Monotonicity.

The first requirement that might be imposed on a hierarchy of any kind is that each of the classes should contain all sets of the preceding classes, that is, for the Kolmogorov hierarchy that

$$\Sigma_\alpha^\Phi \cup \Pi_\alpha^\Phi \subseteq \Delta_\beta^\Phi \quad \text{for} \quad \alpha < \beta.$$

This condition is of course satisfied if $\beta \geqslant 2$, whereas for $\beta = 1$ we obtain the inclusion $\mathbf{G} \subseteq \Phi(\mathbf{F})$ (and the equivalent $\mathbf{F} \subseteq \breve{\Phi}(\mathbf{G})$), which is not satisfied for $\Phi = \bigcap_\omega$, for example. The simplest way of guaranteeing the inclusion is to require that the operation Φ should be stronger (in the non-strict sense, that is, $\geqslant$) than $\bigcup_\omega$, and then we obtain $\mathbf{G} \subseteq \Phi(\mathbf{F})$, since $\mathbf{G} \subseteq \mathbf{F}_\sigma$ in Polish (and in general in all metrizable) spaces. We mention that the inequality $\Phi \geqslant \bigcup_\omega$ is valid for all the operations Φ, except $\bigcap_\omega$, which are considered in practice in the theory of Borel sets, C-sets, and R-sets.

Theorem on the non-vacuousness of the classes.

The second basic question is as follows: does each level of the hierarchy contain sets which are not in the preceding level? This is in general a typical question for hierarchies of any kind, and in the present case the answer is given by the following theorem of Kolmogorov [1].

Theorem. *Suppose that the δs-operation Φ with countable index set is such that the class $\nabla(\Phi)$ is closed relative to countable union $\bigcup_\omega$. Then for all α, $1 \leqslant \alpha < \omega_1$, we have*

$$\bigcup_{\gamma<\alpha} \Sigma_\gamma^\Phi \subsetneqq \Sigma_\alpha^\Phi \quad \textit{and} \quad \bigcup_{\gamma<\alpha} \Pi_\gamma^\Phi \subsetneqq \Pi_\alpha^\Phi.$$

Proof. By induction on α, Kolmogorov constructs a sequence of δs-operations Φ_α, $\alpha < \omega_1$, with countable index sets, having the properties: 1) $\Sigma_\alpha^\Phi \subseteq \Phi_\alpha(\mathbf{F} \cup \mathbf{G})$ and 2) the class $\nabla(\Phi)$ is closed relative to Φ_α. The start of the construction is obvious: we take, for example, $\Phi_0 = \Phi$. Now for the induction step.

Suppose that $\alpha \geqslant 1$ and that all the operations Φ_γ, $\gamma < \alpha$, have already been constructed and satisfy the conditions. Define the auxiliary operation

$$\Theta = \bigcup_\alpha |\{\breve{\Phi}_\gamma\colon \gamma < \alpha\}, \quad \text{where} \quad \alpha = \{\gamma\colon \gamma < \alpha\}$$

and the operation $\Phi_\alpha = \Phi\Theta$. From the induction hypothesis it follows that $\Pi_\gamma^\Phi = \breve{\Phi}_\gamma(\mathbf{F} \cup \mathbf{G})$ for $\gamma < \alpha$, and so the class $\Theta(\mathbf{F} \cup \mathbf{G})$ consists of all countable unions of sets belonging to the classes Π_γ^Φ, $\gamma < \alpha$, one set from each of these classes. By taking the empty set in all except one of these classes, and any selected set in the other, we obtain this selected set as the union. Consequently all sets of the classes Π_γ^Φ, $\gamma < \alpha$, are included in the class $\Theta(\mathbf{F} \cup \mathbf{G})$, and it is now clear that $\Pi_\alpha^\Phi \subseteq \Phi_\alpha(\mathbf{F} \cup \mathbf{G})$. And from the fact that the class $\nabla(\Phi)$ is closed relative to Φ, to $\bigcup_\omega$, and to all the $\Phi_\gamma, \gamma < \alpha$, it follows that it is also closed relative to the new operation Φ_α.

Now, having the desired system of operations Φ_α, we suppose the contrary of the non-vacuousness theorem; that is, let, for example

$$\Sigma_\alpha^\Phi = \bigcup_{\gamma<\alpha} \Sigma_\gamma^\Phi.$$

From this it is easy to deduce that

$$\Sigma_\alpha^\Phi = \Pi_\alpha^\Phi = \Sigma_\beta^\Phi = \Pi_\beta^\Phi \quad \text{for all} \quad \beta,\ \alpha \leqslant \beta < \omega_1,$$

that is, $\Sigma_\alpha^\Phi = \Pi_\alpha^\Phi = \nabla(\Phi)$. But by the construction of Φ_α and the fact that $\mathbf{F} \cup \mathbf{G} \subseteq \mathbf{F}_\sigma$ and $\bigcup_\omega (\nabla(\Phi)) \subseteq \nabla(\Phi)$ we obtain

$$\nabla(\Phi) = \Sigma_\alpha^\Phi = \Pi_\alpha^\Phi = \Psi(\mathbf{F}), \text{ where } \Psi = \Phi_\alpha \bigcup_\omega . \tag{7}$$

This relation is the starting point for the derivation of a contradiction, and the concrete mechanism consists in the use of a further theorem, also proved by Kolmogorov in [1].

Theorem on complements. *For any δs-operation Ψ with countable index set, in the class $\Psi(\mathbf{F})$ there is a set whose complement does not belong to this class.*

Proof. We shall suppose that 1) the index set of Ψ is the set ω of natural numbers, and 2) the underlying space is the Cantor discontinuum $\mathscr{C}$ (this assumption leads to no loss of generality, since every perfect Polish space contains a closed set homeomorphic to $\mathscr{C}$).

The space $\mathscr{C}$, realized as the product of ω copies of a discrete 2-point space, consists of all infinite sequences $c = \langle c_0, c_1, c_2, \ldots\rangle$ of numbers $c_n = c(n) = 0$ or 1. Fixing a countable basis for the topology of consisting of sets C_k, $k \in \omega$, we define the set

$$Y = \{\langle c, x\rangle \in \mathscr{C}^2 \colon x \notin \bigcup_{c(k)=1} C_k\}$$

which is closed in $\mathscr{C}^2$ and universal in the sense that for every closed $X \subseteq \mathscr{C}$ there is a point $c \in \mathscr{C}$ for which

$$X = Y^{(c)}, \text{ where } Y^{(c)} = \{x\colon \langle c, x\rangle \in Y\}.$$

Further, put $Y_m = \{\langle c, x\rangle: \langle (c)_m, x\rangle \in Y\}$ for all m, where the points $(c)_m \in \mathscr{C}$ are given by the formulae $(c)_m(k) = c(2^m 3^k)$. Now, for each countable family of closed sets $X_m \subseteq \mathscr{C}$ there is a point $c \in \mathscr{C}$ such that $X_m = Y_m^{(c)}$ for every m. Consequently the set $H = \Psi\{Y_m: m \in \omega\}$ of the class $\Psi(\mathbf{F})$ is universal for this class, since

$$H^{(c)} = \Psi\{Y_m^{(c)}: m \in \omega\}.$$

For the required example we can take the set

$$Z = H^* = \{c: \langle c, c\rangle \in H\} = \Psi\{Y_m^*: m \in \omega\}$$

which is in the class $\Psi(\mathbf{F})$, since each $Y_m^* = \{c: \langle c, c\rangle \in Y_m\}$ is closed. If the complement $\breve{Z}$ of Z also belonged to the class $\Psi(\mathbf{F})$, then for a suitable point $c \in \mathscr{C}$ we should have

$$\breve{Z} = H^{(c)} = \{x: \langle c, x\rangle \in H\}$$

by the universality of H. This gives a contradiction to the definition of Z: $c \in Z \leftrightarrow \langle c, c\rangle \in H \leftrightarrow c \in \breve{Z}$.

This completes the proof of the theorem on complements.

The proof of the non-vacuousness theorem is also now complete: we consider the relation (7) and recall that $\mathbf{\Pi}_\alpha^\Phi$ consists of the complements of the sets in Σ_α^Φ.

This theorem is, of course, applicable to the Borel hierarchy ($\Phi = \bigcup_\omega$) and the hierarchy of C-sets (Φ is the A-operation), and in these particular cases it had already been established by ad hoc methods (Lebesgue [16] for the first and Selivanovskii [31] for the second). Thus it was Kolmogorov who brought to light the general principle underlying such types of reasoning. The theorem on complements for the A-operation had been proved by Suslin in the form: there exists an A-set whose complement is not an A-set [32].

We mention that for the operations $\Phi = \bigcup_\omega$ and $\Phi = A$, the operations Φ_α, constructed in the course of the proof of the non-vacuousness theorem, satisfy the equation $\mathbf{\Sigma}_\alpha^\Phi = \Phi_\alpha(\mathbf{F} \cup \mathbf{G})$ (in general, a sufficient condition for this is for example that $\bigcup_\omega \leqslant \Phi$ and $\Phi\Phi \approx \Phi$). The construction of such a system of operations for the Borel classes was given by Hausdorff [34], [35], and for the classes of the C-hierarchy by Kantorovich and Livenson [10], [12]. Assuming the above sufficient condition the non-vacuousness can be proved in the stronger form

$$\mathbf{\Sigma}_\alpha^\Phi \not\subseteq \mathbf{\Pi}_\alpha^\Phi \text{ and } \mathbf{\Pi}_\alpha^\Phi \not\subseteq \mathbf{\Sigma}_\alpha^\Phi \text{ for } \alpha < \omega_1,$$

in which it is usually stated for the Borel hierarchy (see [41], §2).

The name of non-vacuousness thoerem may cause some surprise, since in the form in which it is stated the classes are not non-empty, but strictly increasing. The fact of the matter is that the name is inherited from classical descriptive set theory, where the hierarchy normally consisted of pairwise disjoint classes—for example, in the present case, the classes

$$\Sigma_\alpha^\Phi - \bigcup_{\gamma<\alpha} \Sigma_\gamma^\Phi, \quad \Pi_\alpha^\Phi - \bigcup_{\gamma<\alpha} \Pi_\gamma^\Phi$$

whose non-vacuousness is indeed proved in [1] (of course in a different notation).

Historical commentary.
We have already mentioned that the concept of a δs-operation was discovered by Hausdorff independently of Kolmogorov, and was published in the second edition of his Mengenlehre [34] (the relevant chapter of the Russian translation [35] was prepared by Kolmogorov). Naturally a monograph by one of the leading mathematicians of the day had a wider readership than the short paper [1] in the Russian language. For this reason, in certain publications (of some authority, for example in [14], p.345) concerning this topic the δs-operations are called the Hausdorff operations. However, Kolmogorov's contribution to the theory of operations on sets was greater than Hausdorff's. In fact, the German mathematician did not arrive at the notion of the complementary operation, and his book contains neither the theorem on complementation nor the non-vacuousness theorem in their general forms, which constitute the basis of the theory. Neither does Hausdorff have the R-transform, with which our next section is concerned.

§3. The R-transform

Today, when more than half a century has passed since the appearance of "Memoir on the analytical operations and projective sets (II)", in which Kantorovich and Livenson first published material on the R-transform, it can be firmly asserted that this concept was Kolmogorov's most important discovery in the field of operations on sets, and one of the most important achievements in classical descriptive set theory overall. To a significant extent it was in advance of the level of set-theoretical thinking of the day, although it then became the basis and starting-point for very fruitful research by Kantorovich and Livenson, and a little later Lyapunov and others. However its full significance has become clear only more recently, after the development of such branches of the foundations of mathematics as infinite games, the general theory of inductive definitions, and recursions of higher type, which were unheard-of in the 20's and 30's.

Following the unexpected recovery of the second part of Kolmogorov's manuscript "Operations on sets" and its publication in [2] we are able to turn directly to the very origins of all research into the R-transform.

The classical definition.
A δs-operation $\Phi = \Phi_Q$ is fixed, with index set I and basis Q. Its square $\Phi^2 = \Phi\Phi$ (that is, the superposition of Φ with itself) acts on an I^2-indexed family (where $I^2 = I \times I$ is the Cartesian square) according to the formula

$$\Phi^2\{X_{ij}\colon \langle i, j\rangle \in I^2\} = \Phi\{\Phi\{X_{ij}\colon j \in I\}\colon i \in I\},$$

and as a basis for Φ^2 we may take the set of all chains $W \subseteq I^2$ of the form

$$W = \{\langle i, j\rangle\colon i \in U \wedge j \in U_i\},$$

where U and all the U_i belong to Q. Similarly, the δs-operation Φ^3 has index set I^3 and basis consisting of all chains of the form

$$W = \{\langle i, j, k\rangle\colon i \in U \wedge j \in U_i \wedge k \in U_{ij}\},$$

where U, U_i, U_{ij} belong to Q, acting as follows:

$$\Phi^3\ \{X_{ijk}\} = \Phi\{\Phi\{\Phi\{X_{ijk}\colon k \in I\}\colon j \in I\}\colon i \in I\}.$$

The construction can be continued, to give Φ^4, Φ^5, and so on.

But what operation ought to be taken as the limit, so to speak, of the sequence of operations Φ^n? One solution is immediately clear: the union of all the operations Φ^n, $n \in \omega$, via superposition with some simple outer operation—for example, with $\bigcup_\omega$, so that one puts $\Phi^\omega = \bigcup_\omega \mid \{\Phi^n\colon n \in \omega\}$ (construction as in the proof of non-vacuousness in the preceding section).

However Kolmogorov found a different idea, with essentially richer potential. He constructs the δs-operation $R\Phi$, with index set

$$RI = \{\langle i_0, \ldots, i_m\rangle = i \restriction m\colon m \in \omega \text{ and } i_0, \ldots, i_m \in I\}$$

of all finite (of non-zero length) strings of elements of I and with basis RQ comprising all chains $W \subseteq RI$ of the form

$$(8)\quad W = \{\langle i_0, \ldots, i_m\rangle \in RI\colon i_0 \in U \wedge i_1 \in U_{i_0} \wedge \wedge\, i_2 \in U_{i_0 i_1} \text{ and so on up to } i_m \in U_{i \restriction (m-1)}\},$$

where all the chains $U_{i \restriction m}$, $i \restriction m \in RI$, belong to Q.

The key difference between the R-transform and superposition consists in the fact that $R\Phi$ cannot be expressed directly in terms of Φ, and it is necessary to effect a transformation of the basis. Here a question of principle arises: does not the operation $R\Phi$ depend also on the choice of a specific basis Q for Φ? However all is well, because the necessary and sufficient condition (1) of §1 for the validity of $\Phi_{Q_1} = \Phi_{Q_2}$ is preserved in the passage to the bases RQ_1 and RQ_2. Thus $R\Phi$ does not depend on the choice of basis Q for Φ, and the use of the notation $R\Phi$ is justified.

Example. The A-operation is the R-transform of countable union: $A = R \bigcup_\omega$. In its turn, countable intersection $\bigcap_\omega$ is identical with the

R-transform of the unary operation $\bigcap_{\{0\}} = \bigcup_{\{0\}}$ with the one-element index set $\{0\}$ (more precisely, $R\bigcup_{\{0\}}$ becomes $\bigcap_{\omega}$ under a one-to-one correspondence between the index sets). Moreover, one can show that $RA \approx A$ and $R\bigcap_{\omega} \approx \bigcap_{\omega}$, that is in these two examples a double R-transform is equivalent to a single one. Indeed, in general $R\Phi \approx RR\Phi$ for every δs-operation Φ (see the theorem at the end of this section).

Generalized quantifiers.

This concept is needed in order to pass to the modern "game-theoretical" definition of the R-transform. In general, a characteristic feature of modern set-theoretical research has been the replacement of "operational" terms by a more convenient logical symbology in which sets are represented by relations and complementation by the sign of negation, union and intersection by the quantifiers $\exists$ and $\forall$, and so on. In this system, to δs-operations corresponds the concept of a generalized quantifier.

A (*generalized*) *quantifier* over the set I is any $Q \subseteq \mathscr{P}(I)$ (any basis in the sense of § 1), where to avoid triviality it is usually assumed that $\varnothing \notin Q$ and $Q \neq \mathscr{P}(I)$. The *logical function* of Q is given thus:

$$Qi\varphi(i) \leftrightarrow Qi \in I\varphi(i) \leftrightarrow \{i\colon \varphi(i)\} \in Q.$$

The standard quantifiers can be brought within the scope of this definition: $\exists = \{Q \subseteq I\colon Q \neq \varnothing\}$ and $\forall = \{I\}$.

A quantifier Q is called *monotonic* if for $U \subseteq V$, from $U \in Q$ it follows that $V \in Q$; therefore monotonicity of a quantifier corresponds to completeness of a basis. Only such quantifiers are normally considered.

The *dual quantifier* $\breve{Q} = \{V \subseteq I\colon I - V \notin Q\}$ has logical function connected with that of Q by the equivalence

$$\breve{Q}i\varphi(i) \leftrightarrow \neg\, Qi\, \neg\, \varphi(i).$$

For a monotonic Q the dual quantifier $\breve{Q}$ is simultaneously also a basis for the complementary operation: see (4*) of § 1. The quantifiers $\exists$ and $\forall$ are mutually dual.

The definition of a generalized quantifier agrees perfectly with the concept of an analytic operation:

$$x \in \Psi_Q\{X_i\colon i \in I\} \leftrightarrow Qi \in I \quad (x \in X_i),$$

and, if Q is monotonic, with that of a δs-operation:

$$x \in \Phi_Q\{X_i\} \leftrightarrow Qi(x \in X_i) \leftrightarrow \exists U \in Q \forall i \in U(x \in X_i). \tag{9}$$

The notion of R-transform.

We turn to the analysis of the operations Φ^n. Assuming that the basis Q of $\Phi = \Phi_Q$ is complete (this can always be achieved, for any δs-operation, by taking the completion of an arbitrary basis using formula (3) of §1)—that is, Q is a monotonic quantifier, and taking (9) into account, we have

$$x \in \Phi\{X_i:\ i \in I\} \leftrightarrow Qi(x \in X_i) \leftrightarrow \exists U \in Q\ \forall i \in U(x \in X_i);$$
$$x \in \Phi^2\{X_{ij}\}\ \ QiQj(x \in X_{ij}) \leftrightarrow \exists U \in Q\ \forall i \in U\ \exists V \in Q\ \forall j \in V\ (x \in X_{ij});$$
$$x \in \Phi^3\{X_{ijk}\} \leftrightarrow QiQjQk\ (x \in X_{ijk});$$

and so on. From this representation Kolmogorov's idea seems to be both obvious and impossible: to pass to an infinite chain of quantifiers Q and define a δs-operation RQ with index set RI by means of the relations

$$(10)\quad x \in R\Phi\{X_r:\ r \in RI\} \leftrightarrow Qi_0Qi_1Qi_2 \ldots \forall m(x \in X_{i\restriction m}) \leftrightarrow$$
$$\leftrightarrow \exists U_0 \in Q\ \forall i_0 \in U_0\ \exists U_1 \in Q\ \forall i_1 \in U_1 \ldots \forall m(x \in X_{i\restriction m}),$$

where $i \restriction m = \langle i_0, \ldots, i_m \rangle$.

Thus everything is clear and simple, except for one thing: how is one to interpret the infinite sequence of quantifiers in (10)? And this difficulty threatens to take the given definition into the realm of the impossible.

The Kolmogorov game.

The correct interpretation of (10) can be derived on the basis of the following game $\forall_{Qx}\{X_r\}$ which is played by two players, whom we shall call Q and I. These players in turn each make a "move", namely:

player Q (who begins) selects $U_0 \in Q$;
player I, knowing the "move" U_0, selects $i_0 \in U_0$;
player Q, knowing i_0 (the game has perfect information) selects $U_1 \in Q$;
player I, knowing U_1, selects $i_1 \in U_1$;

and so on. The requirements that $U_m \in Q$ and $i_m \in U_m$ constitute the rules of the game, and the first player unable to comply with them on his turn immediately loses. In particular, Q loses if $Q = \varnothing$, and wins with the first move $U_0 = \varnothing$ if $\varnothing \in Q$. If $Q \neq \varnothing$ and $\varnothing \notin Q$, as will henceforth be assumed, then at each stage the players do have the possibility of making a legal move and thus producing an infinite play

$$(11)\qquad U_0,\ i_0,\ U_1,\ i_1,\ U_2,\ i_2,\ \ldots\ (i_m \in U_m \in Q \text{ for all } m).$$

In this main case the outcome of the play is defined as follows: if $x \in X_{i\restriction m}$ for all m, then the winner is taken to be Q, and in the contrary case (that is, if $x \notin X_{i\restriction m}$ for at least one m) the winner is I.

Now we can interpret the second line in (10) as possession by the player Q of a system of moves in the game $\forall_{Qx}\{X_r\}$ which ensures for him a win regardless of the way the opponent plays. This interpretation can be formulated more precisely in terms of the concept of a winning strategy.

In general, a *strategy* for the player Q in a game of the type $\forall_{Qx}\{X_r\}$ is any rule for choosing the moves, depending on the preceding moves of the opponent—that is, any function σ: $RI \cup \{\Lambda\} \to Q$, where Λ is the empty string (which needs to be considered here in order to determine the opening move U_0). A play (11) corresponds to the strategy σ if $U_m = \sigma(i{\restriction}m)$ for all m, that is, when

$$U_0 = \sigma(\Lambda), \quad U_1 = \sigma(i_0), \quad U_2 = \sigma(i_0, i_1), \quad U_3 = \sigma(i_0, i_1, i_2),$$

and so on. Finally, σ is called a *winning strategy* (WS for short) for the player Q in the game $\forall_{Qx}\{X_r\}$ if each player of this game corresponding to the strategy σ ends with a win for Q. Thus a rigorous form of definition (10) is

$$(12) \qquad x \in R\Phi\{X_r\} \leftrightarrow Q \text{ has a WS in the game } \forall_{Qx}\{X_r\}.$$

Let us see that this definition is equivalent to the classical one considered at the beginning of the section. Let $x \in R\Phi\{X_r\}$ in the sense of the classical definition, that is, there exists a chain W of the form (8) giving $x \in X_r$ for all $r \in W$. Then the strategy defined by $\sigma(\Lambda) = U$ and $\sigma(i{\restriction}m) = U_{i{\restriction}m}$ for all strings $i{\restriction}m = \langle i_0, \ldots, i_m\rangle \in RI$ will be winning for Q, that is, $x \in R\Phi\{X_r\}$ in the sense of the definitions (10)–(12). The converse implication is equally simple. (As can be seen from this argument, the chains of the canonical basis RQ for $R\Phi$ are in one-to-one correspondence with the winning strategies for Q in the game $\forall_{Qx}\{X_r\}$.)

Determinacy.
Player I may also posses a winning strategy: in the game $\forall_{Qx}\{X_r\}$ this will be any function τ given on the set of all strings $\langle U_0, \ldots, U_m\rangle$ of chains $U_n \in Q$ such that

1) $\tau(U_0, \ldots, U_m) \in U_m$ for all m and

2) for any play (11), if $i_m = \tau(U_0, \ldots, U_m)$ for all m (corresponding to the strategy τ) then there exists an m such that $x \notin X_{i{\restriction}m}$.

Of course it is impossible for both players to have a winning strategy for one and the same game, because otherwise a play in which both play according to these strategies would give a contradiction.

Therefore there remain two possibilities: either one of the players has a WS (and the other does not)—and then the game is called *determined*; or neither of them has a WS. In the next section we shall prove that all Kolmogorov games are determined.

The complementary transform.
By analogy with the definition of a complementary operation, the transform $\breve{R}$ complementary to R should be so defined that the operation $\breve{R}\breve{\Phi}$ is complementary to $R\Phi$, that is,

$$(13) \qquad \breve{R}\breve{\Phi}\{\breve{X}_r\colon r \in RI\} = \breve{X} \leftrightarrow R\Phi\{X_r\} = X.$$

The explicit "game-theoretical" definition of $\breve{R}$ is given by

$$(10^*) \qquad x \in \breve{R}\Phi\{X_r\} \leftrightarrow Qi_0Qi_1Qi_2 \ldots \exists m(x \in X_{i{\restriction}m}),$$

and a precise interpretation of this definition is obtained on the basis of the game $\exists_{Qx}\{X_r\}$, which is like the game $\forall_{Qx}\{X_r\}$ in all respects except for the definition of the winner: Q wins if $x \in X_{i\upharpoonright m}$ for at least one m. The derivation of (13) from this definition may be carried out by the following argument:

$$\begin{aligned} & x \in R\Phi\{X_r\} \leftrightarrow \\ & \leftrightarrow \exists U_0 \in Q\ \forall i_0 \in U_0\ \exists U_1 \in Q\ \forall i_1 \in U_1\ \ldots\ \forall m(x \in X_{i\upharpoonright m}) \leftrightarrow \\ & \leftrightarrow \neg \forall U_0 \in Q\ \exists\ i_0 \in U_0\ \forall U_1 \in Q\ \exists i_1 \in U_1 \ldots\ \exists m(x \notin X_{i\upharpoonright m}) \leftrightarrow \\ & \leftrightarrow \neg \exists V_0 \in \breve{Q}\ \forall i_0 \in V_0\ \exists V_1 \in \breve{Q}\ \forall i_1 \in V_1\ \ldots\ \exists m(x \notin X_{i\upharpoonright m}) \leftrightarrow \\ & \qquad \leftrightarrow \neg \breve{Q} i_0 \breve{Q} i_1 \breve{Q} i_2\ \ldots\ \exists m(x \in \breve{X}_{i\upharpoonright m}) \leftrightarrow x \in \breve{R}\breve{\Phi}\{\breve{X}_r\}. \end{aligned}$$

The passage from the second line to the third is a formulation of the assertion of determinacy of the game $\forall_{Qx}\{X_r\}$. The passage from the third line to the fourth is valid because the choice in each chain $U \in Q$ of an element with the specified property is equivalent to the choice of one chain V in the complementary basis $\breve{Q}$ which is completely composed of such elements.

Generalizations.

If a sequence of δs-operations Φ_m, $m \in \omega$, is given, then their R-convolution $R\langle \Phi_m : m \in \omega\rangle$ can be defined by writing in the right hand side of definition (10) an infinite sequence of distinct quantifiers: $Q_0 i_0 Q_1 i_1 Q_2 i_2 \ldots$ (where Q_m is a complete basis for Φ_m); the interpretation by means of the corresponding game does not give any difficulty even in the case when the operations Φ_m have different index sets. In [20], Lyapunov defined the R-transform in a still more complicated case: when the next operation depends on the string of indices fixed at the preceding steps, that is, in the game-theoretical form, with a quantifier prefix of the form

$$Q i_0 Q_{i_0} i_1 Q_{i_0 i_1} i_2\ \ldots\ Q_{i\upharpoonright m} i_{m+1}\ \ldots.$$

There are also similar generalized variants of the complementary transform $\breve{R}$, while below in §6 we shall consider a further (and perhaps more interesting) method of generalization.

Transformation of quantifiers.

The construction of the operation $R\Phi$ using formula (10) can be carried over to the definition of the logical function of a monotonic quantifier $[RQ]$ over a set RI:

$$[RQ]r\varphi(r) \leftrightarrow Q i_0 Q i_1 Q i_2\ \ldots\ \forall m \varphi(i\upharpoonright m).$$

Thus $[RQ]$ is the complete basis of the operation $R\Phi$ obtained from the canonical basis RQ by formula (3) of §1, that is,

$$[RQ] = \{S \subseteq RI\colon \exists W \in RQ(W \subseteq S)\}.$$

The quantifier $[\breve{R}Q]$ can also be defined in exactly the same way.

In the unpublished paper [55] (there is a summary in [61]), Harrington studies the complexity of the transformation $Q \to Q^+ = [\breve{R}(Q\breve{Q})]$ from the point of view of recursion in the higher types. It follows from his results that the transformation $Q \to Q^+$ occupies a quite specific position among the objects of type 3.

The R-transform and the power of operations.

We recall that the relations $\leqslant$ and $\approx$, by means of which the power of δs-operations can be compared, have been defined in §1.

Theorem. (a) $\Phi \leqslant R\Phi$, $\Phi^2 \leqslant R\Phi$, *and in general* $\Phi^n \leqslant R\Phi$;

(b) *if* $\Phi \leqslant \Psi$, *then* $R\Phi \leqslant R\Psi$;

(c) $(R\Phi)(R\Phi) \approx R\Phi$—*normality of* $R\Phi$ *in the sense of* [1];

(d) $RR\Phi \approx R\Phi$.

Assertions (a) and (c) were proved by Kolmogorov in [2], and (b) and (d) by Kanotrovich and Livenson in [11]. Here we shall give a proof of (c) which is in essence very close to Kolmogorov's argument, but in a form using infinite sequences of quantifiers that makes the idea of the classical proof completely transparent.

We need to prove that the doubly infinite sequence of quantifiers in the expression

$$(14) \qquad Qi_0Qi_1Qi_2 \ldots \forall m Qj_0Qj_1Qj_2 \ldots \forall l (x \in X_{i\restriction m, j\restriction l})$$

can be replaced by a singly infinite sequence after a re-indexing of the sets $X_{i\restriction m, j\restriction l}$.

Let us denote by $\varphi(m, x)$ the expression after $\forall m$ in (14). The formula $Qi_0Qi_1Qi_2 \ldots \forall m \varphi(m, x)$ obtained by this abbreviation can be written as follows:

$$Qi_0[\varphi(0, x) \wedge Qi_1[\varphi(1, x) \wedge Qi_2[\varphi(2, x) \wedge \ldots]]].$$

Then (14) takes the form

$$\begin{aligned}(15) \quad Qi_0\,[Qj_{00}Qj_{01} \ldots \forall l\,(x \in X_{i\restriction 0, j_0\restriction l}) \wedge \\ \wedge\, Qi_1\,[Qj_{10}Qj_{11} \ldots \forall l\,(x \in X_{i\restriction 1, j_1\restriction l}) \wedge \\ \wedge\, Qi_2\,[Qj_{20}Qj_{21} \ldots \forall l\,(x \in X_{i\restriction 2, j_2\restriction l}) \ldots]]],\end{aligned}$$

where $j_m\restriction l = \langle j_{m0}, \ldots, j_{ml}\rangle$. In this expression the block of quantifiers $Qj_0Qj_1Qj_2 \ldots$ has been repeated, as it were, countably many times, and the copy occurring in the row beginning with Qi_m has been the additional index m, taking j_l to j_{ml}. This has been done so that all the quantifiers Qi_m and Qj_{ml} can be arranged in a sequence $Qk_0Qk_1Qk_2$ ..., preserving 1) the natural order among the quantifiers Qi_m and 2) the natural order within each of the sequences $Qi_mQj_{m0}Qj_{m1}\ \ldots$. After this rearrangement, (15) takes the new form

$$Qk_0Qk_1Qk_2 \ldots (\ldots\ldots\ldots\ldots\ldots), \tag{16}$$

where the string of dots in parentheses stands for the conjunction of all possible expressions $x \in X_{i\restriction m, j_m\restriction l}$ in which $m, l \in \omega$, and each k_n is either i_m or j_{ml}. We now define

$$Y_{k\restriction n} = X_{i\restriction m, j_m\restriction l} \text{ for } k_n = j_{ml},$$

and

$$Y_{k\restriction n} = X_{i\restriction m, j_m\restriction 0} \text{ for } k_n = i_m.$$

It is easy to see that

$$(16) \leftrightarrow Qk_0Qk_1 \ldots \forall n(x \in Y_{k\restriction n})$$

as required.

This proof can be expressed completely rigorously in terms of strategies.

Historical remarks.

1. The classical theory of the R-transform, based on Kolmogorov's definition and his method of indices (see the preceding section), was worked out mainly through the efforts of Lyapunov [20]–[26]; in particular, the idea of the complementary transform $\breve{R}$ is contained in [24]. Lyapunov's most important results in this field concern R-sets, and will be discussed in §§5 and 6 below.

2. Recent research on the R-transform was initiated in Hinman's papers [57], [58], where the construction is described from the point of view of recursion theory. Then at the end of the 60's the intensive development of a general theory of inductive definitions began, including a line of research concerned with the investigation of generalized quantifiers (see [37], [40], [43], [56], [61], [66]). In [44], Aczel gave the definition of the quantifier $Q^+ = [\breve{R}(Q\breve{Q})]$, in our notation, with the logical function

$$Q^+r\varphi(r) \leftrightarrow Qi_0\breve{Q}i_1Qi_2\breve{Q}i_3 \ldots \exists m\varphi(i\restriction m)$$

and indicated how to treat it using the associated game. From this was extracted the 'game-theoretical' definition of the R-transform, which was also published independently by Hinman [59]. The most complete exposition of the technical theory of the R-transform in its modern form is to be found in [45].

3. The general concept of infinite games, which includes the Kolmogorov games, was introduced by Gale and Stewart [54] in 1954. In the 70's, methods associated with these games took over a central role in descriptive set theory, and new results obtained with their help were added to the elegant theory of projective sets, in which all the odd levels of the projective hierarchy are very close in their properties to the first projective level (already rather thoroughly investigated by classical methods in the 20's and 30's), and all the even levels to the second level. The axioms *AD* and *PD*, asserting the determinacy of games of specific types, are quite seriously considered as genuine candidates for addition to the traditional Zermelo-Fraenkel axiomatic systems *ZF* and *ZFC*. (All this is discussed in more detail in [9], [36], [37], [41], and [66].) Thus the question of priority in the invention of infinite games is of some importance.

4. Usually this question is settled with a reference to the work of the Polish mathematicians Banach and Mazur at the beginning of the 30's, written down in what is known as the 'Scottish Book' and published only considerably later [64] (see also [42], [53], and [66]). Banach and Mazur considered certain rather elementary games, without formulating the general concept.

Now, however, after a 'game-theoretical' definition of the R-transform is known, it is possible, and indeed obligatory, to put the discovery of infinite games ten years earlier and to associate it with the research of Kolmogorov, who to all intents and purposes proposed the historically first example of an infinite game.

§4. Inductive analysis. Indices

Two principal investigative techniques are known in connection with the R-transform: the 'direct method' involving the consideration of bases, games, strategies, and the passage from one game to another under various transformations of sequences of quantifiers and so on, and the method of indices or, in modern terminology, inductive analysis.

As applied to the A-operation (which, we recall, is the R-transform of countable union), the apparatus of indices was introduced in the very first papers on A-sets by Suslin [32] and Luzin and Sierpiński [18]. However in [2] Kolmogorov gave an elaboration of the method for a perfectly general situation, where subtleties not arising in the special case acquire an essential significance.

Let us fix a δs-operation $\Phi = \Phi_Q$, given by a complete basis Q, that is, a monotonic quantifier over some set I. Let us fix also a family $\{X_r : r \in RI\}$ of sets X_r situated in some specified underlying space. The aim of inductive analysis is to investigate the sets $X = R\Phi\{X_r\}$ and $\breve{X} = \breve{R}\breve{\Phi}\{\breve{X}_r\}$ (which, as will be shown, is the complement of X in the underlying space; as usual, the sets $\breve{X}_r$ are taken to be the complements of the X_r) by replacing the operations $R\Phi$ and $\breve{R}\breve{\Phi}$ by the special mechanism of multiple application of the simpler initial operations Φ and $\breve{\Phi}$.

The inductive construction.
We first introduce the set $\Xi = RI \cup \{\Lambda\}$, completing the collection of all strings RI by adding the empty string Λ, and for this additional string we denote by X_Λ the underlying space; thus $\breve{X}_\Lambda = \varnothing$. Following Kolmogorov [2], by induction on the ordinal α we define the sets $X_{s\alpha}$ $(s \in \Xi)$ and their complements $\breve{X}_{s\alpha}$:

$$X_{\Lambda 0} = X_\Lambda, \quad \breve{X}_{\Lambda 0} = \varnothing, \quad X_{i\restriction m,0} = \bigcap_{n\leqslant m} X_{i\restriction n}, \quad \breve{X}_{i\restriction m,0} = \bigcup_{n\leqslant m} \breve{X}_{i\restriction n}.$$

$$X_{s\alpha} = \Big(\bigcap_{\gamma<\alpha} X_{s\gamma}\Big) \cap \Phi\Big\{\bigcap_{\gamma<\alpha} X_{si,\gamma} : i \in I\Big\}, \tag{17}$$

$$\breve{X}_{s\alpha} = \Big(\bigcup_{\gamma<\alpha} \breve{X}_{s\gamma}\Big) \cup \breve{\Phi}\Big\{\bigcup_{\gamma<\alpha} X_{si,\gamma} : i \in I\Big\}$$

for $\alpha > 0$; here si denotes the string s extended by the element i: for example $\langle 1, 2\rangle 3 = \langle 1, 2, 3\rangle$.

For a fixed s the sets $\breve{X}_{s\alpha}$ are increasing and consequently the $X_{s\alpha}$ decreasing, with increasing α, and therefore the whole process of inductive construction via (17) must stabilize at some stage. The limiting forms of the sets $X_{s\alpha}$ and $\breve{X}_{s\alpha}$ obtained at this stage are denoted by $X_{s\infty}$ and $\breve{X}_{s\infty}$. (It is easy to see that when I is at most countable we have $X_{s\infty} = X_{s\omega_1}$ and $\breve{X}_{s\infty} = \breve{X}_{s\omega_1}$.)

Interpretation: winning positions.
The game-theoretical approach to the R-transform makes it possible to give an elegant interpretation of the above construction in terms of winning positions. We recall that membership of some point x in the set $X = R\Phi\{X_r\}$ is defined by player Q's possession of a winning strategy in the game $\mathbf{V}_{Qx}\{X_r\}$; and one way of searching for this, by analogy with real-life games such as chess, is to look for winning positions, that is, those choices of some number of initial steps following which a guaranteed win for one of the players becomes obvious.

After player I has made move i_m in the game $\mathbf{V}_{Qx}\{X_r\}$, the subsequent course of the game and its outcome cannot depend on the moves $U_0, \ldots, U_m$ of player Q, that is, what must be considered as a *position* in this game is any string $\langle i_0, \ldots, i_m\rangle = i\restriction m \in RI$, and also the empty string Λ, the initial position.

Lemma 1. *Every position s in the set $S_{x\infty} = \{s \colon x \in X_{s\infty}\}$ is a winning one for player Q in the game $\mathbf{V}_{Qx}\{X_r\}$.*

Proof. The auxiliary sets $S_{x\alpha} = \{s\colon x \in X_{s\alpha}\}$ satisfy certain relations that follow from definition (17). It is convenient to write these in the form of equivalences:

$$s \in S_{x\alpha} \leftrightarrow s \in \bigcap_{\gamma<\alpha} S_{x\gamma} \wedge Qi\,(si \in \bigcap_{\gamma<\alpha} S_{x\gamma}),$$

and in addition $S_{x0} = \{\Lambda\} \cup \{i{\restriction}m\colon \forall n \leqslant m\ (x \in X_{i\restriction n})\}$. Therefore for the limit set $S_{x\infty} = \bigcap_\alpha S_{x\alpha}$ we have

$$s \in S_{x\infty} \leftrightarrow Qi(si \in S_{x\infty}) \leftrightarrow \exists U \in Q\ \forall i \in U\ \ (si \in S_{x\infty}).$$

This means that in any position $s \in S_{x\infty}$ the player Q has a move $U \in Q$ such that for every answering move $i \in U$ by the opponent the extended string si again belongs to $S_{x\infty}$. Thus in the game with position $s \in S_{x\infty}$, player Q has a strategy guaranteeing that no subsequent positions depart from the set $S_{x\infty}$, thus guaranteeing a win for Q, since $S_{x\infty} \subseteq S_{x0}$.

***Lemma* 2.** *All positions in the set $\breve{S}_{x\infty} = \{s\colon x \in \breve{X}_{s\infty}\}$ are winning positions for player I.*

Proof. The sets $\breve{S}_{x\alpha} = \{s\colon x \in \breve{X}_{s\alpha}\}$, the complements in Ξ of the sets $S_{x\alpha}$ defined above, satisfy the relations

$$s \in \breve{S}_{x\alpha} \leftrightarrow s \in \bigcup_{\gamma<\alpha} \breve{S}_{x\gamma} \vee \breve{Q}i\,(si \in \bigcup_{\gamma<\alpha} S_{x\gamma}) \leftrightarrow s \in \bigcup_{\gamma<\alpha} \breve{S}_{x\gamma} \vee \forall U \in Q\, \exists\, i \in U\,(si \in \bigcup_{\gamma<\alpha} \breve{S}_{x\gamma}),$$

where $\breve{Q}$ is the dual quantifier to Q, with logical function $\exists\ V \in \breve{Q}\ \forall i \in V$, or the equivalent $\forall U \in Q\ \exists i \in U$. In addition, for $\alpha = 0$

$$\breve{S}_{x0} = \{i{\restriction}m \in RI\colon \exists n \leqslant m\ \ (x \notin X_{i\restriction n})\}.$$

From these two relations we immediately conclude, by induction on α, that every position $s \in \breve{S}_{x\alpha}$ is a win for player I. But $\breve{S}_{x\infty} = \bigcup_\alpha \breve{S}_{x\alpha}$. The proof is complete.

***Corollary* 1.** *The Kolmogorov game $\forall_{Qx}\{X_r\}$ is determined.*

Indeed, the initial position Λ belongs to one of the mutually complementary sets $S_{x\infty}$ and $\breve{S}_{x\infty}$, and is therefore, by the lemmas, a winning position for one of the players.

We recall that the determinacy of the game $\forall_{Qx}\{X_r\}$ implies the consequential complementation property (13) of the transform $\breve{R}$ defined by formula (10*) of the preceding section.

***Corollary* 2.** *The sets $X = R\Phi\{X_r\}$ and $\breve{X} = \breve{R}\breve{\Phi}\{\breve{X}_r\}$ are mutually complementary.*

Indices.
Parallel with the inductive construction (17), in [2] Kolmogorov gives a definition of the indices. For the system of notation adopted here, $\mathrm{Ind}_s x = \infty$ for $x \in X_{s\infty}$, while if $x \in \breve{X}_{s\infty}$, then $\mathrm{Ind}_s x$ is the least ordinal α such that $x \in \breve{X}_{s\alpha}$.

Kolmogorov's theorem [2]. *We have*

$$X = X_{\Lambda 0} = \bigcap_{\alpha} X_{\Lambda\alpha} = \{x\colon\ \Lambda \in S_{x\infty}\} = \{x\colon\ \mathrm{Ind}_{\Lambda} x = \infty\};$$

$$\breve{X} = \breve{X}_{\Lambda 0} = \bigcup_{\alpha} \breve{X}_{\Lambda\alpha} = \{x\colon \Lambda \notin S_{x\infty}\} = \{x\colon\ \mathrm{Ind}_{\Lambda}\ x < \infty\}.$$

The proof is contained in Lemmas 1 and 2: for example,

$$x \in X \leftrightarrow Q \text{ has a WS in the game } \forall_{Qx}\{X_r\} \leftrightarrow \Lambda \in S_{x\infty}.$$

Remark 1. Kolmogorov-style inductive analysis was the principal technique used by Lyapunov [20]-[25] in developing the classical theory of R-sets. In [20] he introduced the special term 'T-operations' for those given by the inductive construction of formula (17).

Remark 2. With the development of a general theory of inductive definitions (see [37] and [66]) it has become clear that the correspondence, expressed in Kolmogorov's theorem, between a single application of the operations $R\Phi$ and $\breve{R}\breve{\Phi}$ and the one-step inductive process (17) with operations Φ and $\breve{\Phi}$ represents a very deep principle of a general nature, amounting to the following. Suppose that each step of some inductive construction is carried out using a fixed system of operations. Then for a construction of "contracting" type (if the sets being constructed decrease, as for example in the case of the $X_{s\alpha}$), the final result can be obtained from a single application of an operation which is the $\breve{R}$-transform of the superposition of the original operations, while for "expanding" constructions the R-transform must be taken. Moschovakis's theorem to this effect can be found in [66] or in [37], p.255, where possible generalizations are referred to on p.256.

Remark 3. The reader acquainted with the proof of the Gale-Stewart theorem [54] on the determinacy of open games (for a Russian version see [39]) will notice that the proofs of Lemmas 1 and 2 and Corollary 1 (which are direct translations of Kolmogorov's reasoning in [2] via the correspondence between chains and strategies explained in §3) are completely identical in essence with the arguments in [54]. The truth of the matter is that in the game $\forall_{Qx}\{X_r\}$ the set of all "plays" $i = \langle i_0, i_1, i_2, \ldots\rangle$ which conclude with a victory for player I is open in the space I^{ω}.

It can be shown that many known proofs of the determinacy of games of one or another class fall rather naturally into the scheme of inductive analysis of the corresponding operations.

§5. R-sets: construction of the hierarchy

In his concluding remarks following the text of [2], Kolmogorov mentions that any set on the real line can be constructed from open intervals by means of an associated δs-operation, having as basis chains of open intervals contracting on the points of this set—so that there is no particular point in studying completely general operations; on the other hand, it would be more interesting to pass to "special classes of sets, generated by sufficiently simple operations". But which operations are to be regarded as sufficiently simple?

The correct approach here can be deduced from an analogy with Kolmogorov's hierarchy of sets: once we are considering classes of sets closed under a given operation Φ and under complementation (see §2), it is also natural to consider a collection of δs-operations closed under R-transforms and taking the complement of an operation, and also, for convenience, under superposition and containing, together with any operation, all equivalent operations, that is which give exactly the same sets when applied to any families of sets of a fixed class.

Such systems of operations, limited to δs-operations with index sets that are at most countable, were called R-closed by Lyapunov in [20], and among these he picked out the smallest R-closed system $\mathscr{R}_0$ containing the operation $\mathrm{E} = \bigcup_{\{0\}} = \bigcap_{\{0\}}$ as a basis. From the examples considered in §3 it is clear that $\mathscr{R}_0$ includes the operations $\bigcup_\omega$ and $\bigcap_\omega$, the A-operation, and Aleksandrov's Γ-operation [8] complementary to it, together with the other δs-operations obtained from E using various combinations of the transformations referred to in the definition of being R-closed. However in all this variety it is possible to distinguish a definite order, using the same idea of alternating a 'positive' transformation with a passage to the complementary object that leads to Kolmogorov's hierarchy of sets.

The normal series of operations.
This consists of δs-operations R_α, $\breve{R}_\alpha$, indexed by the finite and denumerable ordinals α. The construction is by induction on α:

$R_0 = \bigcup_\omega$ is the initial operation;

$\breve{R}_\alpha$ is the complementary operation to R_α;

$$R_\alpha = R\left(\bigcap_\alpha |\{\breve{R}_\gamma \colon \gamma < \alpha\}\right) \quad \text{for} \quad \alpha \geqslant 1$$

(that is, R_α is the R-transform of the superposition of all the operations R_γ, $\gamma < \alpha$, with $\bigcap_\alpha$ as the outer operation).

The source of this construction is [2], where Kolmogorov constructs the initial operations of the normal series using the scheme

$$\left(\bigcap_\omega = R_0\right) \to \left(\bigcup_\omega = \breve{\bigcap_\omega} = \breve{R}_0\right) \to \left(A = R \bigcup_\omega \approx R_1\right) \to$$
$$\to (\Gamma = \breve{A} = \breve{R}_1) \to (H = R\Gamma \approx R_2).$$

Exactly this definition is given by Lyapunov in [19]. We have used the same definition except for adding $\bigcap_\omega$ and $\bigcup_\omega$ as the zero level; Lyapunov begins the normal series with A and Γ.

The following simple properties of the operations of the normal series are given in [20]; they are easily verified with the help of the theorems on properties of the R-transform in §3 and the complementation theorem of §1.

1. All operations of the normal series are included in $\mathscr{R}_0$.
2. If $\Phi \in \mathscr{R}_0$, then $\Phi \leqslant R_\alpha$ for some $\alpha < \omega_1$.
3. $R_\alpha R_\alpha \approx R_\alpha$ and $\breve{R}_\alpha \breve{R}_\alpha \approx \breve{R}_\alpha$ for all α.
4. $R_\gamma < R_\alpha$, $R_\gamma < \breve{R}_\alpha$, $\breve{R}_\gamma < R_\alpha$, $\breve{R}_\gamma < \breve{R}_\alpha$ for $\gamma < \alpha$.

We remark on two finer points in the construction. It is easy to see that $R_\alpha \approx R\breve{R}_\gamma$ for $\alpha = \gamma + 1$, which means that $R_{\gamma+1} = R\breve{R}_\gamma$ can be taken as the definition of R_α at non-limit stages, as is done for example in [47], leaving the third part of the above definition in force for limit α. And it is also easy to see that

$$R_\alpha \approx R\left(\bigcap_\alpha \|\{R_\gamma \breve{R}_\gamma\colon \gamma < \alpha\}\right),$$

so that this can also be taken as a basis for the construction, as is done in fact by Lyapunov in [20].

R-sets.

According to the definition in [19], [20], these comprise the smallest class of sets, in the space under consideration, containing all open sets (equivalently, in Polish spaces, all closed sets or all Borel sets) and closed under all the operations R_α and $\breve{R}_\alpha$. The hierarchy of R-sets consists of the sequence of mutually complementary classes

$$\mathbf{\Sigma R}_\alpha = R_\alpha(\mathbf{F}), \quad \mathbf{\Pi R}_\alpha = \breve{R}_\alpha(\mathbf{G}),$$

from which the class $\mathbf{\Delta R}_\alpha = \mathbf{\Sigma R}_\alpha \cap \mathbf{\Pi R}_\alpha$ is obtained. (Traditionally these are denoted by R_α, CR_α, BR_α respectively.)

Thus $\mathbf{\Sigma R}_0 = \mathbf{F}$ and $\mathbf{\Pi R}_0 = \mathbf{G}$ are the classes of closed and open sets respectively, and $\mathbf{\Sigma R}_1 = \mathbf{A}$ is the class of Suslin or A-sets, and so on.

The classes $\mathbf{\Sigma R}_\alpha$, $\mathbf{\Pi R}_\alpha$ possess properties which are exactly analogous to those of the corresponding operations described above and are consequences of these (and of the fact that $\mathbf{G} \subseteq \mathbf{F}_\sigma$ in Polish spaces). Thus:

1. Every R-set is in one of the classes $\mathbf{\Sigma R}_\alpha$, $\mathbf{\Pi R}_\alpha$—and then in all later classes.

2. If $\gamma < \alpha < \omega_1$, then $\mathbf{\Sigma R}_\gamma \cup \mathbf{\Pi R}_\gamma \subseteq \mathbf{\Delta R}_\alpha$.

3. The classes $\mathbf{\Sigma R}_\alpha$, $\mathbf{\Pi R}_\alpha$ are closed under the operations R_α and $\breve{R}_\alpha$ respectively. From this and the above it follows that

$$\mathbf{\Sigma R}_\alpha = R_\alpha(\mathbf{B}) \quad \text{and} \quad \mathbf{\Pi R}_\alpha = \breve{R}_\alpha(\mathbf{B}) \quad \text{for} \quad \alpha \geqslant 1.$$

A finer hierarchy of subclasses, defined by $\mathbf{\Sigma R}_{\alpha\xi} = \Sigma_\xi^{R_\alpha}$ (in the sense of §1), can also be considered. Lyapunov [20], [21] established that the class $\nabla(R_\alpha) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma R}_{\alpha\xi}$ formed from these subclasses is identical with $\mathbf{\Delta R}_{\alpha+1}$ only for $\alpha = 0$ (essentially, this is Suslin's theorem [32]), while for $\alpha \geqslant 1$ it is a proper subclass.

We should note that $\nabla(\mathbf{R}_0) = \mathbf{\Delta R}_1 = \mathbf{B}$ is the class of Borel sets, while $\nabla(\mathbf{R}_1) = \nabla(\mathbf{A}) = \mathbf{C}$ is the class of all C-sets in the given space.

The broad field of research on R-sets will be discussed in §7 of this survey, and here we confine ourselves to what is perhaps the most fundamental problem: the relations between R-sets and projective sets.

The position of the R-sets in the projective hierarchy.
This is both a very important question, because Luzin's projective hierarchy is regarded within descriptive set theory as the scale of complexity for point sets, and also a very difficult one, because even for the essentially simpler C-sets the problem of projective classification was solved by Kantorovich and Livenson [10], [12] only a number of years later than their introduction and the appearance of Selivanovskii's paper [31] in which the problem was posed.

The solution for R-sets was found by Lyapunov:

Theorem 1 [20]. *All R-sets belong to the class* Δ^1_2.

(We are using the modern notation Σ^1_n, Π^1_n, Δ^1_n for the projective classes, corresponding to the more traditional A_n, CA_n, B_n: see [9], [33], [41].)

The proof depends on the following lemma.

Lemma 1 [20]. *If the δs-operation* Φ *over a denumerable index set has complete basis* Q *of class* Δ^1_n *where* $n \geqslant 1$, *then* $\Phi(\Delta^1_n) \subseteq \Delta^1_n$.

The proof of this may be written as the double equivalence

$$x \in \Phi\{X_i\} \leftrightarrow \exists U \subseteq I(U \in Q \wedge \forall i \in U(x \in X_i)) \leftrightarrow$$
$$\leftrightarrow \forall U \subseteq I(\forall i(x \in X_i \to i \in U) \to U \in Q),$$

from which the result follows, since when applied to a Δ^1_n-relation the quantifier $\exists U \subseteq I$ gives a Σ^1_n-relation and the quantifier $\forall U \subseteq I$ gives a Π^1_n-relation. The completeness of the basis Q is needed only in deriving the second equivalence.

Now, for the proof of the theorem we need only verify that all the complete bases Q_α of the operations R_α are Δ^1_2-sets in the space $\mathcal{P}(I_\alpha)$, where I_α is the index set of R_α. This was shown by Lyapunov in [20], Ch. III, and he presented the proof of the theorem, in a somewhat different form, in [23].

Lyapunov's theorem does not, however, shed full light on the question under consideration, since it is unclear whether the class of all R-sets is a proper subclass of Δ^1_2 or the whole of it. This additional question greatly interested Lyapunov (see [25]), but it was found impossible to answer it using only the techniques of the classical theory of the R-transform.
A definitive solution was obtained by John Burgess [47]-[50] on the basis of a generalization of the construction of the R-transform that can easily be derived from the "game-theoretical" definition (a classical analogue is not known).

§6. The nature of R-sets

During research at the beginning of the 80's, new approaches to R-sets were discovered. On the whole, these originate from the ideas and constructions in Kolmogorov's paper [2] and by the generalization of one or another feature make it possible to consider the R-sets from a completely new standpoint. We shall discuss three such ideas: programming, game-theoretical Borel operations, and game-theoretical projection or the game-operator.

Programming Borel sets.
The definition of this operation was given by David Blackwell in [46]. Let us fix a denumerable set Ξ. Then a *program* is any function p: $\mathcal{P}(\Xi) \to \mathcal{P}(\Xi)$ with the property that $S \subseteq p(S)$ for every $S \subseteq \Xi$. The sequence of *derived* programs from a given program p is constructed by transfinite induction on α:

$$p_0 = p \text{ and } p_\alpha(S) = p\Big(\bigcup_{\gamma<\alpha} p_\gamma(S)\Big) \text{ for } \alpha > 0.$$

The limit program $p_\infty = p_{\omega_1}$ in this sequence obviously does not then change, that is, $pp_\infty = p_\infty$.

Further, let H be any collection of point sets (for example, the class of Borel sets $\mathbf{B} = \Delta^1_1$). A function F is called *H-measurable* if the F-inverse images of open sets all belong to H, and *H-programmed* if it is a superposition $F = f_1 p_\infty f_2$ for some H-measurable program p and $\mathbf{B}$-measurable functions f_1 and f_2. (The set $\mathcal{P}(\Xi)$ is assumed to have the natural topology of the Cantor discontinuum, relative to which the H-measurability of programs is considered.) Finally, a point set is called H-programmable if its characteristic function is H-programmable; the collection of all such sets is denoted by $\boldsymbol{HP}$.

For example, in the situation of §4 we take $\Xi = RI \cup \{\Lambda\}$ and for $S \subseteq \Xi$ define

$$p(S) = S \cup \{s \in \Xi\colon \breve{Q}i(si \in S)\}.$$

If the quantifier $\breve{Q}$ is Borel as a subset of $\mathscr{P}(\Xi)$(say $\breve{Q} = \exists$ or $\forall$) then this program will be **B**-measurable, and the sets X and $\breve{X}$ **B**-programmable: for example, for $\breve{X}$ we may take $f_1(x) = \{r \in RI\colon x \notin X_r\}$ and $f_2(S) = 1$ or 0 depending on whether or not the empty string Λ belongs to the set $S \subseteq \Xi$. Thus, taking $\breve{Q} = \forall$ (then $\Phi = \bigcup\limits_{\omega}$ and $R\Phi = A$) we conclude that all A-sets and all CA-sets are **B**-programmable. In fact, the class $\mathbf{B}P$ also includes all C-sets but not only these, and is itself a proper subclass of $\mathbf{\Delta R}_2$: see [47], [50]. As we see, programming develops from a Kolmogorov-style inductive analysis.

The operation of programming can be carried out repeatedly, leading to the construction of classes $\mathbf{P}_\alpha$ by induction on $\alpha < \omega_1$: $\mathbf{P}_0 = \mathbf{B}$ and $\mathbf{P}_\alpha = (\bigcup\limits_{\gamma<\alpha} \mathbf{P}_\gamma)P$ for $\alpha > 0$. Sets of the class $\mathbf{P} = \bigcup\limits_{\alpha<\omega_1} \mathbf{P}_\alpha$ are called programmable [50]; it is easy to see that $\mathbf{P}P = \mathrm{P}$ and $\mathbf{P}$ is the smallest class with this property (among those containing the Borel sets). In [50] the following is proved.

Theorem 1. **P** *coincides with the class of all R-sets.*

However the example given above shows that levels in the two hierarchies do not coincide exactly.

Game-theoretical Borel operations.
The starting-point of these is a generalization of the transforms R and $\breve{R}$ involving not the course of the game serving to interpret a row of quantifiers, but the definition of a winner as the outcome. Let λ be a quantifier, not necessarily monotonic, over the set ω of natural numbers. Consider the operation $R\Phi/\lambda$ defined as follows:

$$x \in R\Phi/\lambda\{X_r\} \leftrightarrow Qi_0Qi_1Qi_2 \ldots \lambda m(x \in X_{i\restriction m}),$$

where the right hand side of the equivalence is interpreted in terms of a game completely analogous to the games $\forall_{Qx}\{X_r\}$ and $\exists_{Qx}\{X_r\}$ of §3 except for the fact that the player Q is declared the winner if the condition $\{m\colon x \in X_{i\restriction m}\} \in \lambda$ is satisfied, that is, if $\lambda m(x \in X_{i\restriction m})$.

The rich potential of this generalization is already apparent in the rather simple (but most frequently investigated) case $Q = \forall\exists$. Vaught and Shilling [71], [69] constructed and studied operations $G\lambda$ (G_L^ω in their notation), defining their action in the following way:

$$x \in G\lambda\{X_r\colon r \in R\omega\} \leftrightarrow \forall i_0\exists i_1\forall i_2\exists i_3 \ldots \lambda n(x \in X_{i\restriction n}), \tag{18}$$

where each i_n runs over ω. Thus $x \in G\lambda\{X_r\}$ if the player $\exists$ has a winning strategy in the game $\lambda_x\{X_r\}$, where the players $\forall$ and $\exists$ make alternate moves in the form of the natural numbers $i_0, i_1, i_2, i_3, \ldots$ (beginning, of course, with $\forall$), and $\exists$ wins if the set $\{n\colon x \in X_{i\restriction n}\}$ belongs to λ.

The greatest opportunity for interesting research is given by operations $G\lambda$ which are Borel, that is, when the quantifier λ is Borel (as a subset of the space $\mathscr{P}(\omega)$ with the topology of the Cantor discontinuum); these are the operations considered in [71], [69]. In general, for a given class L each operation $G\lambda$ with $\lambda \in L$ can be called a game-theoretical L-operation. If in addition a specific class K is also fixed, then $GL(K)$ denotes the collection of all sets of the form $G\lambda\{X_r\}$, where $\lambda \in L$ and each X_r belongs to K. Below we shall show that the nature of the sets in $GL(K)$ is determined by the class L, and does not depend effectively on K, at least in the Borel situation.

Game-theoretical projection.
This is an operation of a different type from those considered so far in this survey. It is closely connected with projection in the ordinary sense, and is related to the Borel game-theoretical operations in exactly the same way as projection is to the A-operation; that is, it is stronger in principle, but in the Borel case leads to the same result.

Let $P \subseteq \mathscr{X} \times \mathscr{N}$, where $\mathscr{X}$ is the underlying space under consideration and $\mathscr{N} = \omega^\omega$ is Baire space, consisting of all infinite sequences $\mathbf{i} = \langle i_0, i_1, i_2, \ldots\rangle$ of natural numbers i_n. We put

$$\mathfrak{D}P = \{x \in \mathscr{X}\colon \forall i_0 \exists i_1 \forall i_2 \exists i_3 \ \ldots \ (\langle x, \mathbf{i}\rangle \in P)\}.$$

Thus $x \in \mathfrak{D}P$ when the player $\exists$ has a WS in the game $\mathfrak{D}_x P$, which proceeds similarly to the game $G\lambda_x\{X_r\}$ described above, but $\exists$ is regarded as the winner when the condition $\langle x, \mathbf{i}\rangle \in P$ is satisfied, where $\mathbf{i} = \langle i_0, i_1, i_2, \ldots\rangle$.

If K is some class of point sets, then $\mathfrak{D}K$ denotes the collection of all sets of the form $\mathfrak{D}P$ for $P \in K$.

Proposition (given in [69]). $\mathfrak{D}\mathbf{B} = G\mathbf{B}(\mathbf{B})$.

The implication from left to right is very simple: the set

$$P = \{\langle x, \mathbf{i}\rangle\colon \lambda n(x \in X_{\mathbf{i}\restriction n})\}$$

is Borel, being the inverse image of the Borel λ under the mapping $\langle x, \mathbf{i}\rangle \to \{n\colon x \in X_{\mathbf{i}\restriction n}\}$, which is $\mathbf{B}$-measurable for Borel X_r's. The reverse implication is proved by induction on the Borel structure of P.

Remark. It can be shown that in general

$$\mathfrak{D}L = GL(\mathbf{B}) = GL(\mathbf{F}) = GL(\mathbf{G})$$

for every class L in the Borel hierarchy, that is, the action of the operations of a fixed class $L = \Sigma^0_\alpha$, Π^0_α, or Δ^0_α on 1) the Borel sets (the class $\mathbf{B}$), 2) the closed sets (the class $\mathbf{F}$), and 3) the open sets (the class $\mathbf{G}$) in each case gives exactly the same sets as game-theoretical projection of the sets in L.

Game-theoretical projection and the R-sets.

The following theorem of Burgess [47], [49] is probably the deepest one in the whole of the modern theory of operations on sets.

Theorem 2. $\mathfrak{D}\Delta_1^0 = \mathbf{B}$, $\mathfrak{D}\Delta_2^0 = \mathbf{C}$, $\mathfrak{D}\Delta_3^0 = \mathbf{R}$.

Above all, these three formulae demonstrate the remarkably natural character of the concepts of C-sets and R-sets themselves: we see that a very general kind of process transforms the initial Δ-classes of the Borel hierarchy

$$\Delta_1^0 = \mathbf{F} \cap \mathbf{G}, \quad \Delta_2^0 = \mathbf{F}_\sigma \cap \mathbf{G}_\delta, \quad \Delta_3^0 = \mathbf{F}_{\sigma\delta} \cap \mathbf{G}_{\delta\sigma}$$

into the classes $\mathbf{B}$ (Borel sets), $\mathbf{C}$, and $\mathbf{R}$.

Proviso. We have formulated the first equation of Burgess's theorem in standard form, valid for sets in the Baire space $\mathcal{N}$, in which the open-and-closed sets form a basis for the topology. When it is applied to sets in an arbitrary (complete) Polish space $\mathcal{X}$, the class Δ_1^0 must be replaced by the family of all closed sets (or all open, or all Borel sets, which lead to the same result) $P \subseteq \mathcal{X} \times \mathcal{N}$ with the property that every section $P_x = \{\alpha: \langle x, \alpha\rangle \in P\}$ is a Δ_1^0-set in $\mathcal{N}$. The second and third formulae are true in the general case, but in addition remain valid if Δ_2^0 and Δ_3^0 are replaced by the families of all Borel sets with sections belonging to Δ_2^0 and Δ_3^0 respectively.

Burgess's theorem can be reformulated in terms of operations. For example, for the third formula, 1) all the operations R_α and $\breve{R}_\alpha$ can be represented in an equivalent form $G\lambda$ with a monotonic quantifier $\lambda \subseteq \mathcal{P}(\omega)$ of class Δ_3^0, and conversely 2) R is closed under the action of any game-theoretical Δ_3^0-operation.

Moreover, the formula $\mathbf{R} = \mathfrak{D}\Delta_3^0$ in the theorem leads to a complete solution of the problem of the precise relation of the class of R-sets to the projective class Δ_2^1. The solution is based on three simple facts.

Assertion 1. *If P is a Borel set, then for every point x the game $\mathfrak{D}_x P$ is determined.*

Indeed, the set $P_x = \{i: \langle x, i\rangle \in P\}$ of all plays that are wins for player $\exists$ is a Borel set, and all the games satisfying this condition (called Borel games) are determined, by Martin's theorem [62] (see also [63], [67]).

Assertion 2. $\mathfrak{D}B \subseteq \Delta_2^1$.

Consider an arbitrary Borel P. By definition

$$x \in \mathfrak{D}P \leftrightarrow \exists\sigma(\sigma \text{ is a s. for } \exists)\forall\tau(\tau \text{ is a s. for } \forall)\ \sigma * \tau \in P_x \leftrightarrow$$

$$\leftrightarrow \neg\ \exists\tau(\tau \text{ is a s. for } \forall)\ \forall\sigma(\sigma \text{ is a s. for } \exists)\sigma * \tau \notin P_x,$$

where "s." stands for "strategy", $\sigma * \tau$ denotes the play $\mathbf{i} = \langle i_0, i_1, \ldots\rangle$ of the game $\mathfrak{D}_x P$ uniquely determined by the strategies σ and τ of the two players $\exists$ and $\forall$, and the second equivalence is a consequence of the determinacy of the game. But strategies are here functions on the set $R\omega$ of all finite strings of natural numbers, with values in ω, and so any bijection from $R\omega$ to ω induces a coding of the strategies by the points of $\mathcal{N}$, by means of which the double equivalence reduces to

$$x \in \mathfrak{D}P \leftrightarrow \exists\sigma \in \mathcal{N}\,\forall\tau \in \mathcal{N}\,P_1(x, \sigma, \tau) \leftrightarrow \neg\,\exists\tau \in \mathcal{N}\,\forall\sigma \in \mathcal{N}\,P_2(x, \sigma, \tau),$$

where P_1 and P_2 are Borel relations. Hence it follows that $\mathfrak{D}P \in \Sigma^1_2$ and $\mathfrak{D}P \in \Pi^1_2$, and thus $\mathfrak{D}P \in \Delta^1_2$.

***Assertion* 3.** *In each class $\mathfrak{D}\Pi^0_\alpha$ there exists a set whose complement does not belong to this class.*

This is proved by exactly the same argument as the complementation theorem of §2.

Corollary. *The class of all R-sets is a proper subclass of the projective class Δ^1_2.*

Indeed, if we suppose that $\mathbf{R} = \mathfrak{D}\Delta^0_3 = \Delta^1_2$, we automatically obtain $\mathfrak{D}\Pi^0_3 = \Delta^1_2$, since $\Delta^0_3 \subseteq \Pi^0_3$ and $\mathfrak{D}\Pi^0_3 \subseteq \mathfrak{D}\mathbf{B}$, which however leads to a contradiction with Assertion 3, since Δ^1_2 is closed under complementation.

Thus we have a strictly increasing sequence of classes $\mathfrak{D}\Delta^0_\alpha$, $\alpha < \omega_1$, generated by the game-theoretical projections of the Δ^0_α-sets. The union of these is $\mathfrak{D}\mathbf{B} \subseteq \Delta^1_2$. Here, again, one may ask whether there is equality or strict inclusion, and again the answer is that the inclusion is strict. The truth of the matter is that all the sets of $\mathfrak{D}\mathbf{B}$ are among the "absolute Δ^1_2-sets" of Solovay (this follows from the proof of Assertion 2), and these form a proper subclass of Δ^1_2 (see [50], [52], [68]).

The proof of Burgess's theorem.

The formula $\mathfrak{D}\Delta^0_1 = \mathbf{B}$ should probably be described as "folk-lore": it was known before Burgess's work and has a rather simple proof (see, for example, [67]). The proof of the other two formulae has not so far been published in full. In [47], [49] only an outline is given, from which the general idea can be seen, however: to set up a correspondence between the classes in the hierarchy of R-sets (or of C-sets) and subclasses of the difference hierarchy of the class Δ^0_3 (respectively Δ^0_2). We shall consider this argument for the R-sets.

It is proved in the theory of Borel sets that for every Δ^0_3-set X we can pick an ordinal ξ, $1 \leqslant \xi < \omega_1$, and a decreasing (with respect to $\subseteq$) sequence of sets X_ν, $\nu < \xi$, of the class $\Pi^0_2 = \mathbf{G}_\delta$ such that $X = \bigcup_\nu (X_\nu - X_{\nu+1})$, where the union is taken over all even ordinals $\nu < \xi$, where if ξ itself is odd, then we must also define $X_\xi = \varnothing$. All the sets X that can be defined in this way for a fixed ξ constitute a subclass $\Pi^0_{2,\xi}$ of Δ^0_3. In particular $\Pi^0_{2,1} = \Pi^0_2$, and $\Pi^0_{2,2}$ is the class of all differences of two Π^0_2-sets, and so on, with $\Delta^0_3 = \bigcup_{1\leqslant\xi<\omega_1} \Pi^0_{2,\xi}$. (Of course the construction of the subclasses can be carried out in any Borel Δ-class. The subclasses were first introduced by Lavrent'ev; then they were considered by Luzin [17], Lyapunov, Kuratowski [15], and others.)

Thus Burgess asserts in [49] that $\mathfrak{S}\Pi^0_{2,\xi} = \Pi\mathbf{R}_{1+\xi}$ or $= \Sigma\mathbf{R}_{1+\xi}$ according as ξ is even or odd. We shall present the proof in the simplest case $\xi = 1$ (in [49], $\xi = 2$ is taken) and then add a few words about the general case.

From $\Sigma\mathbf{R}_2$ *to* $\mathfrak{S}\Pi^0_2$. By definition, the operation R_2 is equivalent to $R\breve{R}\bigcap_\omega$, that is, all the sets X of $\Sigma\mathbf{R}_2$ are such that

$$x \in X \leftrightarrow \forall j_{00}\forall j_{01}\forall j_{02} \ldots \exists m_0 \forall j_{10}\forall j_{11} \ldots \exists m_1 \ldots \ldots \forall j_{l0}\forall j_{l1} \ldots \exists m_l \ldots \forall k \; (x \in X_{j\restriction\restriction k}), \tag{19}$$

where $j\restriction\restriction k = \langle j_0\restriction m_0, \ldots, j_k\restriction m_k\rangle$, $j_l\restriction m_l = \langle j_{l0}, \ldots, j_{lm_l}\rangle$ and all the sets X_s, $s \in RR\omega$, are closed. We have to deduce that $X = \mathfrak{S}P$ for a suitable Π^0_2-set P.

The right-hand side of the equivalence (19) is interpreted in terms of the game $\forall^2_x\{X_s\}$ of length $(\omega+1)\omega$, where player $\forall$ makes the moves j_{lq} and player $\exists$ the moves m_l (after completion of the sequence of all moves j_{lq}, $q \in \omega$, by the opponent), and at the end of the game, a win for $\exists$ is defined to occur when $x \in X_{j\restriction\restriction k}$ for all k. The key feature of this game is that each play $j_{l0}, j_{l1}, \ldots, m_l$ can be organized so as to be infinite in only a potential sense, since if player $\exists$ has one or another justification for making some particular move m_l after the completion of all the moves j_{lq}, $q \in \omega$, by player $\forall$ in the round l, then this justification is already operative after the move j_{lm_l}. And if at this point (thus before a decision) $\exists$ makes the move m_l, then all the subsequent moves j_{lq}, $q \geqslant m_l$, by player $\forall$ in round l will have no effect on the outcome, and one can therefore pass immediately to the next round.

This argument leads to the following auxiliary game G'_x of length only ω. The players $\forall$ and $\exists$ (beginning with $\forall$) make the alternative moves $b_0, \varepsilon_0, b_1, \varepsilon_1, \ldots \in \omega$. If the numbers ε_p, $p \in \omega$, include only finitely many zeros, then $\forall$ wins. If there are infinitely many ε_p's equal to zero, let their indices be $p_1, p_2, p_3, \ldots$ in increasing order, and put $p_0 = -1$. Let $m_l = p_{l+1} - p_l - 1$, and also

$$b\restriction\restriction k = \langle r_0, \ldots, r_k\rangle, \text{ where } r_l = \langle b_{p(l)+1}, b_{p(l)+2}, \ldots, b_{p(l+1)}\rangle \tag{20}$$

and specify a win for $\exists$ in the case when $x \in X_{b\restriction\restriction k}$ for all k. This completes the definition of the game G'_x.

The game G'_x is equivalent to the game $\forall_s^{\cdot}\ \{X_s\}$ in the sense that the existence of a WS for player $\exists$ in one of these games is equivalent to the same thing in the other. On the other hand, G'_x is none other than the game $\mathfrak{D}_x P$ where P is the set of all pairs $\langle x, \mathbf{i}\rangle$ such that $x \in \mathscr{X}$, $\mathbf{i} = \langle b_0, \varepsilon_0, b_1, \varepsilon_1, \ldots\rangle \in \mathscr{N}$, and

$$\forall k \exists p \geqslant k\ (\varepsilon_p = 0) \wedge \forall k\ (x \in X_{b \dagger\dagger k}),$$

where the string of strings $b \dagger\dagger k$ is defined by (20). Therefore $X = \mathfrak{D}P$. However, because the set $X_{b \dagger\dagger k}$ is closed the set P belongs to the class $\mathbf{\Pi}^0_2 = \mathbf{G}_\delta$.

From $\mathfrak{D}\mathbf{\Pi}^0_2$ *to* $\Sigma\mathbf{R}_2$. Now we assume that $X = \mathfrak{D}P$ with P in the class $\mathbf{\Pi}^0_2$, and we must prove that $X \in \Sigma\mathbf{R}_2$. First, it is easy to see that P is of the form $P = \{\langle x, \mathbf{i}\rangle : \forall k \exists l (x \in Y^k_{\mathbf{i} \dagger l})\}$, where all the $Y^k_{\mathbf{i}\dagger l} \subseteq \mathscr{X}$ are closed, and without loss of generality we may assume that $Y^k_{\mathbf{i}\dagger p} \subseteq Y^k_{\mathbf{i}\dagger l}$ for $p \leqslant l$ for every string $i \dagger l \in R\omega$ (otherwise, we put $\widetilde{Y}^k_{i\dagger l} = \bigcup\limits_{p \leqslant l} Y^k_{i\dagger l}$).

Further, if after the construction of an infinite sequence of moves $\mathbf{i} = \langle i_0, i_1, i_2, \ldots\rangle$ in the game $\mathfrak{D}_x P$ we obtain $x \in X^k_{\mathbf{i}\dagger l}$ for some $l = l(k)$ (and hence, by hypothesis, also for all $l' \geqslant l$), then this has in fact become clear after the moves $i_0, \ldots, i_l$. Consequently

$$x \in X \leftrightarrow \forall j_{00} \exists j_{01} \forall j_{02} \exists j_{03} \ldots \exists m_0 \forall j_{10} \exists j_{11} \ldots \exists m_1 \ldots$$
$$\ldots \forall j_{l0} \exists j_{l1} \ldots \exists m_l \ldots \forall k (x \in X_{j \dagger\dagger k}),$$

where each of the closed sets $X_{j\dagger\dagger k}$ is defined as follows. The string $j \dagger\dagger k$ is converted to the simple string

$$i \dagger m = \langle j_{00}, j_{01}, \ldots, j_{0m_0}, j_{10}, \ldots, j_{1m_1}, \ldots, j_{k0}, \ldots, j_{km_k}\rangle$$

of length $n = m_0 + \ldots + m_k$, after which we put $X_{j\dagger\dagger k} = Y^k_{i\dagger n}$.

This proves that X is obtained by applying the operation $\Psi = R\breve{R}(\bigcap\limits_\omega \bigcup\limits_\omega)$ to the family of closed sets we have constructed. However, by the theorem on the properties of the transform R at the end of §3, which of course is also valid for the "complementary" transform $\breve{R}$, and because $\bigcap\limits_\omega \bigcup\limits_\omega \leqslant \Gamma = \breve{R} \bigcap\limits_\omega$, we have

$$\breve{R} \bigcap_\omega \leqslant \breve{R}(\bigcap_\omega \bigcup_\omega) \leqslant \breve{R}\breve{R} \bigcap_\omega \approx \breve{R} \bigcap_\omega,$$

from which it follows that $\breve{R}(\bigcap\limits_\omega \bigcup\limits_\omega) \approx \breve{R} \bigcap\limits_\omega$. Thus Ψ is equivalent to the operation $R_2 \approx R\breve{R} \bigcap\limits_\omega$, and therefore $X \in \Sigma\mathbf{R}_2$.

The general case. In the more interesting direction $\mathbf{R} \subseteq \mathfrak{D}\Delta^0_3$ (as regards the projective characterization of the R-sets), the idea of the proof of Burgess's theorem can be clearly seen from the derivation of the inclusion $\mathbf{\Pi R}_3 \subseteq \mathfrak{D}\mathbf{\Pi}^0_{2,2}$ given in [49]. The canonical definition of a $\mathbf{\Pi R}_3$-set can be written as follows:

$$x \in X \leftrightarrow \Box_0\Box_1\Box_2 \ldots \exists n(x \in X_{j\ldots n}),$$

where all the $X_{j\ldots n}$ are open, and each $\Box_p$ is a complete quantifier prefix of the form (19), in which all the variables receive an additional (first) index p, that is, $\Box_p$ is $\forall j_{p00} \ldots \exists m_{p0} \ldots \forall k_p$.

The "modelling game" of length ω is so arranged that 1) the player $\forall$ makes moves corresponding to the j_{plm}, 2) the player $\exists$ makes moves corresponding to the m_{pl} and signifying the end of a round, and 3) player $\forall$ may also make moves corresponding to k_p and signifying the termination of super-plays (one super-play comprises several rounds—m_{pl} of them). A win for $\exists$ in this game occurs when player $\exists$ has made infinitely many moves, corresponding to the m_{pl}, and either $\forall$ has made finitely many moves of the type k_p, or a natural win for $\exists$ occurs, in the sense of the system of sets $X_{j\ldots n}$. A straightforward analysis shows that we now arrive at the action of the operator $\mathfrak{D}$ on a set which is the difference of two Π^0_2-sets (each of which expresses that there are infinitely many of the specified moves).

At higher levels of the hierarchy of R-sets the structure of the rounds of various ranks becomes correspondingly more complicated, but in fact everything reduces to difference combinations of Π^0_2-relations requiring that the number of moves of a specific type is infinite (plus a simple relation for a natural win), upon which $\mathfrak{D}$ is then superposed.

§7. Properties of R-sets

Here we shall consider three lines of research into R-sets: measure and category; partitioning into Borel sets; and the principle of comparison of indices. These may be said to have been inherited from classical descriptive set theory, in which Luzin and his students made a detailed study of the A-operation and its geometric equivalent—the operation of sifting by means of a sieve. However, direct generalization of the results connected with A-sets became possible only after Kolmogorov had isolated the general mechanism lying behind the principal properties of the A-operation, namely the R-transform. Essentially, all three of the lines we have selected are related to a greater or lesser degree to Kolmogorov's paper [2] and in a certain sense spring from it.

Measure and category.

Every Borel set on the real line is Lebesgue measurable and has the Baire property, which means that it coincides with an open set to within sets of first category. This assertion can be derived from the fact that the operations $\bigcup_\omega$ and $\bigcap_\omega$, used in the construction of the Borel sets, preserve measurability (that is, when applied to measurable sets they yield measurable sets) and (in the same sense) the Baire property. Immediately after the introduction of the A-operation by Aleksandrov [7] and the construction by Suslin [32] of the class of A-sets (that is, the Suslin or, in another terminology, Σ_1^1-sets), Luzin and Sierpiński [18] (see also [17]) proved that the A-operation also preserves measurability and the Baire property, and therefore all A-sets on the real line, and their complements the CA-sets, are Lebesgue measurable and possess the Baire property.

Thus the property "preserves measurability and the Baire property" was extended from the operations $\bigcup_\omega$ and $\bigcap_\omega$ (for which it is obvious) to the A-operation. In Theorem VI of [2], Kolmogorov shows that the fact that this extension is possible can be based on the single fact that the A-operation is the R-transform of the operation $\bigcup_\omega$. Here is the formulation of his theorem: if the δs-operation Φ (with countable index set; only such operations are discussed below) preserves measurability, then the operation $R\Phi$ also preserves measurability. This theorem was stated without proof in [2]—a proof was first published by Lyapunov [20] (with acknowledgement to Kolmogorov), together with one for the corresponding theorem concerning the Baire property. Lyapunov deduced that:

1) all R-sets on the real line are measurable (with an appropriate generalization of Lebesgue measurability, this result remains valid for all Polish spaces, as Lyapunov shows); and

2) all R-sets have the Baire property. (This is valid for every Polish space,)

In recent work the results for the Baire property have undergone far-reaching generalizations within the framework of the "game-theoretical" approach to operations. In particular, Kechris [60] established that the Baire property is possessed by all sets of the class $\mathfrak{D}\mathrm{B}$, which is, as we have seen, significantly broader than the class of R-sets. Shilling and Vaught [69] have shown that all game-theoretical Borel operations preserve the Baire property (which implies the Kechris result). The proof of the Shilling-Vaught theorem involves the use of a special game which is described by the quantifier prefix of the following Burgess formula:

$$\forall H_0 \subseteq H \forall i_0 \exists H_1 \subseteq H_0 \exists i_1 \forall H_2 \subseteq H_1 \forall i_2 \ \ldots \ \lambda n \exists m (H_m \leqslant X_{i \restriction n}).$$

Here each H_m must belong to a fixed countable basis for the topology of the given space, $A \leqslant B$ denotes that $A-B$ is a set of first category, $\{X_r\}$ is the family of open sets to which the operation $G\lambda$ is applied, and H is a fixed open set. This game is determined (for a Borel λ) and 1) if the player $\exists$ has a winning strategy, then $H \leqslant X$, where $X = G\lambda\{X_r\}$, but 2) if the player $\forall$ has a WS beginning with some move H_0, then $H_0 \leqslant (H_0 - X)$. This implies that X has the Baire property. See [45], [48]-[50], [69], [71] for further work on the Baire property.

A game-theoretical analysis of measurability demands recourse to essentially more complex games than those associated with the Baire property. This is probably the reason why in recent papers on Borel game-theoretical operations measure is rarely considered separately—usually there is merely a remark to the effect that results can be obtained for measure similar to those valid for category. In particular, it is stated in [69] that Borel game-theoretical operations preserve measurability (with respect to a rather broad class of measures), and consequently all sets of $\mathfrak{H}\mathbf{B}$ are measurable (in the same sense).

Partitions into Borel sets.
Research in this direction was started by Luzin and Sierpiński in [18], where it is shown that every CA-set can be decomposed, in a certain canonical way, into pairwise disjoint Borel sets called constituents, of which there are $\aleph_1$. Later (see [17]) a similar result was established for the A-sets themselves; the resulting constituents are described as 'inner'. In an elementary way this implies that such a partition is possible also for the sets of the wider class $\mathbf{\Sigma}^1_2 = \mathbf{A}_2$, since $\mathbf{\Sigma}^1_2$-sets are projections of CA-sets, and Borel sets project into A-sets. Consequently every R-set can also be partitioned into $\aleph_1$ Borel sets, since $\mathbf{R} \subseteq \mathbf{\Delta}^1_2 \subseteq \mathbf{\Sigma}^1_2$.

However, it turns out that the canonical partitions of the A-sets and CA-sets into constituents have the key property of regularity with respect to measure, meaning that the measure of a given set is concentrated on countably many terms of the partition, and the same property holds with respect to category, which is not in general true for the simplest construction, which we gave earlier, of the partitions for $\mathbf{\Sigma}^1_2$ sets. The problem of constructing partitions of R-sets into Borel sets that are regular with respect to measure and category was solved by Lyapunov in [20] with the help of the following theorem of Kolmogorov.

Theorem* V *of [2]. *Suppose that the δs-operation Φ with countable index set I is such that $\Phi \geqslant \bigcap_\omega$ and $\Phi\Phi \approx \Phi$. Then the complement of any set of $(R\Phi)(\mathbf{F})$ is the union of an increasing sequence (of type ω_1) of $\aleph_1$ sets of the class $\breve{\Phi}(\mathbf{F})$.*

Proof. In the situation considered in §4 we have $\breve{X} = \bigcup_\alpha \breve{X}_{\Lambda\alpha}$, and because I is countable this union is restricted to ordinals $\alpha < \omega_1$, while the sets $\breve{X}_{s\alpha}$ satisfy the equations (17). If we take the initial sets X_r to be closed, then under the hypotheses on Φ (in the form $\breve{\Phi} \geqslant \bigcup_\omega$ and $\breve{\Phi}\breve{\Phi} \approx \breve{\Phi}$) it is easy to show by induction on α that each set $\breve{X}_{s\alpha}$ belongs to $\breve{\Phi}(\mathbf{F})$, that is, it can be obtained from closed sets by a single application of $\breve{\Phi}$.

Removing from each $\breve{X}_{\Lambda\alpha}$ the union of all the $\breve{X}_{\Lambda\gamma}$ with indices $\gamma < \alpha$, we obtain a partition of $\breve{X}$ into the sets

$$\breve{X}_\alpha = \breve{X}_{\Lambda\alpha} - \bigcup_{\gamma<\alpha} \breve{X}_{\Lambda\gamma} = \{x\colon \operatorname{Ind}_\Lambda x = \alpha\}, \quad \alpha < \omega_1.$$

Under the assumptions of the last theorem each $\breve{X}$ is the intersection of a set of class $\breve{\Phi}(\mathbf{F})$ with one of class $\Phi(\mathbf{F})$.

Combining this result with the partition of X itself (which can be obtained with the help of the inner indices: the inner index $\mathrm{Ind}^* x$ of a point $x \in X$ may be defined as the least $\alpha < \omega_1$ such that $S_{x\alpha} = S_{x,\alpha+1}$), and performing an induction on the construction of the operations of the normal series, Lyapunov proved in [20] that each R-set admits a partition into $\aleph_1$ Borel sets, regular with respect to measure and category. Here, the start of the induction is the case $\Phi = \bigcup_\omega$ (consequently $R\Phi = A$), to which Kolmogorov's theorem is inapplicable (since it is not true that $\bigcup_\omega \geqslant \bigcap_\omega$), but the same argument gives the Borel property of sets $\breve{X}_\alpha$ identical with the classical constituents of the CA-sets in the descriptive theory.

The principle of comparison of indices and its applications.

We again return to the situation considered in §4, and we suppose that $R\Phi$ is one of the operations R_α, $\alpha \geqslant 1$, of the normal series, and corresponding to the construction, $\Phi = \bigcup_\omega \bigcup \{\breve{R}_\gamma\colon \gamma < \alpha\}$. We also consider the pair of sets $X = R_\alpha\{X_r\}$, $X' = R_\alpha\{X'_r\}$ of the class $\Sigma\mathbf{R}_\alpha$ (where X_r, X'_r are assumed closed). With the systems of sets X_r and X'_r two Kolmogorov index functions are associated: $\alpha(X) = \mathrm{Ind}_\Lambda x$ and $\alpha'(x) = \mathrm{Ind}'_\Lambda x$, where $\alpha(x) < \omega_1$ for $x \in \breve{X}$ and $\alpha(x) = \omega_1 = \infty$ for $x \in X$ and similarly for α'. What is the nature of the set of points x at which $\alpha(x) \leqslant \alpha'(x)$? The answer is as follows.

Principle of comparison of indices (Lyapunov [21]). *The set $\{x\colon \alpha(x) \leqslant \alpha'(x)\}$ belongs to the class $\Sigma\mathbf{R}_\alpha$.*

In actual fact, Lyapunov considered this question in a significantly more general case (see [20]-[24] and [45]). In descriptive set theory, the principle was first established by Novikov for the case corresponding to $\alpha = 1$ (that is, when R_α is the A-operation; in fact Novikov considered a sieve), and he used it in [27], [28] in order to solve certain problems concerning $\mathbf{B}$-measurable implicit functions and C-sets. But the most natural application is to the proof of separation theorems.

The *first separation principle* is said to hold for some class K (written $\mathrm{Sep}_1(K)$) if any two disjoint sets of this class can be covered by two mutually complementary sets Y, Y' of the same class.

The *second separation principle* $\mathrm{Sep}_2(K)$ is said to hold for K if for any pair of K-sets X, X' the sets $X - X'$ and $X' - X$ can be covered by disjoint sets Y, Y' whose complements belong to K.

Theorem (Lyapunov [22]). *Both separation principles hold for the classes $\Sigma\mathbf{R}_\alpha$ but not for the classes $\Pi\mathbf{R}_\alpha$.*

The 'positive' part for the second principle is proved using the sets

$$Y = \{x:\ \alpha(x) > \alpha'(x)\},\quad Y' = \{x:\ \alpha'(x) > \alpha(x)\},$$

and for the first principle Lyapunov takes the same sets after having arranged that $\alpha(x) \neq \alpha'(x)$ for all x by a special auxiliary construction. The 'negative' part of the theorem is deduced from the 'positive' part using a doubly universal pair of $\Sigma\mathbf{R}_\alpha$-sets.

In the descriptive theory, both of the separation principles for the class $\Sigma\mathbf{R}_1(=\mathbf{A})$, and non-separation for $\Pi\mathbf{R}_1(=\mathbf{CA})$, had been proved by Luzin [17].

In modern research on the R-transform and R-sets the principle of comparison of indices has been used in the derivation of the important property of normability [45] (also called the principle of complete pre-ordering) for the classes $\Pi\mathbf{R}_\alpha$ and for the solution of certain delicate problems about the construction of functions that effect a choice of winning strategy, and concerning sections of plane sets. In particular, in [45], [51], and [70] there is a study of the sets

$$\{x:\ \breve{E}^x = \{y:\ \langle x,\ y\rangle \notin E\} \text{ is of first category }\}$$

for plane sets E of specific classes.

§8. Kolmogorov's problem on the lengths of Borel hierarchies

Early issues of the journal Fundamenta Mathematicae contained a consecutively numbered series of problems in areas related to the foundations of mathematics. Number 65 is a problem of Kolmogorov [4] which, although it did not become so well known as Suslin's problem (number 3, Fund. Math. 1 (1920), 223), nevertheless aroused considerable interest among experts in set theory (see the book [14], p.140).

Suppose that some initial collection of sets F_0 is given. By induction on α we define

$$F_\alpha = \bigcup_\omega \Big(\bigcup_{\gamma<\alpha} F_\gamma \Big) \quad \text{or} \quad F_\alpha = \bigcap_\omega \Big(\bigcup_{\gamma<\alpha} F_\gamma \Big)$$

according as the ordinal α is even or odd. Then $F = F_{\omega_1}$ is the smallest family containing all the sets of F_0 and closed under $\bigcup_\omega$ and $\bigcap_\omega$. In fact this is one of the possible constructions of the hierarchy of Borel sets, considered in papers in the 20's and 30's: the collection $\mathbf{F}$ of all closed sets is then chosen for F_0.

In [65] the least α such that $F_\alpha = F$ is called the Kolmogorov number $K(F_0)$ for F_0. (For $\alpha \geqslant 1$ this is equivalent to $F_{\alpha+1} = F_\alpha$, which is what [4] actually refers to.) Of course $K(F_0) \leqslant \omega_1$, and Kolmogorov's problem is to determine for which values of $\alpha \leqslant \omega_1$ there exists a family F_0 such that $K(F_0) = \alpha$.

Miller's paper [65] contains a survey of both classical and modern results on this problem. The main result given by Miller, and ascribed to Kunen, provides a complete solution: for every $\alpha \leqslant \omega_1$ there is an initial family F_0 for which $K(F_0) = \alpha$. It is interesting that the proof of this given in [65] makes use of the method of forcing, which is usually applied in the derivation of consistency and undecidability results, and so on; here it is used in the solution of a problem in the traditional sense.

Conclusion

In this survey we have tried to exhibit the original form of the ideas introduced by Kolmogorov as a basis for the theory of operations on sets, the development of the theory under the influence of these ideas, and also their enrichment and the new forms they have taken on during the course of this development.

At present this theory is far from having a completely definitive form, and one can easily discern perspectives for further research. Thus it would be very interesting to obtain non-trivial results on the structure of the class of all δs-operations from the point of view of the "stronger" order of § 1. For example, consider the smallest family $\mathscr{B}$ of operations closed under superposition and containing $\underset{\omega}{\cup}$ and $\underset{\omega}{\cap}$. Then $\Phi \leqslant A$ for every $\Phi \in \mathscr{B}$. Is it true that $A \leqslant \Psi$ for every operation Ψ that satisfies $\Phi \leqslant \Psi$ for all $\Phi \in \mathscr{B}$?

A second important objective is the search for an "operational" way of looking at the class $\mathfrak{D}\Delta^0_4$: we know that $\mathfrak{D}\Delta^0_1$ is the class of Borel sets, $\mathfrak{D}\Delta^0_2 = \mathbf{C}$, and $\mathfrak{D}\Delta^0_3 = \mathbf{R}$, but it is not clear what the next step ought to be, and more generally what the general principle should be for all $\mathfrak{D}\Delta^0_\alpha$ giving $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{R}$ at the first three stages.

Another side of this question is the problem of construction of an inductive analysis for game-theoretical Borel operations similar to that given by Kolmogorov in the case that he considered. In view of the connection between inductive analysis and the proof of determinacy for the corresponding games, discussed in §4, a clue to the solution of this second problem might be looked for in the proof of Martin's theorem [62] on the determinacy of Borel games. Certainly, it is possible to carry out an inductive analysis of game-theoretical Σ^0_2-operations parallel to the proof of determinacy of Σ^0_2-games given by Wolfe [72].

Further, in the theory of operations on sets the operations themselves and their transforms are considered. Is there any sense in considering the next higher level, that is, transforms of transforms? Thus it would be interesting to assign some specific meaning to the expression $\breve{R}\breve{R}\breve{R}\breve{R}\breve{R}\breve{R} \ldots \Phi$.

The above list of unsolved problems could easily have been extended, and there is no doubt that the future development of the theory of operations on sets will reveal new facets of this part of the mathematical legacy of the great mathematician Andrei Nikolaevich Kolmogorov.

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