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Valuations and sheaves. On some questions of non-standard analysis

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CONTENTS

In this paper we present some parts of a mathematical theory that is sometimes called *Heyting-valued analysis* (or *non-standard analysis in the broad sense*). While this theory may be regarded as part of general topos theory, it also has applications outside mathematical logic: in algebra, analysis, and even in a broader context, as in Robinson's well-known papers on the application of non-standard analysis to quantum field theory.

In Chapter I we present the actual method of Heyting-valued (in particular, Boolean-valued) analysis. §I.1 is an introduction, where we explain the essence of the method on an intuitive level; an introductory role is also played by those parts of the paper that are headed "example", and also the beginnings of Chapters II–IV and the beginning of the appendix. At the beginning of the appendix we discuss the terms "Heyting-valued and Boolean-valued analysis", "non-standard analysis in the broad sense", and so on. In §I.2 we briefly recall the notions that play an essential part in what follows. §I.3 contains the definition of valuation—the key notion of this method—and all the basic examples of it. The notion of a sheaf on a Heyting algebra appears in §I.7, although it is used implicitly from the very beginning. So the method is presented in §§I.3–I.7. Chapters II–IV contain some concrete examples of applications of the method of Heyting-valued analysis. In Chapter II we mainly consider the question of the existence of a model companion for a locally axiomatizable class of rings. The notion of model companion, a generalization of the notion of algebraic and real closure of a field, was introduced by Robinson, and plays an essential role in model theory. In Example 10 of Chapter II we show the model completeness of a class of atomless rings whose localizations are all central finite-dimensional simple algebras with centre satisfying a model-complete theory. In Chapter III we consider a conjecture of Novikov (see [5], p.127). This chapter is concerned with the transfer from classical to intuitionistic validity in an arbitrary ring. In the cited paper of Novikov it is shown that such a transfer is possible in the case of the ring $\mathbb{Z}$. In Chapter IV we construct, for certain rings of continuous Y-valued functions (as algebras over the ring Y), a non-standard representation $\widetilde{Y}$ such that in a certain sense this algebra is similar to its ring of scalars Y. The appendix contains brief descriptions of applications of Boolean-valued analysis concerned with questions of duality.

Practically all theorems and propositions have full proofs.

All the general theorems of Heyting-valued analysis are contained in Chapter I. Regarding the applications, it is inappropriate, indeed hardly possible, to cover them in full completeness. The author has selected some applications related to his research interests with the intention of avoiding intersections with well-known and easily-available articles. We mention some of these. There are interesting papers of Beidar and Mikhalev ([18], [34]–[35]) (on the study of semiprime rings based on their reduction to prime rings by the method of orthogonal completeness), a series of papers on operator algebras ([20]–[22]), the papers [11], [15] reprinted in [41] (where a theory of Boolean-valued measure and integral representations is constructed), a long series of papers of Kusraev and Kutateladze, see [36]–[38] for example (mainly concerning problems on the K-spaces of Kantorovich and also categorical aspects), and deep papers on Robinson's non-standard analysis by Kanovei ([32]–[33]), Gordon ([39]–[40]), and Zvonkin and Shubin [31].

Further details can be found in papers listed in the references. These papers, in turn, contain numerous references, including some that claim priority. In particular, detailed bibliographies are contained in [1]-[3] and [7]-[10], where one can find references to articles by Robinson, Cohen, Vopěnka, Scott, Solovei, Takeuti, and other well-known authors in this field. An introduction to Heyting-valued analysis is provided by [1], [3] for example.

After a minimal acquaintance with Chapter I one can pass to any other chapter or to the appendix, returning to Chapter I if necessary.

The meaning of the symbol $\rightleftharpoons$ is "equal by definition" or "equivalent by definition". The sign $\square$ marks the end of the proof.

Chapter I

EVALUATION IN ALGEBRAIC SYSTEMS

I.1. The concept of valuation, global validity, and transfer theorem.

An *evaluation* (*valuation*) in a given language for a fixed lattice X is an assignment to every formula φ of an element of X, denoted by $[\![\varphi]\!]_X$ (or briefly by $[\![\varphi]\!]$), in such a way that the logical connectives of the language are modelled by the operations of X. This means that $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$, $[\![\neg\varphi]\!] = \neg [\![\varphi]\!]$, and $[\![\varphi \Rightarrow \psi]\!] = [\![\varphi]\!] \rightarrow [\![\psi]\!]$, where on the left-hand sides the symbols $\vee$, $\wedge$, $\Rightarrow$, $\neg$ are the connectives of the language, and on the right-hand sides $\vee$, $\wedge$, $\rightarrow$, $\neg$ are the operations in X (with the same names). The same holds for all propositional connectives. Quantifiers (and the associated variables) require, in addition, the designation of some fixed set D (usually called *the set of parameters of the given language*). If such a D is fixed, then $[\![\exists x\varphi]\!] = \bigvee \{[\![\varphi(k)]\!] \mid k \in D\}$ and $[\![\forall x\varphi]\!] = \bigwedge \{[\![\varphi(k)]\!] \mid k \in D\}$. The same holds for other predicate connectives. Here the operations $\vee$, $\wedge$ apply to subsets of X, that is, they have the form $\vee\colon \mathcal{P}(X) \to X$, where $\mathcal{P}$ is the operation of taking the power set. In our case the operations $\vee$ and $\wedge$ are the least upper bound and the greatest lower bound in the lattice X of two or, when appropriate, any set of elements of X. The operations $\neg$, $\rightarrow$ can also be expressed in the usual way in terms of the ordering on X.

To emphasize the choice of the set D of parameters, the valuation is sometimes denoted by $[\![\cdot]\!]_{X,D}$ or $[\![\cdot]\!]_D$. We write $1 = \vee X$.

We denote by $L(D)$ the set of all formulae of the language L with parameters from D (without free variables). Then the evaluation is a one-variable function of the form $[\![\cdot]\!]\colon L(D) \to X$ satisfying the inductive conditions mentioned above. Frequently $L(D)$ is denoted briefly by L, and formulae with parameters from D are called *formulae* for short. If the function $[\![\cdot]\!]$ is defined for the atomic formulae of the language, then it extends uniquely to the whole set of formulae.

So, besides the usual validity of a statement φ in a fixed mathematical structure K (notation $K \models \varphi$), there arises a new form of validity (new semantics) $[\![\varphi]\!]_{X,D} = 1$ denoted by $\langle X, D\rangle \models \varphi$ or briefly $X \models \varphi$ (or sometimes $D \models \varphi$), where the valuation $[\![\cdot]\!]_{X,D}$ is also fixed and somehow connected with the structure K. The predicate $X \models \varphi$ is called the *global validity* of φ; sometimes it reads "φ is true in the lattice X". Instead of $X \models \varphi$ the symbol $\Box\varphi$ is also used. This new validity (semantics) has useful properties, and in particular in a number of cases it allows us to solve questions concerning ordinary validity in K. To some extent, it can be compared with (formal-logical) deducibility (denoted in what follows by $\vdash$ in the classical and by $H\vdash$ in the intuitionistic case). The difference is that the proof (verification) of the global validity $X \models \varphi$ reduces in some sense to the computation of "algebraic" functions in the lattice X, and the number of variables of these functions is not greater than the length of the formula φ being verified.

Usually the structure K admits a (non-unique) choice of lattice $X = X(K)$ (for D we usually take the domain of K) and a choice of valuation $[\![\cdot]\!]$ (in this case it is convenient to denote the valuation by $[\![\cdot]\!]_K$) such that for many formulae φ we have $[\![\varphi]\!]_K = 1$ (although, possibly, $K \not\models \varphi$). The class of such formulae is denoted by $\Phi_+(K)$. On the other hand, we denote by $\Phi_-(K)$ the class of formulae ψ that satisfy $([\![\psi]\!]_K = 1) \Rightarrow (K \models \psi)$. Usually the valuation $[\![\cdot]\!]_K$ is closed with respect to some kind of deducibility (depending primarily on the lattice X and so denoted by $\vdash_X$), that is, if $\varphi \vdash_X \psi$ and $[\![\varphi]\!]_K = 1$, then $[\![\psi]\!]_K = 1$.

It is possible to take the view that non-standard analysis in the broad sense is understood as a development based on the study of the valuations $[\![\cdot]\!]_X$ and the semantics $[\![\cdot]\!]_X = 1$ (or of more general semantics $[\![\cdot]\!]_X \in j$, where j is a filter in X, for example $j = \{1\}$). A typical and, probably, the main application of non-standard analysis (in this sense) is the following. If $[\![\varphi_1 \wedge \ldots \wedge \varphi_n]\!]_K = 1$ (but possibly, $K \not\models \varphi_1 \wedge \ldots \wedge \varphi_n$), that is, $\varphi_1, \ldots, \varphi_n \in \Phi_+(K)$, $\varphi_1, \ldots, \varphi_n \vdash_{X(K)} \psi$, and $\psi \in \Phi_-(K)$, then $K \models \psi$. If $\psi \notin \Phi_-(K)$, then an assertion ψ' can nevertheless hold in K, where ψ' is a formula of the initial language equivalent to the global validity of the formula ψ. One can say that the consequences ψ and ψ' of properties $\varphi_1, \ldots, \varphi_n$ mythical for K are valid in K (in the most usual sense). Such a situation is called a *transfer theorem*.

The next question concerns the usual choice of the connectives $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\exists$, $\forall$ and (in the general case) of the infinite sets of parameters D. The latter requires that the operations $\bigvee, \bigwedge\colon \mathcal{P}(X) \to X$ are defined everywhere, or at least on sets of the form $\{[\![\varphi(k)]\!]_X \mid k \in D\}$. Sometimes one can observe which subsets of X have such a form and require the applicability of the operations $\bigvee$ and $\bigwedge$ to them. But it is simpler, to assume (and we do) that X is a complete lattice, that is, the least upper and the greatest lower bounds $\bigvee_\alpha u_\alpha$ and $\bigwedge_\alpha u_\alpha$ exist for any set $\{u_\alpha\} \subseteq X$. It is always true that $u \wedge \bigvee_\alpha u_\alpha \geqslant \bigvee_\alpha (u \wedge u_\alpha)$.

I.2. Heyting algebras and Stone spaces.

A *complete Heyting algebra* is a complete lattice Ω with the property (infinite distributivity) $u \wedge \bigvee_\alpha u_\alpha \leqslant \bigvee_\alpha (u \wedge u_\alpha)$, that is, $u \wedge \bigvee_\alpha u_\alpha = \bigvee_\alpha (u \wedge u_\alpha)$ for any element $u \in \Omega$ and any set $\{u_\alpha\} \subseteq \Omega$. We write $1 = 1_\Omega = \bigvee\Omega$ and $0 = 0_\Omega = \bigwedge\Omega$.

Another equivalent definition of complete Heyting algebra is the following: it is a complete lattice Ω in which a two-place operation $\cdot\to\cdot: \Omega^2 \to \Omega$ is defined and satisfies $w \leqslant (u \to v) \Leftrightarrow u \wedge w \leqslant v$, $\forall u, v, w \in \Omega$. It is convenient to assume (and we do) that $1 \neq 0$.

If H is a lattice (not necessarily complete) with greatest element 1 and least element 0 in which such an operation $\cdot\to\cdot$ is defined, then H is called a *Heyting algebra*. We always regard H as having the structure $\langle H, \vee, \wedge, \to, 0, 1\rangle$, where $\vee, \wedge, \to$ are two-place operations. Every Heyting algebra is distributive in the sense that $(u \vee v) \wedge w = (u \wedge w) \vee (v \wedge w)$ and $(u \wedge v) \vee w = (u \vee w) \wedge (v \vee w)$. In what follows (unless it is explicitly stated otherwise) Ω is a complete Heyting algebra with a fixed structure $\langle \bigvee, \wedge, 0, 1\rangle$, where $\bigvee: \mathcal{P}(\Omega) \to \Omega$ and $\wedge: \Omega^2 \to \Omega$. It is important to fix the structure in an algebra, for example, for the notions of subalgebra or morphism of algebras. In Ω we define

$$\bigwedge_\alpha u_\alpha = \bigvee \{v \mid \forall\alpha\, (v \leqslant u_\alpha)\},$$
$$\neg u = \bigvee \{v \mid v \wedge u = 0\}, \quad \text{and}$$
$$(u \to v) = \bigvee \{w \mid u \wedge w \leqslant v\}.$$

It is easy to see that $(u \leqslant v) \Leftrightarrow u \wedge v = u$, $u \wedge (u \to v) \leqslant v$, $u \wedge \neg u = 0$, $w \leqslant (u \to v) \Leftrightarrow u \wedge w \leqslant v$, and $u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$.

An element $u \in \Omega$ such that $u \vee \neg u = 1$ is called *Boolean*. A *complete Boolean algebra* is an Ω in which all elements are Boolean. A *Boolean algebra* is a Heyting algebra (not necessarily complete) in which all elements are Boolean. A *basis* (*dense subset*) in Ω is a subset $\Omega_0 \subseteq \Omega$ such that $\forall u \in \Omega \exists \{u_\alpha\} \subseteq \Omega_0$ $(u = \bigvee_\alpha u_\alpha)$. Ω is called *zero-dimensional* if it has a basis consisting of Boolean elements. An element $u \in \Omega$ is called *finite* if $\forall \{u_\alpha\} \subseteq \Omega$ $(u \leqslant \bigvee_\alpha u_\alpha \Rightarrow u \leqslant u_{\alpha_1} \vee \ldots \vee u_{\alpha_n})$ for some $\alpha_1, \ldots, \alpha_n$. If Ω has a basis consisting of finite elements, then Ω is called an *algebraic lattice*. It is easy to see that a zero-dimensional algebraic lattice Ω has as a basis the set of finite as well as Boolean elements—it is usually this basis that is used in proofs. A Heyting algebra is called *compact* if the unit is a finite element. A zero-dimensional compact lattice is, of course, algebraic.

Two extensive and in a certain sense mutually complementary classes of complete Heyting algebras are the class of all topologies (a topology $\mathcal{T}$ (X) is the lattice of all open subsets of a non-empty topological space X) and the class of all complete Boolean algebras. In what follows the symbol $\mathbb{B}$ or $\boldsymbol{B}$ always denotes a complete Boolean algebra, and B a Boolean algebra.

Two simple but important classes of complete Heyting algebras (cHa) can be described as all zero-dimensional algebraic complete Heyting algebras and all zero-dimensional compact complete Heyting algebras. The first of these is the class of all lattices of ideals of Boolean rings and the second that of all lattices of ideals of Boolean algebras. The lattice of all ideals of a Heyting algebra H is denoted by $I(H)$.

The *Stone space* $X = X(H)$ of a Heyting algebra H is the set of all prime ideals ("points") in H, that is, those $p \subseteq H$ such that $1 \notin p$, $0 \in p$, $\forall\, u, v \in p\ (u \vee v \in p)$, $\forall\, u \in p\quad \forall\, v \in H\ (u \wedge v \in p)$, $\forall\, u, v \in H\ (u \wedge v \in p \Rightarrow u \in p \vee v \in p)$.

In the set $X(H)$ we fix the topology with basis $\{\{p \mid u \notin p\} \mid u \in H\}$. An element $u \in H$ is identified with the set $\{p \mid u \notin p\}$ and this set is also denoted by u (hence, $p \in u \Leftrightarrow u \notin p$). Thus, this basis can be denoted by H. It is easy to show that in the topological space $X(H)$ the set H consists of all open-and-compact elements (more precisely, open and quasi-compact) and $X(H)$ itself is a T_0-quasi-compact topological space. It is Hausdorff if and only if H is a Boolean algebra. The last assertion is equivalent to the fact that prime and maximal ideals in H coincide.

If H is a Boolean algebra, then $X(H)$ is a totally disconnected compact set and H is the family of all its open-and-closed sets; if H is a complete Boolean algebra, then $X(H)$ is extremally disconnected.

The topology of the space $X(H)$ can be identified, as a lattice, with $I(H)$ according to the rule: $a \mapsto \mathcal{O}_a$, where $a \in I(H)$ and $\mathcal{O}_a = \{p \in X(H) \mid a \not\subseteq p\}$. In this sense $I(H) = \mathcal{T}(X(H))$.

An important property of Stone spaces is the following: if $\{u_\alpha\}$ is a downward-directed family of open-and-compact sets in $X(H)$ and F is any closed set in $X(H)$, then $\forall \alpha\ (u_\alpha \cap F \neq \varnothing) \Rightarrow (\bigcap_\alpha u_\alpha) \cap F \neq \varnothing$. Therefore, in particular, $X(H)$ is a Baire space.

All that has been said above carries over without change to any distributive lattice in the role of H (and after some minor changes also to any distributive upper semilattice with zero).

An important role is played by the possibility of embedding any Heyting algebra H in a complete Boolean algebra $\mathbb{B} = \mathbb{B}(H)$ in such a way that the operations $\vee$, $\wedge$, 0, 1 are preserved and, what is more, of embedding H in a complete Heyting algebra $\Omega = \Omega(H)$ in such a way that all the operations $\vee$, $\wedge$, $\rightarrow$, 0, 1 are preserved. This can be done as follows. We realize H as the algebra of open-and-compact sets in the Stone space $X = X(H)$ and we extend the topology $\mathcal{T} = \mathcal{T}(H)$ to the topology $\mathcal{T}_1$ by adding to $\mathcal{T}$ the complements of all open-and-compact sets in $\mathcal{T}$. Let B be the Boolean algebra generated by H; it consists of the sets in X of the form $\bigwedge_{i=1}^{n} (Cu_i \vee u_i')$, where $u_i, u_i' \in H$, and C denotes set-theoretic complement. B is a basis for the topology $\mathcal{T}_1$. It is convenient to write the elements of B in the form $\bigwedge_i (u_i \underset{B}{\to} u_i')$, where $\underset{B}{\to}$ denotes implication in B. We put

$$I\,(\bigwedge_i (u_i \underset{B}{\to} u_i')) \leftrightharpoons \bigwedge_i (u_i \underset{H}{\to} u_i'),$$

where $\underset{H}{\to}$ denotes implication in H; such an I is of the form $I: B \to H$.

Thus, the *required* $\mathbb{B} = \mathbb{B}\,(H)$ is the Dedekind complement of B, in other words, $\mathbb{B}$ is the family of all open regular subsets (in the sense of the topology $\mathcal{T}_1$) in $X = X(H)$.

We recall that a set $\mathcal{O} \in \mathcal{T}\,(X)$ is called *regular* if $\overset{\circ}{\bar{\mathcal{O}}} = \mathcal{O}$, where $\overset{\circ}{-}$ is the composite of the operators of closure $^-$ and then interior $^\circ$.

Using I it is easy to see that the natural embedding of H in B preserves the operation $\vee$. The embedding of B in $\mathbb{B}$ preserves both operations $\vee$ and $\wedge$.

Next, we extend the operation I to $\mathbb{B}\,(H)$ by the rule

$$I\,(b) \leftrightharpoons \bigvee_{\mathbb{B}} \{I\,(b_1)\,|\,b_1 \in B \wedge b_1 \leqslant b\}.$$

Thus, the *required* $\Omega \leftrightharpoons \Omega\,(H)$ is $\{b \in \mathbb{B}\,(H)|\ \ I(b) = b\}$. Using I it is easy to see that the natural embedding of H in $\Omega(H)$ also preserves the operations $\wedge$, $\to$.

I.3. Definition and examples of valuations.

It is convenient to include with the atomic formulae the following: $\top$ ("truth"), $\bot$ ("falsehood") with the obvious assignment of valuations; $\neg\varphi$ is always understood as $\varphi \Rightarrow \bot$. We repeat the definition of evaluation (valuation) which essentially has already been given in §I.1. A *valuation* for a language L (with equality symbol $\cdot = \cdot$), a family of parameters D, $D \neq \emptyset$, and a complete Heyting algebra Ω is a map of the form $[\![\cdot]\!]$: $L(D) \to \Omega$ (where $L(D)$ is the set of all closed formulae of the language L with parameters from D) satisfying the conditions $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$, $[\![\exists\, x\varphi]\!] = \bigvee \{[\![\varphi\,(k)]\!]|\ k \in D\}$ and similarly for all connectives, and also satisfying the conditions $[\![k = k]\!] = 1$, $\forall k \in D$ ("reflexivity"), and for any atomic formula φ_0 ("consistency with equality") $[\![k = t]\!] \wedge [\![\varphi_0\,(k)]\!] \leqslant [\![\varphi_0\,(t)]\!]$, $\forall k, t \in D$, where the k in φ can take the place of any free variable. It follows from the definition that $[\![k = t]\!] = [\![t = k]\!]$ ("symmetry") and $[\![k = t]\!] \wedge [\![t = l]\!] \leqslant [\![k = l]\!]$ ("transitivity"). that is, $[\![\cdot = \cdot]\!]$ is an equivalence relation on D. By induction with respect to the length of the formula $\varphi\,(x_1, \ldots, x_n)$ we obtain $(\bigwedge_{i=1}^{n} [\![k_i = t_i]\!] \wedge [\![\varphi\,(\bar{k})]\!]) \leqslant [\![\varphi\,(\bar{t})]\!]$. The definition of valuation is not complete if the language contains functional symbols.

Then to every functional symbol $f(x_1, ..., x_n)$ there corresponds a function $f: D^n \to D$ which is "consistent with the valuation" in the sense that $(\bigwedge_{i=1}^{n} [\![k_i = t_i]\!]) \leqslant [\![f(\bar{k}) = f(\bar{t})]\!]$. All terms are, by definition, first computed in D and then substituted into formulae. Such an understanding of the occurrence of terms in formulae is called *operational*.

An Ω-*set* is a triple $\langle \Omega, D, [\![\cdot]\!]\rangle$, where all components are as defined above except that the reflexivity condition is replaced by the symmetry condition (see Example 7).

The constants of the language L are identified with fixed elements of the set D and in this sense we do not distinguish between the constants and the parameters of the language.

All that is required to determine any valuation is to define the function $[\![\cdot]\!]: L_0(D) \to \Omega$, where $L_0(D)$ is the set of all closed atomic formulae of L with parameters from D.

D is frequently the domain of some structure, and therefore a predicate $D \models \varphi$ is defined, where $\varphi \in L(D)$ (in this case we usually write K instead of D). The valuation $[\![\cdot]\!]$ is called *normal* if for any atomic formula $\varphi \in L_0(D)$ we have $(D \models \varphi) \Leftrightarrow ([\![\varphi]\!] = 1)$. If D is a set without any structure, then for any atomic formula φ we put $(D \models \varphi) \Leftrightarrow ([\![\varphi]\!] = 1)$ and obtain a structure in D for which this valuation is normal. The term "normal valuation" is also employed in the case when the given condition is fulfilled only for some atomic formulae. If it is not stated otherwise, we shall assume that this condition refers only to the atomic formula $\cdot = \cdot$.

The valuation $[\![\cdot]\!]_\Omega$ is a *weakly sheaf* valuation if $\forall u \in \Omega$ (u is a Boolean element $\Rightarrow \forall k, t \in D \exists s \in D u \leqslant [\![s = k]\!]_\Omega \wedge \neg u \leqslant [\![s = t]\!]_\Omega$). For a normal valuation this s is unique. We shall denote it by $u \cdot k + \neg u \cdot t$. For a weakly sheaf valuation the following condition is fulfilled: if $\{u_1, \ldots, u_n\}$ is a disjunctive family of Boolean elements from Ω, then for any $\{k_1, \ldots, k_n\} \subseteq D$ there is a $k \in D$ such that $u_1 \leqslant [\![k = k_1]\!], \ldots, u_n \leqslant \leqslant [\![k = k_n]\!]$. We denote this k by $\sum_i u_i \cdot k_i$.

A *sheaf* valuation is a valuation $[\![\cdot]\!]_\Omega$ with the following property: if $\{u_\alpha\} \subseteq \Omega$, $\{k_\alpha\} \subseteq D$, and $\forall \alpha, \beta$ $(u_\alpha \wedge u_\beta \leqslant [\![k_\alpha = k_\beta]\!]_\Omega)$, where $\bigvee_\alpha u_\alpha$ is a Boolean element, then there is a $k \in D$ such that $u_\alpha \leqslant [\![k = k_\alpha]\!]_\Omega \forall \alpha$. A sheaf valuation is weakly sheaf. The corresponding k will be denoted by $\sum_\alpha u_\alpha \cdot k_\alpha$.

A distinctive feature of a sheaf (in the traditional sense of this term) is the possibility of pasting together the separate sections. "Sheaf" valuations and objects are determined exactly by the possibility of pasting together element-sections k_α "defined" on u_α, that is, by the possibility of defining the element-section $\sum_\alpha u_\alpha \cdot k_\alpha$.

We call the valuation $[\![\cdot]\!]_{\Omega, D}$ *accessible* if it satisfies $([\![\exists x\varphi]\!] \geqslant u) \Rightarrow \exists k \in D$ $([\![\varphi (k)]\!] \geqslant u)$ for any formula $\varphi (x)$ with one free variable x and any parameters from D and also for any Boolean element $u \in \Omega$. We usually use this condition with $u = 1$.

For an arbitrary valuation $[\![\cdot]\!]_{\Omega, D}$ in a language including the symbol $\cdot \in \cdot$ and for $d \in D$, we denote by $\hat{d}^{\Omega}$ (or just $\hat{d}$) the set

$$\{x \in D \mid [\![x \in d]\!]_{\Omega, D} = 1\}.$$

The operator $(\cdot)^{\wedge}$ is similar to the operator $^*(\cdot)$ in Robinson's non-standard analysis, which associates with an internal set d the external set $^*d$.

For an arbitrary valuation $[\![\cdot]\!]_{\Omega, D}$ and a prime ideal p in Ω we write $D_p \rightleftharpoons D/{\sim_p}$, where $(x \sim_p y) \Leftrightarrow [\![x = y]\!] \notin p$. For any atomic formula φ_0 and equivalence classes $[k_1]_p, \ldots, [k_n]_p \in D$ we also put

$$(D_p \models \varphi_0 ([k_1]_p, \ldots, ([k_n]_p)) \Leftrightarrow ([\![\varphi_0 (k_1, \ldots, k_n)]\!]_{\Omega, D} \notin p).$$

Such a structure D_p is called a *fibre* over the point p. Instead of $[k]_p$, the symbol $k(p)$ (or just $[k]$) is often used.

Example 1. Let L be the usual language of set theory (the language ZF) with atomic predicates $\cdot \in \cdot$ and $\cdot = \cdot$. As the family of parameters D we choose the class V^{Ω} defined as $\bigcup_{\alpha} V^{\Omega}_{\alpha}$, where α runs over the class On of all ordinal numbers and V^{Ω}_{α} is the set of all functions of the form $f \colon V^{\Omega}_{\beta} \to \Omega$, $\beta < \alpha$. For example, $(f = g) \Leftrightarrow \forall x\, (x \in f \Leftrightarrow x \in g)$ is a closed formula of this language (with parameters $f, g \in V^{\Omega}$). In this case

$$L_0(D) = \{(f \in g) \mid f, g \in V^{\Omega}\} \cup \{(f = g) \mid f, g \in V^{\Omega}\}.$$

We put

$$[\![f \in g]\!]_{\Omega} = \bigvee \{g (h) \wedge [\![f = h]\!]_{\Omega} \mid h \in \mathcal{D} (g)\} \quad \text{and}$$

$$[\![f = g]\!]_{\Omega} = \bigwedge \{(f (h_1) \to [\![h_1 \in g]\!]_{\Omega}) \mid h_1 \in \mathcal{D} (f)\} \wedge \bigwedge \{(g (h_2) \to$$
$$\to [\![h_2 \in f]\!]_{\Omega}) \mid h_2 \in \mathcal{D} (g)\}.$$

This is well-defined, in fact, by induction on α. The valuation extends uniquely to all formulae.

We replace the family of parameters V^{Ω} by $V^{\Omega}/{\sim}$, where

$$f \sim g \rightleftharpoons [\![f = g]\!]_{\Omega} = 1,$$

and we denote this quotient family again by V^{Ω}. The valuation in the language ZF with the family of parameters $V^{\Omega}/{\sim}$ is defined by $[\![\varphi ([f_1], \ldots, [f_n])]\!]_{\Omega} = [\![\varphi (f_1, \ldots, f_n)]\!]_{\Omega}$ (the right-hand side does not depend on the choice of the representatives). It in fact coincides with the valuation in the language ZF with the family of parameters V^{Ω}. This valuation is called the *valuation in the language ZF*.

The Heyting algebra $Z_2 = \{0, 1\}$ can be embedded in any algebra Ω. This embedding induces an embedding $(\cdot)^\vee$: $V \to V^\Omega$ of the class of "all sets" V in V^Ω (where $V = \bigcup_\alpha V_\alpha$ with α running through On and $V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta)$). The embedding $(\cdot)^\vee$ coincides with the identification of V and V^{Z_2}. Usually V, V^{Z_2}, and the image of V under this embedding are not distinguished. In this sense, V is contained in all V^Ω (although V also contains all elements of V^Ω).

If $X \subseteq V^\Omega$ and X is a set, then we denote by $(X)_-$ (or $\underline{X}$) the function defined on X and identically equal to 1. Of course, $\underline{X} \in V^\Omega$.

A function f from V^Ω is *extensional* if $\forall x, y \in \mathcal{D}(f)\,((f(x) \wedge [\![x = y]\!]_\Omega) \leqslant \leqslant f(y)$. More generally, we call a function $f: D \to \Omega$ *extensional*, where D is the set of parameters of a certain valuation $[\![\cdot]\!]_{\Omega, D}$, if it satisfies $\forall d_1, d_2 \in D\ (f(d_1) \wedge [\![d_1 = d_2]\!]_{\Omega, D} \leqslant f(d_2))$.

Some properties of an arbitrary valuation and of the valuation in the language ZF are collected in the following four theorems.

Theorem 1. *Let Ω be zero-dimensional and let $[\![\cdot]\!]_{\Omega, D}$ be a weakly sheaf valuation.*

a) *If Ω is a compact lattice, then the valuation is accessible. If $[\![\cdot]\!]_{\Omega, D}$ is a sheaf valuation and Ω is a Boolean algebra, then this valuation is accessible.*

b) *If $\exists d_1, \ldots, d_n \in D\ ([\![\varphi(d_1, \ldots, d_n)]\!]_{\Omega, D} = 1)$, then we have*

$$[\![\exists \bar{x}\,(\varphi \wedge \psi)]\!]_{\Omega, D} = \bigvee \{[\![\psi(x_1, \ldots, x_n)]\!]_{\Omega, D} \mid x_1, \ldots, x_n \in D,\ [\![\varphi(x_1, \ldots, x_n)]\!]_{\Omega, D} = 1\}$$

and

$$[\![\forall x_1, \ldots, x_n\,(\varphi \Rightarrow \psi)]\!]_{\Omega, D} = \bigwedge \{[\![\psi(x_1, \ldots, x_n)]\!]_{\Omega, D} \mid x_1, \ldots, x_n \in D,\ [\![\varphi(x_1, \ldots, x_n)]\!]_{\Omega, D} = 1\},$$

where φ, ψ contain any parameters from D.

c) *Let $[\![\cdot]\!]_\Omega$ be a valuation in the language ZF. If $\hat{f}^\Omega \neq \varnothing$, then we have $[\![(\hat{f}^\Omega)_- = f]\!]_\Omega = 1$. Let $[\![\cdot]\!]_{\Omega, X}$ be a normal valuation and let the set X be a part of D. If X is a sheaf set with respect to the induced valuation $[\![\cdot]\!]_{\Omega, X}$, then $(\underline{X})^{\wedge\Omega} = X$.*

Proof. a) Suppose that $[\![\exists x \varphi(x, k_1, \ldots, k_n)]\!] \geqslant u$, where u is a finite element of Ω, so we have $\bigvee_{d \in D} [\![\varphi(d, \bar{k})]\!] = \bigvee_{d, \alpha} b_{d, \alpha} \geqslant u$, where the $b_{d, \alpha}$ are Boolean elements. Since u is a finite element, there exist $d_1, \ldots, d_n$ such that $b_{d_1} \vee \ldots \vee b_{d_n} \geqslant u$, where b_{d_i} is the join of the $b_{d, \alpha}$ that correspond to a single d. All the b_{d_i} are Boolean elements. We form $\{b_i\}$ (employing the Boolean property) such that $b_i \leqslant b_{d_i}$, $b_1 \vee \ldots \vee b_n \geqslant u$, and $\{b_i\}$ is a disjunctive family. By the definition of a weakly sheaf valuation, we form $d = \sum_i b_i \cdot d_i$. Then we have $b_i \leqslant [\![d = d_i]\!]$, $b_i \leqslant [\![\varphi(d, \bar{k})]\!]$, and $u \leqslant [\![\varphi(d, \bar{k})]\!]$.

So far we have not used the compactness of Ω. We now note that if u is a Boolean element in a compact Ω, then it also is a finite element.

We verify the second assertion. Suppose that $[\![\exists x\varphi(x, \bar{k})]\!] = \bigvee_d b_d \geqslant b \in \Omega$, where $b_d = [\![\varphi(d, \bar{k})]\!]$. We form a disjunctive family $\{b_d'\}$, $\bigvee_d b_d' \geqslant b$, $b_d' \leqslant b_d$ and by the sheaf condition there is a k such that $b_d' \leqslant [\![k = d]\!]$. The conclusion now follows as above.

b) We have to show that

$$(\bigwedge_{x_1, \ldots, x_n \in D} ([\![\varphi(x_1, \ldots, x_n)]\!] \to [\![\psi(x_1, \ldots, x_n)]\!]) \geqslant$$
$$\geqslant \bigwedge \{[\![\psi(x_1, \ldots, x_n)]\!] \mid x_1, \ldots, x_n \in D, \ [\![\varphi(x_1, \ldots, x_n)]\!] = 1\}.$$

Suppose that $[\![\varphi(x_1, \ldots, x_n)]\!] = \bigvee_\alpha u_\alpha$, where the u_α are Boolean elements of Ω. We put $t_1 = u_\alpha \cdot x_1 + \neg u_\alpha \cdot d_1, \ldots, t_n = u_\alpha \cdot x_n + \neg u_\alpha \cdot d_n$. They all belong to D. Then

$$u_\alpha \leqslant [\![t_1 = x_1]\!] \wedge \ldots \wedge [\![t_n = x_n]\!] \wedge [\![\varphi(x_1, \ldots, x_n)]\!] \leqslant [\![\varphi(t_1, \ldots, t_n)]\!],$$
$$\neg u_\alpha \leqslant [\![t_1 = d_1]\!] \wedge \ldots \wedge [\![t_n = d_n]\!] \wedge [\![\varphi(d_1, \ldots, d_n)]\!] \leqslant [\![\varphi(t_1, \ldots, t_n)]\!].$$

Hence $1 = [\![\varphi(t_1, \ldots, t_n)]\!]$. What is more,

$$[\![\psi(t_1, \ldots, t_n)]\!] \leqslant (u_\alpha \to [\![\psi(x_1, \ldots, x_n)]\!]), \text{ and}$$
$$[\![\psi(t_1, \ldots, t_n)]\!]$$

The case $\exists \bar{x}$ is dealt with in a similar way.

c) It is sufficient to show that $g = (\hat{f}^{\Omega})_{-}$ and f have equal volume (what remains is the question of an arbitrary valuation in a language containing the symbol $\cdot \in \cdot$). Obviously, $[\![h \in g]\!] = \bigvee \{[\![h = h_1]\!] \mid [\![h_1 \in f]\!] = 1\} \leqslant [\![h \in f]\!]$. For the converse we use part b) above. As to the second assertion, in one direction the equality is obvious. If $[\![d \in \underline{X}]\!] = 1$, then $1 = \bigvee_{x \in X} [\![d = x]\!]$. We denote these terms by u_x. Now $u_x \wedge u_y \leqslant [\![x = y]\!]$ and by the sheaf property of the set X there is a $k \in X$ such that $u_x \leqslant [\![k = x]\!]$. Therefore, $1 = [\![k = d]\!]$. By normality we get $d = k \in X$. □

The expression "U-valuation" means that we take into account the following property: $p \in [\![\exists x\varphi]\!] \Rightarrow \exists k \in D \ (p \in [\![\varphi(k)]\!])$, where φ is any formula with parameters from D (in fact, this very property also appears in Theorem 15). It is clear that we are concerned with a weakened accessibility.

***Theorem* 2.** *Let φ be any formula with parameters from D.*

a) *If the valuation is a sheaf valuation and Ω is a complete Boolean algebra, then*

$$[\![\varphi(d_1, \ldots, d_n)]\!] = \{p \in X(\Omega) \mid D_p \models \varphi(d_1(p), \ldots, d_n(p))\}.$$

In b) and c) we assume that the valuation of any atomic formula is a Boolean element.

b) *For a* ∪*-valuation and a formula* φ *without quantifiers in the domain of implication we have*

$$(p \in [\![\varphi(d_1, \ldots, d_n)]\!]) \Rightarrow D_p \models \varphi(d_1(p), \ldots, d_n(p)).$$

c) *If the valuation is weakly sheaf and* φ *is an AE-formula, then we have*

$$(\forall p \in u\, D_p \models \varphi(d_1(p), \ldots, d_n(p)) \Rightarrow [\![\varphi(d_1, \ldots, d_n)]\!] \geqslant u$$

for any Boolean element u *from* Ω.

Proof. a) A valuation with values in a complete Boolean algebra is closed with respect to all the classical transformations of connectives (see Theorem 8, part c) and so of course is the predicate $D_p \models (\cdot)$. Therefore, we can restrict ourselves to the verification of the equivalence $[\![\varphi]\!] \notin p \Leftrightarrow (D_p \models \varphi)$ for atomic formulae and the connectives $\wedge, \neg, \exists$. For atomic φ this equivalence is fulfilled by definition of the fibre D_p. For $\neg$ we have $[\![\varphi]\!] \in p \Leftrightarrow D_p \models \neg\varphi$, $[\![\varphi]\!] \in p \Leftrightarrow \neg[\![\varphi]\!] \notin p$. For $\wedge$ we have

$$[\![\varphi \wedge \psi]\!] \notin p \Leftrightarrow ([\![\varphi]\!] \notin p) \wedge ([\![\psi]\!] \notin p) \Leftrightarrow (D_p \models \varphi) \wedge (D_p \models \psi).$$

For $\exists$ we use the accessibility, which holds by Theorem 1a).

b) For a quantifier-free formula φ we verify the equivalence $[\![\varphi]\!] \notin p \Leftrightarrow D_p \models \varphi$, which is useful in many cases. For an atomic formula and the connectives $\neg$, $\wedge$ this was done in part a) (for $\neg$ it is important that $[\![\varphi]\!] \vee [\![\neg\varphi]\!] = 1$). For the connective $\vee$ we have $[\![\varphi \vee \psi]\!] \notin p \Leftrightarrow$ $\Leftrightarrow ([\![\varphi]\!] \notin p) \vee ([\![\psi]\!] \notin p)$. For the connective $\Rightarrow$ we use $[\![\varphi]\!] \rightarrow [\![\psi]\!] =$ $= \neg[\![\varphi]\!] \vee [\![\psi]\!]$ (employing the fact that these valuations are Boolean). Next, for $\forall$ it is obvious and for $\exists$ we use the condition of ∪-valuation.

c) For a quantifier-free formula this assertion was proved in the previous paragraph, and for $\forall$ it is obvious. We consider the case $\forall p \in u\, (D_p \models \exists x\varphi)$, where φ is a quantifier-free formula. We denote by $d_{p_0}(p_0)$ the element of D_{p_0} such that $D_{p_0} \models \varphi(d_{p_0}(p_0))$, where $d_{p_0} \in D$. By what was proved above and by hypothesis we have $p_0 \in \{p \mid D_p \models \varphi(d_{p_0}(p))\} = [\![\varphi(d_{p_0})]\!]$, a Boolean element. We denote this element by b_{p_0}. Then $\bigvee_{p_0} b_{p_0} \geqslant u$ and by the compactness of Ω we can find $b_{p_1}, \ldots, b_{p_n}$ such that $b_{p_1} \vee \ldots \vee$ $\vee\, b_{p_n} \geqslant u$. We replace b_{p_i} by b_i, where $\{b_i\}$ is a disjunctive family, $b_0 \leqslant b_{p_i}$, and $\bigvee_i b_i \geqslant u$. The valuation is weakly sheaf, and so we can form $k = \sum_i b_i \cdot d_{p_i}$. We now have $b_i \leqslant [\![\varphi(d_{p_i})]\!] \wedge [\![k = d_{p_i}]\!] \leqslant [\![\varphi(k)]\!]$, $[\![\varphi(k)]\!] \geqslant u$, and $[\![\exists x\varphi]\!] \geqslant u$. (If we restrict ourselves to finite u, then the compactness condition for Ω is unnecessary.)

Remark. The connection between Theorem 2a) and Łos's theorem in Robinson's non-standard analysis is obvious.

***Theorem* 3.** *We take $f, g \in V^{\Omega}$ and consider a valuation in the language ZF.*
a) *We have*

$$[\![(\exists x \in f)\varphi]\!]_{\Omega} = \bigvee_{x \in \mathcal{D}(f)} f(x) \wedge [\![\varphi(x)]\!]_{\Omega}, \quad [\![(\forall x \in f)\varphi]\!]_{\Omega} = \bigwedge_{x \in \mathcal{D}(f)} (f(x) \to [\![\varphi(x)]\!]_{\Omega}).$$

b) *For any set $X \subseteq V^{\Omega}$, $\mathcal{D}(f) \subseteq X$, there is an extensional function g, $\mathcal{D}(g) = X$, such that $[\![f = g]\!]_{\Omega} = 1$, that is, $f = g$ in V^{Ω}. For extensional functions f, g we have $[\![x \in f]\!]_{\Omega} = f(x)$ (if $x \in \mathcal{D}(f)$) and*

$$[\![f = g]\!] = \bigwedge_{x \in \mathcal{D}(f)} (f(x) \leftrightarrow g(x))$$

(*if the domains of f and g coincide*).
c) *We have $f(x) \leqslant [\![x \in f]\!]_{\Omega}$ (if $x \in \mathcal{D}(f)$). Moreover, $[\![f = g]\!]_{\Omega} \geqslant$ $\geqslant u \Leftrightarrow f_1 = g_1$ if f and g are extensional with the same domain and $[\![f_1 = f]\!]_{\Omega} \geqslant u$ for any $u \in \Omega$, where $f_1(x) \rightleftharpoons f(x) \wedge u$ and $g_1(x) \rightleftharpoons g(x) \wedge u$.*
d) *We have*

$$[\![\varphi(\check{x}_1, \ldots, \check{x}_n)]\!]_{\Omega} = \begin{cases} 1, & \varphi(x_1, \ldots, x_n), \\ 0, & \neg\varphi(x_1, \ldots, x_n), \end{cases}$$

where $x_1, \ldots, x_n \in V$ and φ is any bounded formula (that is, all quantifiers of φ appear in the form $\exists x \in u$ or $\forall x \in u$ and u is a free or a bound variable). In this sense $(\cdot)^{\vee}\colon V \to V^{\Omega}$ is an "almost elementary" embedding.

The following relations between V^{Ω} and V^{B} are often used (see Ch. III), where $\mathbb{B} = \mathbb{B}(\Omega)$ was defined at the end of §I.2.
Firstly, V^{Ω} is a part of V^{B}, and secondly we have the following result.

***Theorem* 3e).** *We have $\forall f, g \in V^{\Omega} [\![f \in g]\!]_{\Omega} \leqslant [\![f \in g]\!]_{B}$, $[\![f = g]\!]_{\Omega} \leqslant [\![f = g]\!]_{B}$, and also "$\{f, g\}_{-}$, the pair of f and g" and "$\{\{f\}_{-}, \{f, g\}_{-}\}_{-}$, the ordered pair of f and g" make sense in Ω and in $\mathbb{B}$, and, finally, $[\![x = \{f, g\}]\!]_{\Omega} \leqslant$ $\leqslant [\![x = \{f, g\}]\!]_{B}$, $[\![x = \langle f, g\rangle]\!]_{\Omega} \leqslant [\![x = \langle f, g\rangle]\!]_{B}$, where f and g are from V^{Ω}.*

The proof of Theorem 3 is contained in essence in [3] for example, and in fact also in [2].

***Theorem* 4.** *The valuation in the language ZF is a normal and sheaf valuation. If Ω is a complete Boolean algebra, then this valuation is accessible.*

Proof. The fact that the valuation is normal follows at once from the definition of the valuation in the language *ZF*. Suppose that $\{u_\alpha\} \subseteq \Omega$, $\{f_\alpha\} \subseteq V^{\Omega}$, and that these families are consistent, that is, $u_\alpha \wedge u_\beta \leqslant [\![f_\alpha = f_\beta]\!]$. We put $f(\cdot) = \bigvee_{\alpha} (u_\alpha \wedge f'_\alpha(\cdot))$, where each f'_α is an extensional function equal to f_α and they have the same domain for all α. Then f has the same domain and is also extensional. Next,

$$f \wedge u_\beta = \bigvee_{\alpha} u_\alpha \wedge u_\beta \wedge f'_\alpha = \bigvee_{\alpha} u_\alpha \wedge u_\beta \wedge f'_\beta = u_\beta \wedge f'_\beta;$$

$$u_\beta \leqslant [\![f = f'_\beta]\!], \quad u_\beta \leqslant [\![f = f_\beta]\!],$$

using Theorem 3, part c twice. Therefore $f = \sum_\alpha u_\alpha \cdot f_\alpha$. To obtain the second assertion we apply Theorem 1, part a. □

The construction of V^Ω and its simple properties appeared gradually. This also applies to the concept of arbitrary valuation. To some extent it can be found in papers of Gödel, Church, and later Cohen, and it appears explicitly in papers of Vopenka, Scott, Solovay, Grayson, and others.

Example 2. Let $\langle K, +, -, 0, 1\rangle$ be an arbitrary associative ring with identity. In what follows we denote by $B(K)$ the set of all central idempotents of the ring K, that is, $k \in B(K) \Leftrightarrow k^2 = k \wedge \forall t\, (k\cdot t = t\cdot k)$. In $B(K)$ we have the canonical structure of a Boolean algebra: $e_1 \wedge e_2 = e_1 \cdot e_2$, $\neg e = 1 - e$, $e_1 \vee e_2 = e_1 + e_2 - e_1 \cdot e_2$, where $e_1, e_2 \in B(K)$. In what follows we denote by $X(K)$ the Stone space of the Boolean algebra $B(K)$ and by $\mathcal{T}(K)$ its topology. A point from $X(K)$ is called a *point of the ring* K. Next, the letter O always denotes an open set from the corresponding topology: in the case in hand, O is an element of $\mathcal{T}(K)$. Obviously, $B(K) \subseteq \mathcal{T}(K)$. This $\mathcal{T}(K)$ is sometimes called the *topological completion* of the Boolean algebra $B(K)$.

We denote by $\mathbb{B}(K)$ or $\boldsymbol{B}(K)$ the complete Boolean algebra that is the Dedekind completion of the Boolean algebra $B(K)$. It is convenient to identify it with the lattice of all regular open sets in $X(K)$, that is, $\mathbb{B}(K) \subseteq \mathcal{T}(K)$. We note some simple properties: $\neg_{\mathcal{T}} O = (CO)^0$, where C denotes set-theoretic complement, $\overset{\circ}{\bar{O}} = \neg_{\mathcal{T}}\neg_{\mathcal{T}} O$, and also $\overset{\circ}{\bar{O}}$ is the smallest regular open set containing O, $\bigvee_{\mathcal{T}\,\alpha} O_\alpha = \bigcup_\alpha O_\alpha$, $\bigvee_{\boldsymbol{B}\,\alpha} O_\alpha = (\bigcup_\alpha O_\alpha)^{\bar{\circ}}$, $\bigwedge_{\mathcal{T}\,\alpha} O_\alpha = (\bigcap_\alpha O_\alpha)^0$, $\bigwedge_{\boldsymbol{B}\,\alpha} O_\alpha = (\bigcap_\alpha O_\alpha)^0$, and $\neg_{\mathcal{T}} O = \neg_{\boldsymbol{B}} O$, that is, the operations in $\mathcal{T}$ and $\mathbb{B}$ differ only in the join.

Let L be the language of rings, that is, the language containing the symbols $=, +, -, \cdot, 0, 1$, one kind of variable (running over K), and all the usual connectives. For the family of parameters we take $D \leftrightharpoons K$. Then $L_0(D) = \{(k = t) \mid k, t \in K\}$. It is presumed that for atomic formulae with terms, that is, for the formulae of the form $(p = q)$, where p, q are polynomials with the parameters $k_1, \ldots, k_n \in K$ substituted in place of free variables, we first compute p and q in K and then calculate the valuation of $[\![k = t]\!]$, where $k, t \in K$. Such an understanding of terms that are parts of formulae is called *operational*.

So we put $[\![k = t]\!] = \bigvee \{e \in B(K) \mid e\cdot k = e\cdot t\}$, where $\bigvee$ is computed in one of the two completions of the algebra $B(K)$—the topological completion $\mathcal{T}(K)$ or the Dedekind completion $\mathbb{B}(K)$. In this way we define two valuations $[\![\cdot]\!]_{\mathcal{T}(K)}$ and $[\![\cdot]\!]_{\boldsymbol{B}(K)}$. They are respectively called the $\mathcal{T}$- and $\mathbb{B}$-*valuation in the language of rings*. The symbol $[\![\cdot]\!]_K$ is sometimes used instead of $[\![\cdot]\!]_{\mathcal{T}(K)}$. The valuation $[\![\cdot]\!]_{\mathcal{T}(K)}$ was defined by Scott (see [1]).

For any $p \in X(K)$, let $\bar{p}$ denote the smallest ideal in K containing p. It is easy to check that $\bar{p} = p \cdot K$ and $1 \notin \bar{p}$, that is, $\bar{p}$ is a proper ideal. Depending on the conditions imposed on the ring K, the class of ideals $\{\bar{p} \mid p \in X(K)\}$ can be described in terms of the ring K itself without mentioning the Boolean algebra $B(K)$. We put $K_p \rightleftharpoons K/\bar{p}$. We mention the valuable function $p \mapsto K_p$ and also the function $k \mapsto k(\cdot)$, where $k \in K$ and $k(\cdot): X(K) \to \bigcup_{p \in X(K)} K_p$, $k(p) \rightleftharpoons [k]_{\bar{p}}$ and $[k]_{\bar{p}}$ is the equivalence class in K_p of the element k with respect to the ideal $\bar{p}$. This construction (in fact, of the covering space $\langle \bigcup_{p \in X(K)} K_p, X(K)\rangle$ with fibres K_p) was obtained by Pierce.

Of course, if $k = t$, then $[\![k = t]\!]_{\mathcal{T}(K)} = [\![k = t]\!]_{B(K)} = 1$.

***Theorem* 5.** *The valuation $[\![\cdot]\!]_{\mathcal{T}(K)}$ is a normal and sheaf valuation and hence it is accessible.*

***Lemma* 1.** *If $e \leqslant [\![k = t]\!]_{\mathcal{T}(K)}$, where $e \in B(K)$, then $e \cdot k = e \cdot t$.*

Proof. By hypothesis $e \subseteq \bigcup_\alpha e_\alpha$, where $e_\alpha \cdot k = e_\alpha \cdot t$. By the compactness of e we get $e \subseteq e_{\alpha_1} \cup \ldots \cup e_{\alpha_n}$ and since the components are Boolean, $e = e_1' \sqcup \ldots \sqcup e_n'$, where $e_i' \in B(K)$, $e_i' \leqslant e_{\alpha_i}$, and $\sqcup$ denotes the disjunctive join. Therefore,

$$e = e_1' + \ldots + e_n', \qquad e \cdot k = e_1' \cdot k + \ldots = e_1' \cdot e_{\alpha_1} \cdot k + \ldots = e_1' \cdot e_{\alpha_1} \cdot t = e \cdot t.$$

(A simple argument also shows that the map $k \mapsto k(\cdot)$ is injective. The values of this map are called *sections*. Hence the elements of an arbitrary ring can be identified with sections.)

Proof of Theorem 5. By Lemma 1 we find at once that the valuation is normal. By Theorem 1, part a, the accessibility will follow from the fact that it is a normal and sheaf valuation, since $\mathcal{T}(K)$ is a zero-dimensional compact lattice. It remains to verify the fact that the valuation is a sheaf valuation. We first note that it is a weakly sheaf valuation, since we can paste together k and t on $e \in B(K)$ into the element $(e \cdot k + (1 - e) \cdot t)$ of the ring.

Suppose that $\bigcup_\alpha O_\alpha = e \in B(K)$. Then $\bigcup_{\alpha, \beta} e_{\alpha, \beta} = e$, where $O_\alpha = \bigcup_\beta e_{\alpha, \beta}$, $e_{\alpha_1} \cup \ldots \cup e_{\alpha_n} = e$, and e_{α_1} is the join of a finite number of $e_{\alpha_1, \beta_1}, \ldots, e_{\alpha_1, \beta_m}$. We form the disjunctive system $\{e_i\}$, where $\bigcup_i e_i = e$ and $e_i \leqslant e_{\alpha_i}$. Now we paste $\sum_i e_i \cdot k_{\alpha_i} \rightleftharpoons k$. This k is the required one: $e_i \leqslant [\![k = k_{\alpha_i}]\!]$,

$$O_\alpha = O_\alpha \wedge \bigvee_i e_i \leqslant \bigvee_i (O_\alpha \wedge O_{\alpha_i}) \wedge [\![k = k_{\alpha_i}]\!]) \leqslant \bigvee_i ([\![k_\alpha = k_{\alpha_i}]\!] \wedge [\![k = k_{\alpha_i}]\!]) \leqslant$$
$$\leqslant [\![k = k_\alpha]\!]. \ \square$$

The ring K is *normal* if $\forall k \in K \exists e_0 \in B(K)\ \forall e \in B(K) (e \cdot k = 0 \Leftrightarrow e \leqslant e_0)$. This formula written in the language of rings will be denoted by Φ_1. The intuitive meaning is clear: in a normal ring the set of annihilators from $B(K)$ of any fixed element $k \in K$ contains the greatest element. Examples of normal rings are the rings mentioned in Theorem 14, including the strictly bi-Rickartian rings defined by the condition $\forall k \in K \exists e \in B(K)(\langle k\rangle^* = e\cdot K)$, where $\langle k\rangle$ is the principal (two-sided) ideal generated by k, and a^* is the right annihilator of the ideal a, that is, $a^* = \{t \in K \mid a\cdot t = 0\}$. The condition that a ring be atomless will be denoted by Φ_2, that is,

$$\Phi_2 \rightleftharpoons \forall e \exists t\ \exists e_0 \forall t_1\ (e^2 = e \wedge e\cdot t = t\cdot e \Rightarrow e_0^2 = e_0 \wedge e_0\cdot t_1 = t_1\cdot e_0 \wedge \\ \wedge (e_0 = 0 \Rightarrow e = 0) \wedge (e_0 = e \Rightarrow e = 0) \wedge (e_0\cdot e = e_0).$$

The fact that K is atomless is the condition of the absence of atoms in the Boolean algebra $B(K)$, and is equivalent to the condition of the absence of isolated points in the topological space $X(K)$. Let Φ_3 denote the condition that the ring K is Abelian, that is, $\Phi_3 \rightleftharpoons \forall e \forall t\ (e^2 = e \Rightarrow e\cdot t = t\cdot e)$. For example, in a ring without nilpotent elements (that is, $x^2 = 0 \Rightarrow x = 0$) Φ_3 is fulfilled. Other properties of rings frequently mentioned in this paper are introduced in §I.6, and are denoted there by Φ_4, Φ_5.

A ring K will be called a *sheaf ring* if it is normal and, in addition,

$$\forall \{e_\alpha\} \subseteq B(K)\ \forall \{k_\alpha\} \subseteq K \exists k \in K\ (1 = \bigvee_B e_\alpha \wedge \forall \alpha, \beta\ (e_\alpha\cdot e_\beta\cdot k_\alpha = \\ = e_\alpha\cdot e_\beta\cdot k_\beta) :\Rightarrow: \forall \alpha\ (e_\alpha\cdot k = e_\alpha\cdot k_\alpha)).$$

The normality condition ensures the uniqueness of such a k; we shall denote it by $\sum_i e_\alpha\cdot k_\alpha$. In the case when $B(K)$ is a complete Boolean algebra, the notion of a sheaf ring (module) coincides with that of an orthogonally complete ring (module) introduced by Beidar and Mikhalev. The term used here is consistent with the theory of sheaves and with [1].

***Theorem* 6.** *If K is a normal ring, then the valuation $[\![\cdot]\!]_{B(K)}$ is normal. If K is a sheaf ring, then this valuation is a normal and sheaf valuation and hence accessible.*

Proof. If K is normal, then $\bigcup \{e \mid e\cdot k = 0\} = e_0$, where e_0 comes from the normality condition for k. Then $[\![k = 0]\!]_B = [\![k = 0]\!]_{\mathcal{T}} = e_0$, and $[\![k = 0]\!]_B = 1$ means that $e_0 = 1$. It follows by Lemma 1 that $k = 0$. We verify that the valuation is a sheaf valuation. Suppose that $b = \bigvee_\alpha b_\alpha$ and $b_\alpha \wedge b_\beta \leqslant [\![k_\alpha = k_\beta]\!]_{B(K)}$. We write $b_\alpha = \bigcup_\gamma e_{\alpha\gamma}$, where $e_{\alpha\gamma} \in B(K)$. Then we have $e_{\alpha\gamma_1} \wedge e_{\beta\gamma_2} \leqslant [\![k_\alpha = k_\beta]\!]_{B(K)}$; from the fact that the ring is normal we have $[\![k_\alpha = k_\beta]\!]_{B(K)} = [\![k_\alpha = k_\beta]\!]_{\mathcal{T}(K)}$, and by Lemma 1 we have $e_{\alpha\gamma_1}\cdot e_{\beta\gamma_2}\cdot k_\alpha = e_{\alpha\gamma_1}\cdot e_{\beta\gamma_2}\cdot k_\beta$. We adjoin to the family $\{e_{\alpha\gamma}\}$ all e_δ, where $\neg b = \bigcup_\delta e_\delta$, $e_\delta \in B(K)$, and correspondingly we adjoin to the family $\{k_\alpha\}$ any fixed element of K. By the condition that the ring K is a sheaf ring, we can find a k such that $e_{\alpha\gamma} \leqslant [\![k = k_\alpha]\!]_{\mathcal{T}(K)} \leqslant [\![k = k_\alpha]\!]_{B(K)}$. Therefore $b_\alpha \leqslant [\![k = k_\alpha]\!]_{B(K)}$. We now apply Theorem 1, part a. □

Example 3. Let $\mathscr{X}$ be a left module over a ring K (as in Example 2). The language of modules consists of two sorts of variable, k (over K) and x (over $\mathscr{X}$), the atomic symbols of the language of rings and, moreover, of the symbol $\cdot \underset{1}{=} \cdot$ (for elements of $\mathscr{X}$), the symbols $+$, $-$, 0 (for operations in $\mathscr{X}$), the functional symbol $k \cdot x$, and also the usual connectives for the variables of both sorts. The definition of the valuation in Example 2 is supplemented by the condition $[\![x = y]\!] = \bigvee \{e \in B(K) \mid e \cdot x = e \cdot y\}$, where $\bigvee$ is taken in $\mathscr{T}(K)$ or $\mathrm{B}(K)$. We call these valuations $\mathscr{T}$- and B-*valuations*, respectively, in the language of modules.

A module over a normal ring is called *normal* if $\forall x \exists e_0 \forall e (e \cdot x = 0 \Leftrightarrow e \leqslant e_0)$. *Sheaf modules* are defined in the same way as in Example 2.

Theorem 7. a) *The valuation* $[\![\cdot]\!]_{\mathscr{T}(K)}$ *in the language of modules is a normal and sheaf valuation, and therefore it is accessible.*

b) *If the module* $\mathscr{X}$ *is normal, then the valuation* $[\![\cdot]\!]_{B(K)}$ *in the language of modules is normal. If* $\mathscr{X}$ *is a sheaf module, then the valuation* $[\![\cdot]\!]_{B(K)}$ *is a normal and sheaf valuation and therefore it is accessible.*

The proof is the same as in the case of rings.

Proposition 1. *For a normal ring (module) K and a formula φ in the language of rings not containing the quantifier $\exists$ in the premises of any implication nor the quantifier $\forall$ in the domain of any connective $\bigvee$, we have* $[\![\varphi]\!]_{\mathscr{T}(K)} \leqslant [\![\varphi]\!]_{B(K)}$. *For a sheaf ring (module) K and any formula φ in the prenex form we have* $[\![\varphi]\!]_{\mathscr{T}(K)} = [\![\varphi]\!]_{B(K)}$. *(These yield the corresponding relations between the global validities.)*

Example 4. One of the foremost examples of valuation is the function associating with every formula the corresponding element of the Lindenbaum-Tarski algebra. However, such a valuation is "made" in an essentially different way from the valuations of Examples 1, 2 (see [17], p.211).

Example 5. The modal connectives can be interpreted by means of the corresponding evaluation. Then the modal logics are well described by means of common validity in suitable classes of presheaves and sheaves (defined in §I.7), that is, by means of the corresponding evaluation. These classes can be described extensively, in particular, in terms of the homology of sheaves. There are possibly interesting examples of evaluation in more general lattices such as semidistributive and Dedekind lattices, in particular in connection with quantum logic. By constructing the algebra Ω one can compute the valuation in an algorithmic way (under certain assumptions) in a time which is of the order of the logarithm of the number of elements of this algebra.

I.4. The connection between deducibility and global validity.

As defined in §I.1, Φ_+ is the set of all closed formulae φ with parameters from D such that $\Omega \models \varphi$ (for a fixed language L, a family of parameters D, a complete Heyting algebra Ω, and a valuation $[\![\cdot]\!]_{\Omega, D}$). How can we find this set in practice? Direct computation of the valuation is sometimes workable but may be a complicated task. We usually know beforehand a list T (finite or infinite) of formulae ("axioms") belonging to Φ_+. It is useful to find a notion of deduction $\underset{\Omega}{\vdash}$ (depending on the valuation) such that $(T \underset{\Omega}{\vdash} \varphi) \Rightarrow (\Omega \models \varphi)$. In certain cases we have $(T \underset{\Omega}{\vdash} \varphi) \Leftrightarrow (\Omega \models \varphi)$ and we can even have T and $\underset{\Omega}{\vdash}$ such that the set of all deducible formulae is recursively axiomatizable. For example, this happens in the case when $\Omega = [0;\ 1]^h$. In this context we shall need in what follows some (very traditional) axiomatics and a notion of deduction.

The axiomatics of the set theory *ZFC* consisting of the axioms of *ZF* and the axiom of choice *AC* is well known (see [2]). Considering it together with the axioms and the rules of inference of the classical predicate calculus in the language *ZF*, we obtain the theory *ZFC*, deducibility in which we shall denote by $ZFC \vdash$. The theory *ZF* is obtained from the theory *ZFC* by removing the axiom *AC*. Deducibility in the classical predicate calculus is denoted by $\vdash$.

The intuitionistic predicate calculus is obtained from the classical predicate calculus by removing the "law of the excluded middle"—the axiom $\varphi \vee \neg\varphi$ (and the equivalent axiom $\neg\neg\varphi \Rightarrow \varphi$), and we denote deducibility in this calculus by $\underset{H}{\vdash}$. Combining this calculus in the language *ZF* with the axioms of the theory *ZF*, where exactly one of them is reformulated in the classically equivalent way, we obtain the intuitionistic theory of sets HZF^-. The axioms of *ZF* mentioned above are the axioms of equality (for $\cdot\in\cdot$ and $\cdot=\cdot$, which, however, follow from the other axioms), extensionality, pairing, unions, power-set, $\in$-induction (instead of the classical axiom of founding which is equivalent to it): $\forall x\, (\forall y \in x \varphi(y) \Rightarrow \varphi(x) \Rightarrow \forall x \varphi(x))$, separation, replacement: $\forall x \in u \exists! y \varphi(x, y) \Rightarrow \exists v\, \forall x \in u \exists y \in v \varphi(x, y)$, and infinity. If in HZF^- we substitute the replacement axiom by a stronger axiom of "collection": $\forall x \in u \exists y \varphi(x, y) \Rightarrow \exists v \forall x \in u \exists\, y \in v \varphi(x, y)$, then we obtain a theory which we denote by *HZF*. Of course, $ZF \vdash HZF$. In certain cases the theory *HZF* is made even stronger by adding Zorn's lemma *ZL* as an axiom and the axiom *HAC* (in *ZF* the axiom *ZL* is equivalent to *HAC*, which is equivalent to the axiom *AC*):

$$\forall x \in u \exists y\, (y \in x) \wedge \forall x, y \in u\, (x = y \vee \neg (x = y)) \Rightarrow \exists f\colon u \to \bigcup u\, (\forall x \in u\, (f(x) \in x)).$$

It is interesting that in the theory $HZF^+ \rightleftharpoons (HZF + ZL + HAC)$ the law of the excluded middle is still not deducible, so this theory also is to some extent intuitionistic. For the theory HZF^- the properties of disjunctiveness and extensionality are fulfilled, and this underlines its intuitionistic status (see, for example, [9], [10]).

Theorem 8. a) *If* **ZFC** $\vdash \varphi$, *then for any complete Boolean algebra* $\mathbb{B}$ *we have* $\mathbb{B} \models \varphi$ (*here we consider the valuation in the language ZF defined in Example* 1).

b) *If* **HZF** $\vdash \varphi$, *then for any complete Heyting algebra* Ω *we have* $\Omega \models \varphi$ (*here we consider the same valuation in the language ZF*).

c) *For any valuation* $[\![\cdot]\!]_B$ *with values in a complete Boolean algebra* $\mathbb{B}$, *if* $\varphi_1, \ldots, \varphi_n \vdash \psi$, *then* $[\![\varphi_1]\!]_B \wedge \ldots \wedge [\![\varphi_n]\!]_B \leqslant [\![\psi]\!]_B$. *So* $(B \models \varphi_1 \wedge \ldots \wedge \varphi_n) \Rightarrow (\mathbb{B} \models \psi)$, *where* $n \geqslant 0$.

d) *For any valuation* $[\![\cdot]\!]_\Omega$ *with values in a complete Boolean algebra* Ω, *if* $\varphi_1, \ldots, \varphi_n \vdash_H \psi$, *then* $[\![\varphi_1]\!]_\Omega \wedge \ldots \wedge [\![\varphi_n]\!]_\Omega \leqslant [\![\psi]\!]_\Omega$, *so* $(\Omega \models \varphi_1 \wedge \ldots \wedge \varphi_n) \Rightarrow (\Omega \models \psi)$, *where* $n \geqslant 0$.

e) *Let* $\Omega_1 = \mathcal{T}(X_1)$, *where* X_1 *is the Baire space* (*that is,* $X_1 = \omega^\omega$, *where* ω *is the sequence of natural numbers*). *If for any sheaf valuation* $[\![\cdot]\!]_{\Omega_1}$ *we have* $\Omega_1 \models \varphi$ *then* $\vdash_H \varphi$ ("*the completeness theorem for intuitionistic predicate calculus in the language ZF*").

In this way, the traditional classical and the intuitionistic predicate calculi, as well as the axiomatics of set theory, are closely connected with valuations. We have the same situation in modal logics.

The following two (mutually dependent) problems are closely connected with non-classical logics, in particular with intuitionistic logic: firstly, the development within the framework of the given logic of ordinary mathematics (algebra, topology, analysis, and so on) and, secondly, for a given logic the quest for a syntactical translation of the formulae $\varphi \mapsto \varphi'$ for which the deducibility (or validity) of the formula φ in the theory with classical logic implies the deducibility (respectively, validity) of the formula φ' in the same theory with this non-classical logic, and the meanings of the statements φ and φ' are close. For example, the famous Kolmogorov (Gödel negative) translation $\varphi \mapsto \varphi^-$ is such that φ^- is obtained from φ by adding $\neg\neg$ before the atomic formulae and the connectives $\vee$, $\exists$. Then $\vdash \varphi$ implies $\vdash_H \varphi^-$. The Gödel negative translation that solves, for example, the question of the consistency of the theories with both classical and intuitionistic logics apparently gives little from the point of view of developing intuitionistic mathematics. The point is that the theorems φ of, for example, Galois theory mean little in the form φ^-. A fundamental result of Novikov ([5], p.127) gives another example. If in the ring of integers $\mathbb{Z}$ the formula $\varkappa \leftrightharpoons \forall x \exists y \varphi(x, y)$ (where $\varphi(x, y)$ defines a decidable predicate) is (classically) valid, then in the same ring the formula $\varkappa$ is intuitionistically valid. The intuitionistic validity is defined, in essence and substantially, without axiomatization. Intuitionistic logic is connected with the idea of "effectiveness": if $\exists y \varphi'(y)$ or $\forall x \exists y \varphi'(x, y)$ is intuitionistically valid, then we reckon that we can effectively produce a y such that $\varphi'(y)$ or a function $f(x)$ such that $\forall x (\varphi'(x, f(x)))$ (as a term or in another explicit way).

Hence the importance of the results of the following type is clear: if $\exists y\varphi(y)$ or $\forall x\exists y\varphi(x, y)$ is classically valid in a ring K, then $\exists y\varphi'(y)$ or $\forall x\exists y\varphi'(x, y)$ is intuitionistically valid in the same ring, where φ' is close in meaning to φ and does not contain "meaningless" connectives like $\neg\neg$. In the case of the results of Novikov we have $K = \mathbb{Z}$ and $\varphi = \varphi'$. In Ch. III we shall consider Novikov's conjecture (stated in his paper) that his result is true "under very broad conditions"; see also the definition in §II.1.

I.5. The connection between validity and global validity.

We shall find some classes of formulae useful. We say that a formula is in "*weakly E-normal form*" if none of its implications contains the connective $\exists$ in the premise and if there is no connective $\forall$ in the domain of the connective $\bigvee$. We have a "*weakly A-normal form*" if neither $\forall$ nor $\Rightarrow$ occurs in the premises of implications. A formula is in a "*normal form*" if every implication occuring in it contains only the connectives $\bigwedge$ and $\bigvee$ in the premise and in the conclusion. We recall that by $\neg\varphi$ we mean $\varphi \Rightarrow \bot$.

A formula is positive if it does not contain the connective $\Rightarrow$ (nor, of course, the connective $\neg$).

A *Horn* formula is obtained by applying the connectives $\forall, \exists, \bigwedge$ to a formula of the form $\varphi_1 \bigwedge \ldots \bigwedge \varphi_n \Rightarrow \varphi$, where $n \geqslant 0$ and $\varphi_1, \ldots, \varphi_n, \varphi$ are atomic formulae (including $\top$ and $\bot$). This class can be written graphically in the form $\forall, \exists, \bigwedge[\varphi_1 \bigwedge \ldots \bigwedge \varphi_n \Rightarrow \varphi]$. The last notation is also convenient in other cases, and therefore we denote by $Q_1, \ldots, Q_k[\chi_1, \ldots, \chi_m]$ the class of formulae obtained by the application in arbitrary order and number of the connectives $Q_1, \ldots, Q_k$ to the formulae ("blocks") of the form $\chi_1, \ldots, \chi_m$.

A formula is *almost positive* if it is of the form $\exists, \forall, \bigwedge, \bigvee$ $[\forall \bar{t}\, (\varphi(\bar{t}) \Rightarrow \psi(\bar{t}))]$, where φ is any Horn formula and ψ is any positive or almost positive formula. The variables $\bar{t}$ are called *block* variables and φ is called the *premise of the block*. We note that, besides $\bar{t}$, φ can also contain other block variables. For example, in the formula

$$\zeta \rightleftharpoons \forall w\ (\exists k\ (w \leqslant k) \,{:}{\Rightarrow}{:}\, \exists t\ (w \leqslant t \bigwedge \forall s\ (w \leqslant s \Rightarrow t \leqslant s))$$

(this is an important formula expressing the conditional completeness of a lattice; w is an arbitrary subset and k, t, s are arbitrary elements of the lattice), both w and s are block variables in $\varphi \rightleftharpoons (w \leqslant s)$.

The rank of an almost positive formula is defined by induction. An almost positive formula whose blocks all have positive formulae in the conclusions is of rank 0. An almost positive formula whose blocks all have almost positive formulae of rank up to and including $(n-1)$ in the conclusions is of rank n.

We fix an arbitrary valuation $[\![\cdot]\!]_{\Omega, D}$ and also an almost positive sentence $\varkappa \rightleftharpoons \varkappa(k_1, \ldots, k_n)$ with parameters from D. For an almost positive formula $\varkappa$ we define an auxiliary predicate $P[\cdot]$. If $\varkappa$ is positive, then $P[\varkappa] \rightleftharpoons T$.

If $\varkappa$ is a block of the form $\forall \bar{t}\ (\varphi(\bar{t}) \Rightarrow \psi(\bar{t}))$, then

$$P[\varkappa] \rightleftharpoons \exists \bar{t}_0 \in D\ ([\![\varphi(\bar{t}_0)]\!] = 1) \wedge \forall \bar{t} \in D\ (D \models \varphi(\bar{t}) \Rightarrow P[\psi(\bar{t})]);$$

next, $P[\exists x\varphi] \rightleftharpoons \exists k \in D(D \models \varphi(k) \wedge P[\varphi(k)])$, $P[\forall x\varphi] \rightleftharpoons \forall k \in D(P[\varphi(k)])$, $P[\varphi \wedge \psi] \rightleftharpoons P[\varphi] \wedge P[\psi]$, $P[\varphi \vee \psi] \rightleftharpoons (P[\varphi] \wedge D \models \varphi) \vee (P[\psi] \wedge D \models \psi)$. The sentence $\varkappa$ is *decidable* if $D \models \varkappa$ implies $P[\varkappa]$. For example, the sentence ζ has rank 1, and is decidable since we can choose $w \rightleftharpoons \varnothing$ and k by the condition $D \models \exists k\ (w \leqslant k)$. The valuation we consider here is such that $[\![w \leqslant k]\!] \rightleftharpoons [\![\forall x \in w\ (x \leqslant k)]\!]$.

The choice of the term "decidable" can be explained as follows: if in the language of rings an almost positive formula $\varkappa$ has the form $\varkappa = \forall \bar{t}\ (\varphi(\bar{t}) \Rightarrow \psi(\bar{t}))$, where $\varphi(\bar{t}) = p_1(\bar{t}) = 0 \wedge \ldots \wedge p_m(\bar{t}) = 0$ and $p_1, \ldots, p_m$ are polynomials, and ψ is decidable, then the decidability of $\varkappa$ (for a normal valuation $[\![\cdot]\!]_{\Omega, D}$) is equivalent to the fact that the system of equations $p_1 = 0, \ldots, p_m = 0$ can be solved in D. Various criteria, including syntactical ones, can be given for the decidability of almost positive formulae.

***Theorem* 9.** *Suppose that a valuation $[\![\cdot]\!]_{\Omega, D}$ is normal for all atomic formulae and is accessible. The formulae we consider contain the parameters from D.*

a) *For any Horn formula φ, $\Omega \models \varphi$ implies $D \models \varphi$.*

b) *For any positive formula ψ, $D \models \psi$ implies $\Omega \models \psi$.*

c) *Suppose, in addition, that $[\![\cdot]\!]_{\Omega, D}$ is a weakly sheaf valuation and Ω is zero-dimensional. For any almost positive formula $\varkappa$, $D \models \varkappa$ implies $\Omega \models \varkappa$.*

Proof. For an atomic formula φ, the fact that the valuation is normal yields $(D \models \varphi) \Leftrightarrow [\![\varphi]\!] = 1$. If the valuation of the formula $\varphi_1 \wedge \ldots \wedge \varphi_n \Rightarrow \varphi$ is equal to 1, then $[\![\varphi_1 \wedge \ldots \wedge \varphi_n]\!] \leqslant [\![\varphi]\!]$. If $[\![\varphi_i]\!] < 1$ for some i, $1 \leqslant i \leqslant n$, then $D \not\models \varphi_i$ and the proof is complete. Otherwise $[\![\varphi]\!] = 1$ and the proof is complete. If $[\![\varphi \wedge \psi]\!] = 1$ or $[\![\forall x\varphi]\!] = 1$, then we use induction. If $[\![\exists x\varphi]\!] = 1$, then by accessibility we have $[\![\varphi(k)]\!] = 1$, where $k \in D$, and we use induction.

b) For the atomic formulae we proceed in the same way and then use induction.

c) Suppose that a decidable almost positive formula $\varkappa$ of rank n with parameters from D is valid in D. By the condition of decidability we obtain $P[\varkappa]$. By induction with respect to the structure of $\varkappa$ we shall show that if $D \models \varkappa(\bar{k})$ and $P[\varkappa(\bar{k})]$, then $[\![\varkappa(\bar{k})]\!] = 1$. If $\varkappa$ is a block, then we compute the valuation only for $\bar{t}$ such that $[\![\varphi(\bar{t})]\!] = 1$. Whence $D \models \psi(\bar{t}, \bar{k})$ and also $P[\psi(\bar{t})]$, where ψ is an almost positive formula of rank less than or equal to $n-1$. For ψ we perform induction with respect to the rank, whence $[\![\psi]\!] = 1$. If $\varkappa \rightleftharpoons \exists x\varphi$, then $D \models \varphi(k_0, \bar{k})$ and $P[\varphi(k_0, \bar{k})]$, that is, we can apply the induction step. If $\varkappa \rightleftharpoons \forall x\varphi$, then $D \models \varphi(k_0, \bar{k})$ and $P[\varphi(k_0, \bar{k})]$ for any $k_0 \in D$. If $\varkappa \rightleftharpoons \varphi \wedge \psi$ or $\varkappa \rightleftharpoons \varphi \vee \psi$, then (second case) $D \models \varphi$ and $P[\varphi]$. □

Suppose that the Heyting algebras Ω_1 and Ω_2 are embedded in a lattice H and H is a Heyting algebra, though we do not assume that the embedding preserves the operation $\to$, but instead assume that $u \underset{\Omega_1}{\to} v \leqslant u \to v \leqslant u \underset{\Omega_2}{\to} v$, where $\to$ is implication in H. If $\Omega_1 \subseteq \Omega_2$ or $\Omega_2 \subseteq \Omega_1$, then for H we take Ω_2 or Ω_1 respectively. We write $\Omega_1 \leqslant \Omega_2$ if $\forall\alpha\,(u_\alpha \leqslant v_\alpha)$ implies $\bigvee_\alpha {}_{\Omega_1} u_\alpha \leqslant \bigvee_\alpha {}_{\Omega_2} v_\alpha$ and $\bigwedge_\alpha {}_{\Omega_1} u_\alpha \leqslant \bigwedge_\alpha {}_{\Omega_2} v_\alpha$. We write $\Omega_1 \lesseqgtr \Omega_2$ if $\Omega_1 \leqslant \Omega_2$ but with $u \vee_{\Omega_1} v \leqslant u \vee_{\Omega_2} v$ (that is, $\leqslant$ instead of $=$) and $u \underset{\Omega_1}{\to} v = u \underset{\Omega_2}{\to} v$ (that is, $=$ instead of $\leqslant$). For example, for any ring K we have $\mathrm{B}(K) \subseteq \mathcal{T}(K)$ and $\mathcal{T}(K) \lesseqgtr \mathrm{B}(K)$ and, what is more, we have $\bigwedge_{\mathcal{T}} u_\alpha = \bigwedge_B u_\alpha$. The second important example is as follows. For any complete Heyting algebra Ω we have $\Omega \subseteq \mathrm{B}(\Omega)$, $\Omega \leqslant \mathrm{B}(\Omega)$ and, what is more, $\bigvee_\Omega u_\alpha = \bigvee_{B(\Omega)} u_\alpha$, where $\mathrm{B}(\Omega)$ was defined at the end of §I.2. These properties easily follow from the definitions of the corresponding algebras.

***Theorem* 10.** a) *Suppose that $\Omega_1 \leqslant \Omega_2$, $[\![\cdot]\!]_{\Omega_1, D_1}$ and $[\![\cdot]\!]_{\Omega_2, D_2}$ are two valuations coinciding on atomic formulae, and a formula φ has parameters from $D_1 \cap D_2$ and the premises of all its implications contain only the connectives $\vee, \wedge$. Then $[\![\varphi]\!]_{\Omega_1, D_1} \leqslant [\![\varphi]\!]_{\Omega_2, D_2}$.*

b) *Suppose that φ is a formula in the language of rings with parameters from the ring K. If K is a normal ring and φ is in weakly E-normal form, then $[\![\varphi]\!]_{\mathcal{T}(K)} \leqslant [\![\varphi]\!]_{B(K)}$.*

Part b) of this theorem is closely related to Proposition 1.

I.6. Cancellation of idempotents and transfer theorems in the case of rings. Expressibility of global validity.

The ring K is "*without idempotents*" if it has the property

$$\Phi_4 = \forall e\,(e^2 = e \Rightarrow e = 0 \vee e = 1),$$

and "*indecomposable*" if it has the property

$$\Phi_5 = \forall e \exists t\,(e^2 = e \wedge e\cdot t = t\cdot e \Rightarrow e = 0 \vee e = 1);$$

we also write

$$\Phi_5' = \forall e\,((e^2 = e \wedge \forall t\,(e\cdot t = t\cdot e)) \Rightarrow e = 0 \vee e = 1).$$

Classically (but not intuitionistically) Φ_5 and Φ_5' are equivalent.

***Theorem* 11.** a) *For any ring K and any $\mathcal{T}(K)$-valuation in the language of rings the property Φ_5' is globally valid. If K is a normal ring, then for a $\mathrm{B}(K)$-valuation the property Φ_5 (and Φ_5') is globally valid.*

b) *Let T be an arbitrary collection of almost positive decidable sentences in the language of rings with parameters from the ring K, and suppose that $K \models T$. If T and $\Phi_5' \underset{H}{\vdash} \varphi$, then $\mathcal{T}(K) \models \varphi$. If K is a sheaf ring and $T, \Phi_5 \vdash \varphi$, then $\mathrm{B}(K) \models \varphi$.*

Remark. This theorem explains why the passage to global validity is useful: all idempotents cancel and we can use this, since global validity turns into ordinary validity (see, in particular, Theorem 12).

Let $\varphi \mapsto \varphi'$ be the following translation of the formulae in the language of rings with parameters from K. If $\varphi(k_1, \ldots, k_n)$ is any formula in prenex disjunctive form and $e \in B(K)$, then $\varphi'(k_1, \ldots, k_n, e)$ is obtained from φ by a shift of the quantifier prefix occurring in φ and by the assignment

$$\exists \bar{e}_s \forall t \, \forall e_0 \exists t_0 [e_1^2 = e_1 \wedge e_1 \cdot t = t \cdot e_1 \wedge \ldots \wedge \prod_s (1 - e_s) \leqslant \leqslant (1 - e) \wedge e_1 \cdot k_1 = e_1 \cdot t_1 \wedge \ldots \wedge (e_0^2 = e_0 \wedge e_0 \cdot t_0 = t_0 \cdot e_0 \wedge e_0 \cdot k_2 = = e_0 \cdot t_2 \Rightarrow e_0 \leqslant 1 - e_1) \wedge \ldots],$$

where the index s runs over the number of the disjunctive parts of φ, $k_1 = t_1$ is the first equality, and $k_2 \neq t_2$ is the first inequality in the first disjunctive part of φ. We note that φ' is always a Horn formula.

***Theorem* 12.** *Suppose that φ is in disjunctive normal form.*

a) *For any Boolean element $e \in B(K)$ we have*

$$([\![\varphi\ (\bar{l})]\!]_{\mathcal{T}(K)} \geqslant e) \Leftrightarrow (K \models \varphi'\ (l,\ e)).$$

b) *For any sheaf ring K and $e \in B(K)$ we have*

$$([\![\varphi\ (\bar{l})]\!]_{B(K)} \geqslant e) \Leftrightarrow (K \models \varphi'\ (l,\ e)).$$

In the same way we define the translation $\varphi'(l) \mapsto \varphi''(l, e)$ for formulae φ in conjunctive normal form. Namely, for a quantifier-free φ we put

$$\varphi'' = \exists e_1, \ldots, e_r \, \forall t \, \forall e_0 \, \exists t_0 \, [e_1^2 = e_1 \wedge e_1 \cdot t = t \cdot e_1 \wedge \ldots \wedge \prod_s (1 - e_s) \leqslant \leqslant 1 - e \wedge \ldots \wedge e_1 \cdot k_1 = e_1 \cdot t_1 \wedge \ldots \wedge (e_0^2 = e_0 \wedge e_0 \cdot t_0 = t_0 \cdot e_0 \wedge e_0 \cdot k_2 = = e_0 \cdot t_2 \Rightarrow e_0 \leqslant 1 - e_1) \wedge \ldots],$$

where s runs through the number of the factors, r is the total number of disjunctive parts, and the quantifier prefix is shifted as before.

In the same way, such a translation can also be defined for other forms of presentation of φ.

Instead of $\varphi'(\bar{k}, 1)$ and $\varphi''(\bar{k}, 1)$ we shall write $\varphi'(\bar{k})$ and $\varphi''(\bar{k})$.

***Theorem* 12.** c) *Suppose that φ is in conjunctive normal form. The equivalence in parts* a) *and* b) *of Theorem* 12 *can be replaced by an equivalence of the form* $([\![\varphi\ (\bar{k})]\!] \geqslant e) \Leftrightarrow (K \models \varphi''\ (\bar{k},\ e))$.

Remark. In Theorems 11, 12, and in the next corollary, it is sufficient to consider, instead of sheaf rings, normal rings and accessible valuations.

***Corollary* 1.** *Any set of sentences (in a corresponding language extending the initial language of rings) of the form $\mathcal{T}(K) \models \varphi$ ("φ is globally valid"), where φ runs through the set of sentences in the language of rings with* *parameters from K, is equivalent in the class of all rings to a Horn theory T in the language of rings with the same range of parameters from K, namely, to the theory* $T' = \{\varphi' \mid \varphi \in T\}$; *instead of* T' *we can take the theory* $T'' = \{\varphi'' \mid \varphi \in T\}$ *(here we assume that T is, respectively, in disjunctive or conjunctive normal form). If global validity is understood in the sense of* $\mathbb{B}(K)$*-valuation, then the same assertion is true in the class of all sheaf rings.*

A class $\mathcal{K}$ of rings is *internally axiomatizable* if there is a theory T (in the language of rings) such that $K \in \mathcal{K} \Leftrightarrow [\![T]\!] = \mathbf{1}$. Corollary 1 says that in a number of cases the "internal" axiomatizability coincides with the "external" (that is, the usual) axiomatizability. Similar assertions are also true for certain other structures and the corresponding valuations. Other forms of axiomatizability, such as local axiomatizability, are considered in Ch. II. We note that internal axiomatizability is "simpler", for example, in the sense that the theory T can be formulated in the class of indecomposable rings.

Corollary 2. *If* φ *is a Horn formula in the language of rings with parameters from an arbitrary ring K, then* $(K \models \varphi''(\bar{k}))$ *or* $(K \models \varphi'(\bar{k}))$ *implies* $K \models \varphi(\bar{k})$.

Theorem 13. *Under the conditions of Theorem* 11b) *we have* $K \models \varphi' \wedge \varphi''$ *(if* φ *is a Horn formula, then also* $K \models \varphi$*).*

Example 6. By Theorem 11b) K-spaces and $L\mathbb{R}$-algebras (that is, algebras of measurable $\mathbb{R}$-valued functions) for the corresponding valuations can be described as the field $\mathbb{R}$. Theorem 13 gives the transfer of the corresponding properties of the ordered field $\mathbb{R}$ to K-spaces and $L\mathbb{R}$-algebras. (The description of K-spaces as the field $\mathbb{R}$ was given by Gordon in 1977 and the connection between $L\mathbb{R}$-algebras and real closed fields was discovered by Scott in 1969).

The classes of rings mentioned in the next theorem are defined, for example, in [3], p.389. Theorem 14 enables us to apply Theorem 11, part b (and consequently Theorem 13) by adding to Theorem 11, part b the property i') before the symbol of deducibility $\vdash$ in the case when the ring K has the property i), where the number i takes the values 1, 2, 3, 4. We note that, compared with i), i') is a strong property. This extends the remark after Theorem 12c).

Theorem 14. *For a ring K which is* 1) *strictly bi-Rickartian,* 2) *biregular,* 3) *Abelian regular, or* 4) *strictly Rickartian, the following properties are* $\mathcal{T}(K)$*-globally valid respectively:* 1$'$) *prime,* 2$'$) *quasisimple,* 3$'$) *division ring, or* 4$'$) *without zero divisors.*

A proof of the theorem is contained in [3].

I.7. Universal valuation. The notion of sheaf on a Heyting algebra. One of the important merits of the language of *ZF* set theory and also of *ZFC* set theory is that almost all the usual mathematical languages can be interpreted in the language ZF, and the corresponding theories can be reduced to the theory ZFC, and often even to the weaker theory ZF. A second important merit of the language of ZF theory is that, working within the framework of this theory, we can use in the arguments such notions as ideal, family of ideals, the rings $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and so on, which are not expressible in the language of rings.

Therefore, it is of interest to reduce the valuation $[\![\cdot]\!]_{\Omega, D}$ in a certain specific language L with a specific set of parameters D^{Ω} to a valuation in the language ZF (with a set of parameters Ω fixed once and for all). Such a reduction is possible (see Theorem 16) and in this sense the valuation in the language ZF (defined in Example 1) is sometimes called a *universal valuation*. Of course, this does not mean that other valuations are not necessary.

To work with arbitrary valuations, the notions of a presheaf and a sheaf on a Heyting algebra are convenient and we recall them now. These are connected with the names of Leray, Grothendieck, Lawvere, and Tierney (see [1]).

Any partially ordered set, in particular a Heyting algebra H, can be considered as a trivial category: the set of objects is H itself, and the set of morphisms $\operatorname{Hom}(u, v)$, where $u, v \in H$, consists of one element if $u \leqslant v$ and is empty if $u \nleqslant v$. A contravariant functor $\mathcal{F}(\cdot)$ from H into any category is called a *presheaf*. For the sake of definiteness we consider presheaves with values in the category of sets, that is, in fact with values in the class of all sets V. By definition, the sets $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are disjoint if $u \neq v$. Suppose that H is embedded in a complete Heyting algebra Ω with respect to the operations $\wedge$, $\vee$, 0, 1 (but not necessarily with respect to $\rightarrow$). Then Ω induces in H a predicate $u = \bigvee_{\alpha}{}_{\Omega} u_\alpha$, where $u \in H$, $\{u_\alpha\} \subseteq H$. A presheaf $\mathcal{F}(\cdot)$ on H is a *sheaf* (with respect to the given Ω) if

$$\forall u \in H \forall \{u_\alpha\} \subseteq H \forall \{k_\alpha\}\ \exists!\ k \in \mathcal{F}(u)\, (u = \bigvee_{\alpha}{}_{\Omega} u_\alpha \wedge \forall \alpha\, (k_\alpha \in \mathcal{F}(u_\alpha)) \wedge \forall \alpha,$$

$$\beta\, (\rho^{u_\alpha}_{u_\alpha \wedge u_\beta}\, (k_\alpha) = \rho^{u_\beta}_{u_\alpha \wedge u_\beta}\, (k_\beta) \Rightarrow \forall \alpha\, (\rho^{u}_{u_\alpha}\, (k) = k_\alpha),$$

where ρ^u_v is the morphism corresponding to the case $v \leqslant u$.

A presheaf $\mathcal{F}(\cdot)$ on H is *normal* (*Hausdorff*) if in the previous property instead of $\exists!\ k$ we only require the uniqueness of k.

It is easy to prove (by means of inverse limits) that a sheaf can be canonically extended from H to a sheaf on Ω. If H is a complete Heyting algebra, then for Ω we take H itself (unless explicitly stated otherwise).

If $\mathcal{F}(\cdot)$ is a presheaf on Ω, then we put $\mathcal{F} = \bigcup\{\mathcal{F}(u) \mid u \in \Omega\}$ and we put Ek equal to the unique u from Ω such that $k \in \mathcal{F}(u)$. We note that $E\colon \mathcal{F} \to \Omega$ On $\mathcal{F}$ we define the operation $k \restriction u = \rho^{Ek}_{Ek \wedge u}\, (k)$. Instead of $\mathcal{F}(1)$ we write $\mathcal{F}_1$. The elements of the set $\mathcal{F}(1) = \mathcal{F}_1$ are called *global*. Of course, $\rho^1_u\, (k) = k \restriction u$ for global k.

Example 7. The valuation connected with a presheaf. Stratified valuation. For any presheaf $\mathscr{F}(\cdot)$ on Ω, we choose as the set of parameters $\mathscr{F}(1) = \mathscr{F}_1$ and we put $[\![k = t]\!]_{\mathscr{F}_1} = \bigvee \{u \in \Omega \mid k{\restriction}u = t{\restriction}u\}$, where $k, t \in \mathscr{F}_1$, that is, $[\![k = t]\!]_{\mathscr{F}_1} = \bigvee\{u \in \Omega \mid \rho^1_u(k) = \rho^1_u(t)\}$.

If the language under consideration contains (besides $\cdot = \cdot$) an atomic formula (that is, for example, binary), then according to the general definition of valuation we ought to define a function of the form $[\![\varphi_0(\cdot, \cdot)]\!]_{\mathscr{F}_1}\colon \mathscr{F}_1^2 \to \Omega$ consistent with $[\![\cdot = \cdot]\!]_{\mathscr{F}_1}$, in other words, an extensional function for the valuation $[\![\cdot]\!]_{\mathscr{F}_1}$. In a similar way we interpret the functional symbols $[\![f(\cdot, \ldots, \cdot) = \cdot]\!]_{\mathscr{F}_1}\colon \mathscr{F}_1^{n+1} \to \Omega$. In fact (for example, for a two-place functional symbol f), we ought to define a function $f\colon \mathscr{F}_1^3 \to \Omega$ with the property

$$\forall k, k_1, t, t_1, y, y_1\, (f(k, t, y) \wedge f(k_1, t_1, y_1) \wedge [\![k = k_1]\!]_{\mathscr{F}_1} \wedge \wedge [\![t = t_1]\!]_{\mathscr{F}_1} \leqslant [\![y = y_1]\!]_{\mathscr{F}_1})$$

and

$$\bigvee_{y \in \mathscr{F}_1} f(k, t, y) = 1, \ \forall k, t \in \mathscr{F}_1.$$

The structure of a presheaf can be used to obtain an interpretation of atomic formulae and functional symbols. Namely, if on each $\mathscr{F}(u)$ we are given a predicate $P_u(\cdot, \cdot)$ consistent with the "structure of the presheaf" (that is, $((v \leqslant u) \wedge P_u(k{\restriction}u, t{\restriction}u)) \Rightarrow P_v(k{\restriction}v, t{\restriction}v)$, where $k, t \in \mathscr{F}_1$), then we put $[\![\varphi_0(k, t)]\!]_{\mathscr{F}_1} = \bigvee\{u \in \Omega \mid P_u(k{\restriction}u, t{\restriction}u)\}$.

The function obtained in this way is extensional. For an interpretation of a functional symbol (for example, two-place), any natural map of functors $f\colon (\mathscr{F}(\cdot))^2 \to \mathscr{F}(\cdot)$ is suitable: the symbol $f(\cdot, \cdot)$ corresponds to $f\colon \mathscr{F}_1^2 \to \mathscr{F}_1$. So we extend the valuation from the atomic formulae to all formulae according to the general definition of valuation. The valuation we get is called a *valuation of the presheaf* $\mathscr{F}(\cdot)$ and is denoted by $[\![\cdot]\!]_{\mathscr{F}_1}$. It is easy to see that for any formula φ we have the possibility of substituting $[\![\varphi(l_1, \ldots, l_n)]\!]_{\mathscr{F}_1}$ with respect to $[\![\cdot = \cdot]\!]_{\mathscr{F}_1}$.

If everywhere in the definition of the valuation $[\![\cdot]\!]_{\mathscr{F}_1}$ we replace $\mathscr{F}_1$ by $\mathscr{F}$, then we obtain another example of a valuation. However, this valuation seems not to find important applications, since it does not take into account the additional structure that we have in $\mathscr{F}$ (but not in $\mathscr{F}_1$). Namely, the set $\mathscr{F}$ is "stratified" by the function $E\colon \mathscr{F} \to \Omega$.

We put

$$[\![k \equiv t]\!]_{\mathscr{F}} = \bigvee \{u \in \Omega \mid k{\restriction}u = t{\restriction}u\} \quad \text{and}$$
$$[\![k = t]\!]_{\mathscr{F}} = \bigvee \{u \in \Omega \mid \rho^{Ek}_u(k) = \rho^{Et}_u(t)\}.$$

Of course, for $k, t \in \mathscr{F}_1$ we have $[\![k = t]\!]_{\mathscr{F}} = [\![k \equiv t]\!]_{\mathscr{F}}$. The structure $\langle \Omega, \mathscr{F}, [\![\cdot = \cdot]\!]_{\mathscr{F}} \rangle$ is a typical example of an Ω-set (see the definition in §I.3). The valuation so far defined only for atomic formulae extends to all formulae: for propositional connectives as usual and for quantifier connectives in a new fashion: $[\![\exists x\varphi]\!] = \bigvee \{Ek \wedge [\![\varphi(k)]\!]_{\mathscr{F}} \mid k \in \mathscr{F}\}$, $[\![\forall x\varphi]\!] = \bigwedge \{Ek \to [\![\varphi(k)]\!]_{\mathscr{F}} \mid k \in \mathscr{F}\}$. The atomic formulae and the functional symbols (if we have any besides $\cdot = \cdot$) are interpreted in a way consistent with the function $[\![\cdot \equiv \cdot]\!]_{\mathscr{F}}\colon\ \mathscr{F}^2 \to \Omega$. We call this valuation a *stratified valuation of the presheaf* $\mathscr{F}(\cdot)$ and we denote it by $[\![\cdot]\!]_{\mathscr{F}}$.

Two valuations $[\![\cdot]\!]_{\Omega, D_1}$ and $[\![\cdot]\!]_{\Omega, D_2}$ are *equal* if there is a bijection $\psi : D_1 \to D_2$ such that for all formulae φ we have

$$[\![\varphi(k_1, \ldots, k_n)]\!]_{\Omega, D_1} = [\![\varphi(\psi(k_1), \ldots, \psi(k_n))]\!]_{\Omega, D_2}$$

and if in D_1 or D_2 we have certain structures, then we assume they are the same and the bijection preserves them.

Example 8. A large stock of valuations can be obtained from the valuation $[\![\cdot]\!]_{\Omega}$ in the language ZF (see Example 1) in the following canonical way. Suppose that in a certain language we have, besides $\cdot = \cdot$, exactly one atomic binary predicate φ_0 (to be concrete). We choose a pair of functions $h_1,\ h \in V^{\Omega}$. The symbol φ_{h, h_1} denotes the relativization of the bound variables to k ($\exists x \in h$ and $\forall x \in h$) and the replacement of $\varphi_0(x, y)$ by $\langle x, y\rangle \in h_1$. We put $[\![\varphi]\!]_{\Omega, h, h_1} = [\![\varphi_{h, h_1}]\!]_{\Omega}$. Instead of $h,\ h_1$ we can write h as is usually done when only the "support of the structure" $\langle h,\ h_1\rangle$ is indicated. For the set of parameters D of the valuation $[\![\cdot]\!]_{\Omega, h, h_1}$ we take $\mathscr{D}(h)$. Intuitively, $\langle h,\ h_1\rangle$ is not simply a non-standard and Ω is a non-standard model in the corresponding language. Instead of $[\![\varphi]\!]_{\Omega, h}$ the symbol $[\![h \models \varphi]\!]_{\Omega}$ is sometimes used, since the relativization of φ to k coincides with the definition of the predicate $h \models \varphi$. □

In the case of the stratified valuation it is convenient to consider the notions of E-normality and E-fibre instead of normality and fibre.
A valuation $[\![\cdot]\!]_{\mathscr{F}}$ is *E-normal* if

$$\forall u \in \Omega\ \forall k,\quad t \in \mathscr{F}\ (Ek = Et = u \wedge [\![k = t]\!]_{\mathscr{F}} \geqslant u \Rightarrow k = t)$$

and an *E-fibre*, denoted by D_p, is the structure defined as in §I.3, with the only difference that for the factorization with respect to $\sim_p$ we do not take the whole of $D = \mathscr{F}$ but $\{k \in \mathscr{F} \mid Ek \notin p\}$.

A *singleton* (with respect to the valuation $[\![\cdot]\!]_{\mathscr{F}}$ or $[\![\cdot]\!]_{\mathscr{F}_1}$) is any extensional function of the form $p\colon\ \mathscr{F} \to \Omega$ (or, respectively, of the form $\mathscr{F}_1 \to \Omega$) having the property $p(k) \wedge p(t) \leqslant [\![k = t]\!]_{\mathscr{F}}$. Intuitively, a singleton is an Ω-one-element subset in $\mathscr{F}$ or $\mathscr{F}_1$, respectively. For example, the function $p_k(l) = [\![k = l]\!]_{\mathscr{F}}$, where k is fixed, $k \in \mathscr{F}$, and l runs through $\mathscr{F}$, is a singleton with respect to the valuation $[\![\cdot]\!]_{\mathscr{F}}$. If in this example we replace $\mathscr{F}$ by $\mathscr{F}_1$, then we obtain an example of a singleton with respect to the valuation $[\![\cdot]\!]_{\mathscr{F}_1}$. A valuation is called *complete* if any singleton p is equal to a singleton of the form p_k for some k. The notions of singleton and complete valuation are defined in [1].

We recall that a fibre $\mathcal{F}_p$ of a presheaf $\mathcal{F}(\cdot)$ is defined as the direct limit of the system $\{\mathcal{F}(u) \mid p \in u \in \Omega\}$. If $k \in \mathcal{F}(u)$, $u \in \Omega$, then the value of k at the point p (if it is defined) is denoted by $k(p)$.

The following proposition is an analogue of Lemma 1.

***Proposition* 2.** *For a normal presheaf, if $u \leqslant [\![k = t]\!]_{\mathcal{F}}$, then $k{\restriction}u = t{\restriction}u$ for all $k, t \in \mathcal{F}$.*

***Theorem* 15.** *Let $[\![\cdot]\!]_{\mathcal{F}_1}$ and $[\![\cdot]\!]_{\mathcal{F}}$ be two valuations defined by a presheaf $\mathcal{F}(\cdot)$ on a complete Heyting algebra Ω.*

a) *The presheaf $\mathcal{F}(\cdot)$ is normal for any global elements if and only if the valuation $[\![\cdot]\!]_{\mathcal{F}_1}$ is normal. It is normal if and only if the valuation $[\![\cdot]\!]_{\mathcal{F}}$ is E-normal.*

b) *The presheaf $\mathcal{F}(\cdot)$ is a sheaf if and only if the valuation $[\![\cdot]\!]_{\mathcal{F}}$ is E-normal and complete.*

Suppose that for the atomic formulae the valuation $[\![\cdot]\!]_{\mathcal{F}}$ has the property $p \in [\![k = t]\!]_{\mathcal{F}} \Rightarrow \exists\, u\,(p \in u \wedge k{\restriction}u = t{\restriction}u)$ (in fact it is a property of Ω of the form $p \in \bigvee_\alpha u_\alpha \Rightarrow \exists\alpha\,(p \in u_\alpha)$).

c) *We have $[\![k = t]\!]_{\mathcal{F}_1} = \{p \in X(\Omega) \mid k(p) = t(p)\}$.*

d) *The fibre $\mathcal{F}_p$ of the presheaf and the E-fibre of the valuation $[\![\cdot]\!]_{\mathcal{F}}$ coincide. If Ω is zero-dimensional and $[\![\cdot]\!]_{\mathcal{F}}$ is a weakly sheaf valuation, then the fibre $\mathcal{F}_p$ of the normal presheaf $\mathcal{F}(\cdot)$ and the fibre of the valuation $[\![\cdot]\!]_{\mathcal{F}_1}$ coincide.*

Proof. a) We know that if $1 = \bigvee_\alpha u_\alpha$ and $\rho^1_{u_\alpha}(k) = \rho^1_{u_\alpha}(t)$ for all α, where $k, t \in \mathcal{F}_1$, then $k = t$. If $[\![k = t]\!]_{\mathcal{F}_1} = 1$ then $1 = \bigvee \{u \mid \rho^1_u(k) = \rho^1_u(t)\}$, so $k = t$. Conversely, if $1 = \bigvee_\alpha u_\alpha$ and $\rho^1_{u_\alpha}(k) = \rho^1_{u_\alpha}(t)$ for all α, then $[\![k = t]\!]_{\mathcal{F}_1} = 1$, so $k = t$. Suppose that $\mathcal{F}(\cdot)$ is normal, $Ek = Et = u$, and $[\![k = t]\!]_{\mathcal{F}} \geqslant u$. Then we have

$$u \leqslant \bigvee \{v \mid k{\restriction}v = t{\restriction}v\} = \bigvee \{v \mid \rho^u_{Ek \wedge v}(k) = \rho^u_{Et \wedge v}(t)\},$$

and $k = t$. Conversely, if $k, t \in \mathcal{F}(u)$, $u = \bigvee_\alpha u_\alpha$, and $\rho^u_{u_\alpha}(k) = \rho^u_{u_\alpha}(t)$, then $[\![k = t]\!]_{\mathcal{F}} \geqslant u$ and $k = t$.

Part b) is proved in [1].

c) If $k(p) = t(p)$, that is, $k{\restriction}u = t{\restriction}u$ and $p \in u$, where $u \in \Omega$, then $[\![k = t]\!]_{\mathcal{F}_1} \geqslant u \ni p$. If $p \in [\![k = t]\!]_{\mathcal{F}_1} = \bigvee \{u \in \Omega \mid k{\restriction}u = t{\restriction}u\}$, then by hypothesis we get $p \in u$ and $k{\restriction}u = t{\restriction}u$, that is, $k(p) = t(p)$.

d) The sets that are factorized in the cases of the presheaf $\mathcal{F}(\cdot)$ and the valuation $[\![\cdot]\!]_{\mathcal{F}}$ are the same, and equal to $\{k \in \mathcal{F} \mid p \in Ek\}$. The equivalence relations are also the same:

$$k \sim_p t \Leftrightarrow \exists u\,(p \in u \wedge k{\restriction}u = t{\restriction}u) \Leftrightarrow p \in [\![k = t]\!]_{\mathcal{F}}.$$

If Ω is zero-dimensional, then we can take $\mathcal{F}_p$ as the set for factorization in the definition of $\mathcal{F}_1$, and the same goes for the valuation $[\![\cdot]\!]_{\mathcal{F}_1}$. □

Let $\mathcal{F}(\cdot)$ be a presheaf on Ω. We recall that p_k is the function defined on $\mathcal{F}$ by $p_k(l) = [\![k = l]\!]_{\mathcal{F}}$, $p_k\colon \mathcal{F} \to \Omega$. Similarly, p_k' is the function defined on $\mathcal{F}_1$ by $p_k'(l) = [\![k = l]\!]_{\mathcal{F}_1}$. Of course $p_k, p_k' \in V^\Omega$, where l from the domain of p_k or p_k' is identified with $\check{l} \in V^\Omega$. We write $\Pi = \{p_k \mid k \in \mathcal{F}\}$ and $\Pi' = \{p_k' \mid k \in \mathcal{F}_1\}$. We put $\mathcal{F}'(p_k) = Ek$ and $\mathcal{F}_1'(p_k') \equiv 1$, where $\mathcal{F}'$ is defined on Π and $\mathcal{F}_1'$ on Π'. The prime in the symbols p_k' and Π' is usually omitted, and we keep in mind that $\mathcal{F}'$ and $\mathcal{F}_1'$ are defined in different ways; $\mathcal{F}', \mathcal{F}_1' \in V^\Omega$.

Theorem 16. *Let $\mathcal{F}(\cdot)$ be a presheaf on a complete Heyting algebra Ω.*

a) *We have $[\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{F}_1} = [\![\varphi(p_{k_1}', \ldots, p_{k_n}')]\!]_{\Omega, \mathcal{F}_1'}$ for any formula φ in the language of rings with parameters from $\mathcal{F}_1$ (on the right-hand side we have the valuation from Example* 8). *For a normal presheaf the valuations $[\![\cdot]\!]_{\mathcal{F}_1}$ and $[\![\cdot]\!]_{\Omega, \mathcal{F}_1'}$ are equal (at least on global elements).*

b) *We have $[\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{F}} = [\![\varphi(p_{k_1}, \ldots, p_{k_n})]\!]_{\Omega \mathcal{F}'}$ for any formula φ in the language of rings with parameters from $\mathcal{F}$. For a normal presheaf the valuations $[\![\cdot]\!]_{\mathcal{F}}$ and $[\![\cdot]\!]_{\Omega, \mathcal{F}'}$ are equal.*

c) *If $\mathcal{F}(\cdot)$ is a sheaf, then $\mathcal{F}_1 \simeq (\mathcal{F}_1')^{\wedge\Omega}$ and $\mathcal{F}_1 \simeq (\mathcal{F}')^{\wedge\Omega}$, where $\simeq$ is an isomorphism of the form $k \mapsto p_k'$ and $k \mapsto p_k$, respectively.*

d) *If $f_i\colon (\mathcal{F}(\cdot))^{n_i} \to \mathcal{F}(\cdot)$ is a natural transformation of presheaves (n_i-ary operation on $\mathcal{F}_1$, where $1 \leqslant i \leqslant m$), then $\langle \mathcal{F}_1', \{f_i\}_-\rangle$ and $\langle \mathcal{F}_1, \{f_i\}\rangle$ are algebraic systems of the same type (and also for $\mathcal{F}'$).*

Remark. This theorem remains true if we extend the language of rings by new predicative and functional symbols. In d) we have in mind positively-defined algebraic systems, for example groups or rings.

Proof. a) The functions p_k' and $\mathcal{F}_1'$ are obviously extensional. If the formula φ is atomic, that is, it is an equality, then we have

$$[\![p_k' = p_t']\!]_{\Omega, \mathcal{F}_1'} = \bigwedge_{l \in \mathcal{F}_1} (p_k'(\check{l}) \leftrightarrow p_t'(\check{l})) = \bigwedge_{l \in \mathcal{F}_1} ([\![k = l]\!]_{\mathcal{F}_1} \leftrightarrow [\![t = l]\!]_{\mathcal{F}_1}) = [\![k = t]\!]_{\mathcal{F}_1}.$$

Obviously, the connectives $\wedge$, $\vee$, and $\Rightarrow$ preserve equality. The connective $\forall$ also preserves equality, since

$$[\![\forall x \varphi]\!]_{\mathcal{F}_1} = \bigwedge_{k \in \mathcal{F}_1} [\![\varphi(k)]\!]_{\mathcal{F}_1} \quad \text{and} \quad [\![\forall x \varphi]\!]_{\Omega, \mathcal{F}_1'} = \bigwedge_{k \in \mathcal{F}_1} [\![\varphi(p_k')]\!]_{\Omega, \mathcal{F}_1'}.$$

So we consider the case of the connective $\exists$. For a normal sheaf the map $k \mapsto p_k'$ is a bijection, which means that the valuations coincide. (Therefore for a normal sheaf we need not distinguish between k and p_k', so we identify them.)

b) The functions p_k are extensional and the cases of atomic formulae and propositional connectives are dealt with as in a). Quantifiers are handled by taking into account the definition of a stratified valuation. (We note that the function $\mathcal{F}'$ is extensional.)

The parts c)–d) are proved in [3].

Example 8a). We apply Theorem 16 to one particular presheaf which we shall need in Ch. II. Namely, we fix a ring K and put $\mathcal{F}(e) = e \cdot K$, where e runs through $B(K)$, and for $e_1 \leqslant e_2$ we put $\rho_{e_1}^{e_2}(k) = e_1 \cdot k$ (where k runs through $\mathcal{F}(e_2)$, that is, $e_2 \cdot k = k$). Pierce essentially proved that the presheaf $\mathcal{F}(\cdot)$ is a sheaf with respect to the topological completion $\mathcal{T}(K)$ of the algebra $B(K)$. Therefore, as always, $\mathcal{F}(\cdot)$ extends to a sheaf on $\mathcal{T}(K)$. We restrict $\mathcal{F}(\cdot)$ to $\mathbb{B}(K)$, which is the algebra of regular open sets in $X(K)$ and also the Dedekind completion of the algebra $B(K)$. We denote this restriction by $G(\cdot)$. Such a $G(\cdot)$ is a presheaf but not necessarily even normal. The fact that K is a sheaf ring in fact means that $G(\cdot)$ is a sheaf. We can again define the valuations of Example 2 as $[\![\varphi]\!]_{\mathcal{T}(K)} = [\![\varphi]\!]_{\mathcal{T}_1}$ and $[\![\varphi]\!]_{\mathbb{B}(K)} = [\![\varphi]\!]_{G_1}$. We write $K' = \mathcal{F}'_1$, that is, K' is the identity function defined on all p_k of the form $p_k(l) = [\![k = l]\!]_{\mathcal{T}_1}$. Intuitively K' is a non-standard image of the ring K in the universe of sets $V^{\mathcal{T}(K)}$. Using the presheaf $G(\cdot)$, we can also form $G'_1 \in V^{\mathbb{B}(K)}$, where G'_1 is similar to K', that is, G'_1 is the identity function on all L_k of the form $L_k(\check{l}) = [\![k = l]\!]_{G_1}$. But in general p_k is not equal to L_k, since $\mathcal{F}(\cdot)$ is not equal to $G(\cdot)$.

Theorem 16e). *Suppose that K is a normal ring. Then $G(\cdot)$ is a normal presheaf on global elements (in particular, $K' = \mathcal{F}'_1 = G'_1$) and the map $k \mapsto p_k$ is injective, that is, K can be embedded in $\hat{K}'^{\mathcal{T}}$ and $\hat{K}'^{\mathbb{B}}$, and also $[\![\varphi]\!]_{\mathcal{T}(K)} = [\![\varphi]\!]_{\mathcal{T},K'} \leqslant [\![\varphi]\!]_{\mathbb{B},K'} = [\![\varphi]\!]_{\mathbb{B}(K)}$ for any φ in weakly E-normal form.*

Proof. Suppose that $1 = \bigvee_B e_\alpha$, $e_\alpha \in B(K)$, and $e_\alpha \cdot k = e_\alpha \cdot t\ \forall \alpha$. Then $\bigcup_\alpha e_\alpha \subseteq e_0$, where the e_0 comes from the normality condition. Therefore, $e_0 = 1$ and $k = t$. Then $p_k(l) = L_k(l) = e_0$, where e_0 comes from the normality condition. Therefore, K' equal by definition to $\mathcal{F}'_1$ is also equal to G'_1, that is,

$$K' \in V^{\mathbb{B}(K)} \subseteq (V^{\mathcal{T}(K)} \cap V^{\mathbb{B}(K)}).$$

By Theorem 16, part a, the map $k \mapsto p_k$ is an embedding, and taking into account the fact that $[\![p_k \in K']\!] = 1$ (for the $\mathcal{T}$ and $\mathbb{B}$ cases) we obtain the required embedding of K. The last assertion follows from Theorem 16, part a and Theorem 10, part b.

We recall that $\Omega_1 = \mathcal{T}(X_1)$, where X_1 is the Baire line ω^ω.

Theorem 17. *Let φ be a closed formula in the language ZF. If for any $h, h_1 \in V^\Omega$ we have $\Omega_1 \models \varphi_{h,h_1}$, where h is an identity function with non-empty domain and h_1 is an extensional function on $(\mathcal{D}(h))^2$, then $\vdash_H \varphi$.*

This theorem, and also the article [5] (p.127), stimulate the following definition given by Takeuti (see [10]). Let φ be a formula in the language ZF, possibly with parameters from $V \simeq V^{Z_2}$. We denote by $cHa \models \varphi$ the predicate $\forall \Omega\ (V^\Omega \models \varphi)$, where Ω, as always, runs through all complete Heyting algebras. This predicate can be called *Heyting validity*. Heyting validity does not imply deducibility in the theory HZF, since bounded formulae can have the same valuation in all V^{Ω}, for example, the sequence of natural numbers ω is exactly the same object $\check{\omega}$ in all V^{Ω}. However, we can suppose that for unbounded formulae Heyting validity is in better correspondence with intuitionistic validity in set theory. In particular, we have certain properties of disjunctivity and extensionality for Heyting validity. Of course if $HZF \vdash \varphi$, then $cHa \models \varphi$.

CHAPTER II

LOCALIZATIONS AND VALUATIONS

II.1. Local axiomatizability of a class of algebraic systems.

Let $\mathcal{K}$ be the class of sheaves $\mathcal{F}(\cdot)$ on complete Heyting algebras Ω, where Ω runs through a certain class $\mathcal{K}_0$; each value $\mathcal{F}(u)$ is an algebra of the same signature. Usually the fibres (algebras) $\mathcal{F}_p$ of the sheaf $\mathcal{F}(\cdot)$ (here and in what follows p runs through $X(\Omega)$) are simpler than the algebra $\mathcal{F}(1)$. Therefore, we try to reduce properties of $\mathcal{F}(1)$ to properties of the family $\{\mathcal{F}_p \mid p \in X(\Omega)\}$. This is one manifestation of the "method of localization".

The class of sheaves $\mathcal{K}$ is *locally axiomatizable* if there is a theory T in the language corresponding to the signature of the algebras such that $\mathcal{F}(\cdot) \in \mathcal{K} \Leftrightarrow (\{\mathcal{F}_p\} \models T)$, where here and in what follows $\{\mathcal{F}_p\} \models T$ means that $\forall p\ (\mathcal{F}_p \models T)$. The theory T is called a *local theory of the class* $\mathcal{K}$. The notion of local axiomatizability, as well as the well-known notion of axiomatizability, expresses a certain "closure (completeness)" property of the class $\mathcal{K}$. For a sheaf $\mathcal{F}(\cdot)$ on Ω, valuations $[\![\cdot]\!]_{\mathcal{F}_1}$ and $[\![\cdot]\!]_{\mathcal{F}}$ with values in Ω were defined in Example 7. These valuations give a useful tool for passing from the local theory T of the locally axiomatizable class $\mathcal{K}$ to the class of global objects $\{\mathcal{F}(1) \mid \mathcal{F} \in \mathcal{K}\}$ and the theory $Th\{\mathcal{F}(1) \mid \mathcal{F} \in \mathcal{K}\}$. An important role is played by the possibility of expressing the global validity (with respect to $[\![\cdot]\!]_{\mathcal{F}_1}$ or $[\![\cdot]\!]_{\mathcal{F}}$) in the algebra $\mathcal{F}(1)$. This means the existence of a translation $\varphi \mapsto \varphi'$ (independent of the sheaf $\mathcal{F}(\cdot)$) such that $(\mathcal{F}(1) \models \varphi') \Leftrightarrow ([\![\varphi]\!]_{\mathcal{F}_1} = 1)$ for all $\mathcal{F}(\cdot)$ in $\mathcal{K}$ (and similarly for the valuation $[\![\cdot]\!]_{\mathcal{F}}$). In this case we call $\mathcal{K}$ a *class with global validity*. In this chapter we apply such an approach in the case when all the global objects of $\mathcal{F}(1)$ are associative rings with identity.

In this case the expressibility of global validity is guaranteed by Theorem 12. In the presence of such an expressibility, some concrete questions about the class $\{\mathcal{F}(1) \mid \mathcal{F}(\cdot) \in \mathcal{K}\}$ or the theory $Th\{\mathcal{F}(1) \mid \mathcal{F} \in \mathcal{K}\}$ can be reduced to the theory T in the uniform way. This can be also done for model-theoretic questions. For example, we can reduce decidability, completeness, model completeness, categoricity, stability and similar properties to the analogous properties of the theory T.

We note that the properties of axiomatizability and local axiomatizability of the class $\mathscr{K}$ do not imply each other and their relation to the property of internal axiomatizability (see the definition in §I.6) is also non-trivial.

In Example 8a) we considered the following (Pierce) sheaf defined for any associative ring K with identity. For $e \in B(K)$ we put $\mathscr{F}(e) \leftrightharpoons e \cdot K$, and if $e_1 \leqslant e_2$ we take $\rho^{e_2}_{e_1}$: $\mathscr{F}(e) \to \mathscr{F}(e_1)$ to be multiplication by e_1, that is, $\rho^{e_2}_{e_1}(k) \leftrightharpoons e_1 \cdot k$. Of course, $\mathscr{F}(1) = K$. As a rule, $B(K)$ is not complete. We have: if $e = \bigcup_\alpha e_\alpha$, where $e \in B(K)$, $\{e_\alpha\} \subseteq B(K)$, and $k_\alpha \in \mathscr{F}(e_\alpha)$, that is, $e_\alpha \cdot k_\alpha = k_\alpha$ and $e_\alpha \cdot e_\beta \cdot k_\alpha = e_\alpha \cdot e_\beta \cdot k_\beta$, then $\exists! k \in \mathscr{F}(e) \forall \alpha\, (e_\alpha \cdot k = k_\alpha)$. The existence of such a k is obvious because e is compact and the uniqueness follows from Lemma 1 for Theorem 5. Therefore, in a standard way we can extend $\mathscr{F}(\cdot)$ from $B(K)$ on $\mathscr{T}(K)$ putting $\mathscr{F}(O)$ equal to the inverse limit of the system $\{\mathscr{F}(e) \mid e \subseteq O\}$. We obtain a (Pierce) sheaf on $\mathscr{T}(K)$. Any class of rings $\mathscr{K}$ can be identified with the class of corresponding Pierce sheaves and we need not distinguish K and $\mathscr{F}(\cdot)$ on $\mathscr{T}(K)$. The fibre $\mathscr{F}_p$ of this sheaf coincides with $K_p = K/\bar{p}$, where $\bar{p} = p \cdot K$, that is, we are in the situation of Examples 2 and 8a). In place of $\mathscr{F}'_1$ (see the definition before Theorem 16), where $\mathscr{F}(\cdot)$ is the Pierce sheaf of the ring K, we shall write K'. Of course, K' can be defined without mentioning the Pierce sheaf. The fibre K_p in the algebraic context is also called the *Pierce localization of the ring K* at the "ring point" p. Other forms of localization of rings and the corresponding sheaves may also be considered. The results that follow basically carry over to them but we restrict ourselves here to the Pierce localization.

Theorem 15c) in fact implies the following result.

***Proposition* 3.** *We have* $[\![k = t]\!]_{\mathscr{T}(K)} = \{p \in X(K) \mid k(p) = t(p)\}$.

Proof. If $p \in [\![k = t]\!]_{\mathscr{T}(K)}$, then $p \in e$ and $e \cdot k = e \cdot t$. If $k(p) = t(p)$, then $p \in e$ and $e \cdot k = e \cdot t$. (Proposition 3 is also true for any positive quantifier-free formula.)

II.2. Valuation and model completeness. Boolean absoluteness.

We apply the approach described in §II.1 to the following problem of Macintyre: if T has a model companion, then does $\mathscr{K}$ have a model companion (and of what kind) (see [7], p.173)? We recall the corresponding definitions.

A class $\mathscr{K}$ of rings is Σ-*model complete* if for any $K, L \in \mathscr{K}$, $K \subseteq L$, and any formula φ from the class of formulae Σ (here and in what follows in the language of rings) with any parameters $k_1, \dots, k_n$ from K we have $(K \models \varphi(k_1, \dots, k_n)) \Leftrightarrow (L \models \varphi(k_1, \dots, k_n))$. If Σ is the class of all formulae, then $\mathscr{K}$ is called *model complete*. If a class $\mathscr{K}$ is model complete and axiomatizable, then the theory $Th\mathscr{K}$ is called *model complete*. If the class $\mathscr{K}$ is axiomatizable, then the model completeness of the class $\mathscr{K}$ and the theory $Th\mathscr{K}$ is equivalent to the fact that any system of equations and inequalities with coefficients from K that has a solution in L also has a solution in K ("Robinson's criterion"). The latter is equivalent to the fact that any formula $\varphi(x_1, ..., x_n)$ is equivalent to an A-formula $\psi(x_1, ..., x_n)$ in the theory $Th\mathscr{K}$ ("Robinson's theorem"). A formula expressing in a natural way the assertion of the existence of a solution of a concrete system of equations and inequalities is called *primitive*.

A class $\mathscr{K}^*$ is a *model companion of the class* $\mathscr{K}$ if these two classes can be embedded in one another, that is, $\forall K \in \mathscr{K} \exists L \in \mathscr{K}^* (K \mathrel{\widetilde{\subseteq}} L)$ and $\forall L \in \mathscr{K}^* \exists K \in \mathscr{K} (L \mathrel{\widetilde{\subseteq}} K)$, and $\mathscr{K}^*$ is model complete. If these classes are axiomatizable, then the theory $Th\mathscr{K}^*$ is called a *model companion of the theory* $Th\mathscr{K}$. It is well known that, for example, the class of all algebraically closed fields is a model companion of the class of all fields and the class of real closed fields is a model companion of the class of all ordered fields.

We fix a theory T having a model companion T^* and also the class $\mathscr{K} \rightleftharpoons \{K | \{K_p\} \models T\}$. When does the class $\mathscr{K}$ have a model companion and of what sort? In particular, for the class $\mathscr{K}_0$ of biregular rings (in other words $\mathscr{K}_0 \rightleftharpoons \{K \mid \{K_p\} \models$ "simple ring"$\}$), which of its subclasses are model complete? These are questions of Macintyre (see [7], p.173 and [8], p.88). In [8] he found a model companion for the class $\mathscr{K}$ of the form $\mathscr{K} \rightleftharpoons \{K \mid \{K_p\} \models$ "field"$\}$, in other words, he found a model companion for the subclass of $\mathscr{K}_0$ consisting of all commutative regular rings. This result has also been obtained by other authors, as is discussed in [8]. Proofs of the fact that the classes $\mathscr{K}_0$ and $\mathscr{K}$ can indeed be characterized in this way by their localizations, and also other examples of locally axiomatizable classes, can be obtained, for example, from the theorems of [3], p.389 (see also Example 9). Some answers to these questions of Macintyre are contained in the next section. Namely, we actually produce the following class $\mathscr{K}^* = \{K \mid \{K_p\} \models T^*,\ K \models \Phi_1 \wedge \Phi_2\}$, where Φ_1 is the normality condition and Φ_2 is the condition that the ring K has no atoms, and we find a sufficient (and in fact necessary) condition for $\mathscr{K}^*$ to be a model companion for $\mathscr{K}$. It is easy to show that $\forall K \in \mathscr{K}_0 (K \models \Phi_1)$. Therefore, if $T^* \models$ "simple", then $\mathscr{K}^* \subseteq \mathscr{K}_0$ and the condition Φ_1 in the definition of the class $\mathscr{K}^*$ is unnecessary. In Example 10 we produce a new model-complete subclass of the class $\mathscr{K}_0$.

In what follows, essential use is made of the following result.

Proposition 4. *If K is a normal ring and $\{\mathscr{K}_p\} \models T^*$ (where T^* is a model complete theory), then for any formula* φ,

$$[\![\varphi (k_1, \ldots, k_n)]\!]_{\mathscr{T}(K)} = \{p \in X(K) \mid K_p \models \varphi (k_1 (p), \ldots, k_n (p))\}$$

and $[\![\varphi (k_1, \ldots, k_n)]\!]_{\mathscr{T}(K)}$

is an open-and-closed set, where $k_1, ..., k_n \in K$.

Proof. For atomic formulae we have $[\![k = t]\!] = e_0$, where e_0 comes from the definition of normality for the element $k - t$, and we employ Proposition 3. For the connectives $\vee$, $\wedge$, $\neg$ everything is obvious. For the connective $\exists$ we note that $[\![\exists x\varphi]\!] = \{p \in X \mid K_p \models \exists x\varphi\}$. Using the model completeness for the formula $(\neg\exists x\varphi)(x_1, \ldots, x_n)$, we obtain the reducing E-formula $\psi(x_1, \ldots, x_n)$ and since the ring is normal the valuation $[\![\exists x\varphi]\!]$ is open-and-closed. For the connective $\forall$ we note that

$$[\![\forall x\varphi]\!] \subseteq \{p \in X \mid K_p \models \forall x\varphi\} \subseteq \bigcap\{[\![\varphi(k)]\!] \mid k \in K\}$$

and we employ the fact that $\{p \in X \mid K_p \models \exists x \neg\varphi\}$ is open-and-closed.

The next two propositions are special cases of Theorem 2, parts b and c (see also [3], p.388).

***Proposition* 5.** *Let φ be any formula in normal or prenex form (in the latter case K is normal). If $p \in [\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{T}(K)}$, then $K_p \models \varphi(k_1(p), \ldots, k_n(p))$.*

Proof. We consider the case of the normal form. If $p \in [\![k = t]\!]$, then $p \in e \leqslant [\![k = t]\!]$, and by Lemma 1 we have $k(p) = t(p)$. The cases $\varphi = \varphi_1 \vee \varphi_2$ and $\varphi = \varphi_1 \wedge \varphi_2$ are obvious. If $\varphi = \varphi_1 \Rightarrow \varphi_2$ and $p \in [\![\varphi_1 \Rightarrow \varphi_2]\!]$, then in the case when $p \in [\![\varphi_1]\!]$ we get $p \in [\![\varphi_2]\!]$ and the proof is complete. In the case when $p \notin [\![\varphi_1]\!]$ and $K_p \models \varphi_1$ we arrive at a contradiction by induction on the number of connectives in φ_1.

Remark. In connection with Proposition 5 (and Theorem 2, part b, which is also true for the normal form), we turn our attention to the following: by Theorem 11, part a, it is $\mathcal{T}$-globally valid that "K is without non-trivial central idempotents", that is, $[\![\Phi_5']\!]_{\mathcal{T}} = 1$. We must not infer from this that $K_p \models (\Phi_5' \wedge \Phi_5)$, because Φ_5' is not in normal form and we must not take Φ_5, which is in normal form, since $\mathcal{T}$-global validity is subordinate to intuitionistic logic. Indeed, K_p can have numerous central idempotents (see [27]). This shows that Theorem 11 is non-trivial.

***Proposition* 6.** *If an AE-formula φ is in normal form or K is normal, then*

$$([\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{T}(K)} \geqslant e) \Leftrightarrow \forall p \in e\ (K_p \models \varphi(k_1(p), \ldots, k_n(p))),\ e \in B(K).$$

A class $\mathcal{K}$ is *Boolean regular* if for $K, L \in \mathcal{K}$ we have $K \subseteq L \Rightarrow B(K) \subseteq \subseteq B(L)$. For example, it is sufficient to verify $Z(K) \subseteq Z(L)$, where $Z(K)$ is the centre of the ring K.

***Proposition* 7.** a) *A model-complete class $\mathcal{K}$ is Boolean regular.*

b) *Boolean regularity is equivalent to $K \subseteq L \Rightarrow \forall p_1 \in X(L)\,(p_1 \cap K) \in X(K)$, and also to $K \subseteq L \Rightarrow \exists p_1 \in X(L)\,((p_1 \cap K) \in X(K))$.*

Proof. a) For any $e \in B(K)$ we carry the formula $\forall x\,(e \cdot x = x \cdot e)$ over from K to L.

b) We verify the implications in the order given. Of course, $p_1 \cap K = p_1 \cap \cap B(K) \subseteq B(K)$, does not contain 1, contains 0, is closed with respect to $\vee$, and by Boolean regularity is downwards transitive and prime. If there is an $e \in B(K)\backslash B(L)$, then $p_1 \cap K$ is not prime.

A class $\mathcal{K}$ is *Boolean prime* if for $K, L \in \mathcal{K}$ we have

$$K \subseteq L \Rightarrow \forall e_1 \in B(L)\,(e_1 \neq 0 \Rightarrow \exists p_1 \in e_1 \exists p \in$$
$$\in X(K)\,[p \supseteq p_1 \cap K \wedge \forall k \in K\,(\exists e \in p_1\,(e \cdot k = k) \Rightarrow \exists e \in p\,(e \cdot k = k))].$$

If $\mathcal{K}$ is a Boolean regular class and $\forall K \in \mathcal{K}\,(K \models \Phi_1)$, then the condition for the class $\mathcal{K}$ to be Boolean prime takes the form

$$K \subseteq L \Rightarrow \forall p_1 \in X(L)\,(p_1 \cdot L) \cap K \subseteq (p_1 \cap K) \cdot K$$

(see the proof of Proposition 8, part b).

***Proposition* 8.** *Suppose that $\mathcal{K}$ is a Boolean regular class and $\forall K \in \mathcal{K}\,(K \models \Phi_1)$.*

a) *From the AE-model completeness of the class $\mathcal{K}$ it follows that it is Boolean prime.*

b) *The property of the class $\mathcal{K}$ of the form*

$$K \subseteq L \Rightarrow \forall k \in K \forall e \in B(K)\,(K \models \psi(k, e) \Rightarrow L \models \psi(k, e)),$$

where

$$\psi \leftrightharpoons \forall e_0 \exists t\,(e_0^2 = e_0 \wedge e_0 \cdot t = t \cdot e_0 \wedge e_0 \cdot k = 0 \Rightarrow e_0 \cdot e = e_0),$$

is equivalent to the fact that it is Boolean prime.

Proof. Part a follows immediately from the equivalence in b.

b) Suppose that this property holds and (using the hypothesis) $e \leftrightharpoons [\![k = 0]\!]_K$. Then $L \models \psi(k, e)$, that is, $[\![k = 0]\!]_L \leqslant [\![k = 0]\!]_K$. Hence

$$[\![k \neq 0]\!]_K \leqslant [\![k \neq 0]\!]_L \forall p \in X(L)\,\forall k \in K\,[\exists e_1 \in$$
$$\in p_1\,(e_1 \cdot k = k) \Rightarrow [\![k \neq 0]\!]_K \leqslant e_1,$$
$$[\![k \neq 0]\!]_K \in p_1 \cap K,\ [\![k \neq 0]\!]_K \cdot k = k],$$

that is, $\exists e \in (p_1 \cap K)\,(e \cdot k = k)$. This is stronger than the assertion that the class $\mathcal{K}$ is Boolean prime. Conversely, suppose that $e_0 \in B(L)$ and $e_0 \cdot k = 0$. Let us suppose that $e_0 \cap (1-e) \neq 0$. By the condition that our class is Boolean prime we can choose a point $p_1 \in e_0 \cap (1-e)$ in $X(L)$ such that $\overline{p}_1 \cap K \subseteq \overline{p}$, where $p = p_1 \cap K$ is a prime ideal in $B(K)$. Since $(1-e_0) \cdot k = k$, there is an $e_1 \in p$ such that $e_1 \cdot k = k$. Therefore, $(1-e_1) \notin p_1$, $(1-e_1) \cdot k = 0$, $(1-e_1) \leqslant e$, $e \notin p_1$, but on the other hand $(1-e) \notin p_1$. This is a contradiction. Here $k \in K$ and $e \in B(L)$.

Remark. The property in Proposition 8, part b follows from the fact that the class is Boolean prime without using the condition $\forall K \in \mathcal{K}\,(K \models \Phi_1)$.

Example 9. If the rings in $\mathcal{K}$ have property Φ_3, then $\mathcal{K}$ is Boolean regular. The class of strictly Rickartian rings in the class of normal rings has for a local theory the axiom $\forall k, t\,(k \cdot t = 0 \Rightarrow k = 0 \vee t = 0)$, that is,

$\mathcal{K} = \{K \mid K \models \Phi_1 \wedge \{K_p\} \models T\}$. It consists of normal rings and it is Boolean regular. The last assertion is true because $T \vdash \Phi_4$, and consequently Φ_3 holds in $\mathcal{K}$.

The class $\mathcal{K}_1$ of Abelian regular rings has for a local theory the axiom "division ring". This class is also Boolean regular and, in addition, Boolean prime. The latter is true by virtue of the fact that any Boolean regular class of rings $\mathcal{K}$, the localizations of whose elements are simple rings (that is, any Boolean regular subclass of the class of biregular rings), is Boolean prime. We verify the last assertion. Suppose that $K, L \in \mathcal{K}$, $K \subseteq L$, and we have $p_1 \in X(L)$, $k \in \overline{p}_1 \cap K$. We have to show that $k \in \overline{p}$ for some $p \in X(K)$. We put $p \rightleftharpoons p_1 \cap K \subseteq B(K)$. By Boolean regularity we have $p \in X(K)$. We consider the homomorphism $[t]_{\overline{p}} \mapsto [t]_{\overline{p}_1 L}$, $K/\overline{p} \to L/\overline{p}_1$, whose kernel is a two-sided ideal in $K/\overline{p}$ and contains the element $[k]_{\overline{p}}$. If $a = K/\overline{p}$, then $[1]_{\overline{p}_1 L} = [0]_{\overline{p}_1 L}$, that is, $1 = e_1 \cdot l$, where $e_1 \in p_1$. Hence $e_1 = 1$, a contradiction. If $a = 0$, then $[k]_{\overline{p}} = 0$, so $k = e \cdot k$, where $e \in p$.

The class $\mathcal{K}_0$ of biregular rings is, as we have noticed, characterized by the condition $K \in \mathcal{K}_0 \Leftrightarrow \{K_p\} \models$ "simple". It is not locally axiomatizable (in the language of rings).

Another interesting class of rings $\mathcal{K}_2$ consists of all strictly bi-Rickartian rings. In the class of normal rings it is locally axiomatizable by the theory T with one axiom: "primitive ring", that is, $\mathcal{K}_2 = \{K \mid K \models \Phi_1 \wedge \{K_p\} \models T\}$. We recall that a primitive ring is defined by the condition that for any (two-sided) ideals a, b, $a \cdot b = 0$ implies that $a = 0 \vee b = 0$. This condition can be expressed in the language of rings:

$$\forall k, k_1 \exists t \, (k \cdot t \cdot k_1 = 0 \Rightarrow k = 0 \vee k_1 = 0).$$

We note that all the local theories mentioned in this example are such that all their models have only two central idempotents 0 and 1, that is, they satisfy Φ_5.

Example 10. The next example is the most important one in this chapter. We present it for associative rings with identity, although it remains correct for rings without an identity and also for the non-associative rings that are discussed in [23]. We shall assume that the reader has read the next section up to the end of the statement of Theorem 18. The assertions related to this example are numbered with the prefix 10 because of their special nature. We denote by $\mathcal{L}$ the class of all primitive *PI*-rings A (over a commutative ring R with identity) of a fixed degree s. The centre of A, which we everywhere denote by $F \rightleftharpoons Z(A)$, is an integral domain (that is, a commutative ring without zero divisors). We recall that an algebra A_R is a *PI-ring* if at least one polynomial (with non-commutative variables) and leading coefficient equal to 1 is identically equal to zero in it; the least degree of such a polynomial is called the *degree of the algebra* A_R (see, for example, [29], vol. 2, p.43). Then A_F (regarded as a central algebra) can be embedded in its classical ring of fractions $S_A \rightleftharpoons A \otimes_F F_{cl}$, where F_{cl} is the field of fractions of F; the field F_{cl} can also be embedded in S_A. This can be done by the formulae $a \mapsto a \otimes 1$ and $f \cdot g^{-1} \mapsto 1 \otimes f \cdot g^{-1}$ respectively (since A_F and ${}_F F_{cl}$ are torsion-free as modules). Any maximal linearly independent system $\{a_i\}$ in A over F forms a basis $\{a_i \otimes 1\}$ for S_A over F_{cl}, and any basis of S_A can be transformed into this form (for details see [28], pp.46–49). Therefore, certain properties of A and S_A can be expressed in terms of one another. Pozner's theorem (see [29], vol. 2, p.48) says that S_A is an m-dimensional simple algebra (over its centre F_{cl}). For any simple algebra A that is m-dimensional (over its centre) we have $m = n^2$ for some n. In what follows we call such an A an *n-algebra* (see [25]). In our case $n = [s/2]$. In the class of prime rings A the condition "*PI*-algebra of degree s" is axiomatizable in the form $A \models S_{2n} \wedge \neg S_{2n-2}$, where S_k is the standard identity of degree k (see [25], p.404). This condition can also be axiomatized in a different way: in A there is a maximal system of m elements linearly independent over its centre F. Thus, the class $\mathcal{L}$ is axiomatizable by these three axioms, and we denote them by $T_{\mathcal{L}}$.

We consider the class $\mathcal{L}_{T_0}$ consisting of all n-algebras with centre F that satisfy a fixed theory T_0. We have $\mathcal{L}_{T_0} \subseteq \mathcal{L}$, since in A_F precisely those identities are fulfilled that hold in the matrix ring $M_n(F)$ (see [25], p.403) and, in particular, the standard identity S_{2n} holds in A_F.

Proposition 10.1. *The class $\mathcal{L}_{T_0}$ is axiomatizable by the theory T given below.*

Proof. It follows easily from the Artin–Wedderburn theorem that $A_F \cong M_k(D)$, where D is a division ring and an algebra over its centre $F = Z(D)$ of dimension s (see [25], p.227). It is clear that $n^2 = k^2 \cdot s$. Therefore, we can axiomatize $\mathcal{L}_{T_0}$ by the theory containing the axiom

$$\bigvee_{k^2 \cdot s = n^2} \exists \{e_{ij}\}^k \forall x \exists y \{\sum_{i=1}^{k} e_{ii} = 1 \wedge e_{ij} \cdot e_{pq} = \delta_{jp} \cdot e_{iq} \wedge [x \cdot e_{ij} = e_{ij} \cdot x \Rightarrow y \cdot e_{ij} =$$
$$= e_{ij} \cdot y \wedge (x = 0 \vee x \cdot y = y \cdot x = 1)]) \wedge \exists x_1, \ldots, x_s [\forall x \exists y_1, \ldots, y_s$$
$$\forall n (x_1 \cdot e_{ij} = e_{ij} \cdot x \wedge \ldots \wedge y_1 \cdot u = u \cdot y_1 \wedge \ldots \wedge x = x_1 \cdot y_1 + \ldots + x_s \cdot y_s) \wedge$$
$$\wedge (x_1, \ldots, x_s \text{ are linearly independent over the centre })]\},$$

where δ_{jp} is the Kronecker delta, and also the list of relativizations to the centre F of all axioms of the theory T_0. Let A be a model of the theory T. It is easy to see that $A \cong M_k(Z(\{e_{ij}\}))$, where $Z(\{e_{ij}\})$ is the centralizer of the system of "matrix units" $\{e_{ij}\}$, and it always forms a ring (see [24], p.52). In our case it is a division ring of dimension s over the centre of A, by hypothesis.

***Proposition* 10.2.** *The class $\mathcal{K}_{T_\bullet} \rightleftharpoons \{K \mid \{K_p\} \models T\}$ is Boolean regular (and so is any of its subclasses). What is more, for K, $L \in \mathcal{K}_{T_\bullet}$ we have $K \subseteq L \Rightarrow Z(K) \subseteq Z(L)$.*

Proof. We note that $\mathcal{L}_{T_\bullet} \subseteq \mathcal{K}_{T_\bullet}$. We verify the second assertion. A multilinear polynomial (with non-commuting variables) is *distinguished* if its coefficients are ± 1, its values on any central algebra $M_n(F)$ lie in the centre, and it is not identically equal to zero, where F is any prime field, that is, $\mathbb{Q}$ or any field of residues. Such a polynomial exists, for example Razmyslov's polynomial ψ. It is clear that it has the same list of properties as all $n \times n$-matrix rings over fields. Since any n-algebra $A_{\mathcal{F}}$ satisfies exactly the same identities as $M_n(F)$, where $F \rightleftharpoons Z(A)$, ψ has the same list of properties for all n-algebras (see [29], vol. 2, p.47). For any algebra A the image of ψ on A coincides with the whole centre F. Suppose that $f \in Z(K)$ and $K \subseteq L$, where K, $L \in \mathcal{K}_{T_\bullet}$. For any point $p_0 \in X(K)$ we have $k(p_0) \in Z(K_{p_0})$. Therefore, there exist $t_1^{p_0}, \ldots, t_m^{p_0} \rightleftharpoons \bar{t}^{p_0}$ such that $k(p_0) = \psi(\bar{t}^{p_0}(p_0))$. This equality also holds in an open-and-closed neighbourhood e_{p_0} of the point p_0, that is,

$$\forall p \in e_{p_0} (k(p) = \psi(\bar{t}_{p_0}(p))).$$

We choose a disjunctive subcover $e_{p_1}, \ldots, e_{p_l}$ of the whole of $X(K)$. We paste together $t_1^{p_1}, \ldots, t_1^{p_l}$ on $e_{p_1}, \ldots, e_{p_l}$ (that is, we form $t_1 \rightleftharpoons \sum_{i=1}^{l} t_1^{p_i} \cdot e_{p_i}$) and we also form $t_2, \ldots, t_m$. Then $k = \psi(\bar{t})$. Since $\bar{t} \subseteq L$, $k = \psi(\bar{t})$ "in L" and for any point $p \in X(L)$ we have $k(p) = \psi(\bar{t}(p))$, so $k(p) \in Z(L_p)$. Thus, $k \in Z(L)$.

***Theorem* 10.3.** *If T_0 is a model-complete theory, then so is T.*

Proof. Suppose that A, $B \in \mathcal{L}_{T_\bullet}$ and $A \subseteq B$, and let $F \rightleftharpoons Z(A)$ and $G \rightleftharpoons Z(B)$. Then Proposition 10.2 shows that $F \subseteq G$ and it is easy to see that any basis $\{x_1, \ldots, x_n\}$ for A over F is a basis for B over G. In fact, we form the subalgebra $A \cdot G$ in B over G generated by the subring A. We have $Z \rightleftharpoons Z_{A \cdot G}(A) \supseteq F, G$; $F, G \subseteq Z(A \cdot G)$, and $A \cdot G$ is an algebra over F. According to [25], p.232, we find that $A \cdot G \cong A \otimes_F Z_{A \cdot G}(A)$, where the isomorphism is given by $a \otimes c \mapsto a \cdot c$. But this is also an isomorphism of G-algebras. So

$$n = \dim_G B \geqslant \dim_G A \cdot G = \dim_G (A \otimes_F Z)_G = (\dim_F A) \cdot (\dim_G Z) = n \cdot [Z : G].$$

Hence $Z = G$. Thus, $\dim_G A \cdot G = n$ and $A \cdot G = B$. Therefore, $B_G \cong (A \otimes_F G)_G$. Now any basis $\{a_i\}$ of A_F becomes the basis $\{a_i \otimes 1\}$ of $(A \otimes_F G)$, that is, of B_G. Arguing as in [23], p.23, we complete the proof.

Examples of model completeness are: the theory of quaternions, the class of n-algebras with centres real or algebraically closed, the class of rings that are elementarily equivalent to the quaternions or to the ring $M_2(\mathbb{R})$, and so on.

Corollary 10.4. *Suppose that T_0^* is a model companion of T_0. Then the theory T^* (corresponding to the class $\mathcal{L}_{T_0^*}$) is a model companion of the theory T.*

Theorem 10.5. *Suppose that T_0 is a model-complete theory. Then the class $\mathcal{K}_{T_0} \leftrightharpoons \{K \mid K \models \Phi_2 \wedge \{K_p\} \models T\}$ is model complete and axiomatizable by Horn formulae. What is more, it is Boolean absolute.*

The proof consists in verifying the Boolean absoluteness (by applying 10.2 and Example 9) and in applying 10.3 on the basis of Theorem 18, part b.

For example, the following classes are model complete:

$$\{K \mid \forall p\, (K_p \equiv \mathbb{H}, K \models \Phi_2\} \quad \text{and}$$

$$\{K \mid (\forall p\, (K_p \equiv \mathbb{H} \vee K_p \equiv M_2(\mathbb{R})) \wedge K \models \Phi_2\}.$$

Theorem 10.6. *Suppose that T_0^* is a model companion of T_0. The class $\mathcal{K}_{T_0^*} \leftrightharpoons \{K \mid K \models \Phi_2 \wedge \{K_p\} \models T^*\}$ is Horn axiomatizable and is a model companion of the class $\mathcal{K}_{T_0}$.*

We dwell briefly on completeness and decidability. We denote by $\mathcal{L}_{T_0, \bar{c}}$ a subclass of $\mathcal{L}_{T_0}$ with a fixed $k-s$ relation and fixed structural constants $\bar{c}$ from $\mathbb{Q}$. It is axiomatizable by some theory $T_{\bar{c}}$.

Proposition 10.7. *Suppose that T_0 is a complete and model-complete theory of rings. Then the class $\mathcal{K}_{T_0, \bar{c}} \leftrightharpoons \{K \mid K \models \Phi_2 \wedge \{K_p\} \models T_{\bar{c}}\}$ is Horn-axiomatizable, model complete, and complete, and also decidable if T_0 is recursively axiomatizable.*

Theorem 10.8. *Suppose that T_0^* is the theory of algebraically closed fields. Then the Horn-axiomatizable class $\mathcal{K}_{T_0^*}$ is a model companion of the class $\mathcal{K}_{\mathcal{L}} \leftrightharpoons \{K \mid \{K_p \models T_{\mathcal{L}}\}$.*

Example 11. We now present Example 10 in the far simpler case of matrix rings. We denote by T the set of all propositions (in the language of rings) that are valid in the matrix ring $M_n(F)$, where F is a fixed ring, that is, $T \leftrightharpoons Th\,(M_n\,(F))$. Similarly, we put $T^* \leftrightharpoons Th\,(M_n\,(F_1))$, where F_1 is an algebraically closed field. In [23] it was noticed that the theory T^* is model complete and we assume that the theories $Th(F)$ and $Th(F_1)$ are mutually model embeddable. Then the theories T and T^* are also mutually model embeddable, that is, T^* is a model companion for T. These theories are normal and totally autonomous, since they contain the proposition Φ_5. We form the classes

$$\mathcal{K} \leftrightharpoons \{K \mid \forall p\, (K_p \equiv M_n\,(F))\} \quad \text{and}$$

$$\mathcal{K}^* = \{K \mid K \models \Phi_2 \wedge \forall p\, (K_p \equiv M_n\,(F_1))\},$$

which correspond, as usual, to the local theories T and T^*. The class $\mathcal{K}^*$ is Boolean regular and in addition for $K, L \in \mathcal{K}^*$ we have $K \subseteq L \Rightarrow Z(K) \subseteq Z(L)$. We verify the last assertion. In $M_n(F_1)$ (where in all arguments F_1 can be replaced by any commutative ring) there is a collection of elements e_{ij}, $1 \leqslant i, j \leqslant n$ ("matrix units") such that $\sum_{i=1}^{n} e_{ii} = 1$ and $e_{ij} \cdot e_{pq} = \delta_{jp} \cdot e_{iq}$ (where δ_{jp} is the Kronecker delta) and this can be expressed as an E-formula. By Proposition 6 and Theorem 9, part a, we find that this formula is true in K; the resulting system of matrix units in K will be denoted by $\{e_{ij}\}$. It is also a system of matrix units in L. It is easy to see ([24], p.52) that $L \cong M_n(G)$, where $G \rightleftharpoons Z_L(\{e_{ij}\})$ is the centralizer of the family $\{e_{ij}\}$ (that is, $Z_L(X) \rightleftharpoons \{l \in L \mid \forall x \in X\ (l \cdot x = x \cdot l)\}$); here G is a ring and the isomorphism sends an l from G into $l \cdot E$. By the Amitsur-Levitzki theorem ([25], p.405), and again by Proposition 6 and Theorem 9, part a, we find that the standard identity of order $2n$ holds in L, which by a theorem of Leron and Vapne ([26], p.135), means that G is a commutative ring. Now, if $k \in Z(K)$, then $k \in G$ and for any l in L (denoting the corresponding matrix by m) we get $kE \cdot m = m \cdot kE$, which completes the proof. All the localizations of a ring K in the class $\mathcal{K}^*$ are simple rings. For if $K \equiv M_n(F)$, where F is a field, then in $M_n(F)$, and consequently in K, we have

$$\exists \bar{e}_{ij} \forall x, y \left(\sum_{=1}^{n} e_{ii} = 1 \wedge e_{ij} \cdot e_{pq} = \delta_{jp} \cdot e_{iq} \wedge e_{ij} \cdot x = x \cdot e_{ij} \right).$$

As above $K \cong M_n(G)$, where $G \rightleftharpoons Z(\{e_{ij}\})$ and G is the centre of K. It is clear that $G \equiv F$ and, in particular, G is a field. Two-sided ideals in $M_n(G)$ are of the form $M_n(a)$, where a is a two-sided ideal in G, and therefore K is a simple ring. Thus, the class $\mathcal{K}^*$ is Boolean regular and Boolean prime (according to Example 9). As we noticed in Example 9, it consists of biregular rings, and therefore $\forall K \in \mathcal{K}^*\ (K \models \Phi_1)$, that is, $\mathcal{K}^*$ is of the required form. Consequently, $Th\mathcal{K}^*$ is a model-complete Horn theory, and it is a model companion for the class $\mathcal{K}$. From this we can pass in the usual way to the completeness and decidability of the theory $Th\mathcal{K}^*$. For example, if the field F_1 has a decidable theory, then the theory of $M_n(F_1)$ is decidable ([23], p.36) and by Theorem 21, part b, the theory $Th\mathcal{K}^*$ is, in addition to what has been said, complete and (if ThF_1 is recursively axiomatizable) decidable.

In the same way we see that if T is the theory of n-matrix rings over commutative regular algebraically closed atomless rings, then the theory T^* of n-matrix rings over commutative regular algebraically closed atomless rings is a model companion of T, and the same is true for the corresponding classes $\mathcal{K}^* \rightleftharpoons \{K \mid K \models \Phi_2 \wedge \{K_p\} \models T^*\}$ and $\mathcal{K} \rightleftharpoons \{K \mid \{K_p\} \models T\}$.

A *Boolean absolute* class is a Boolean regular and Boolean prime class (see Example 9).

II.3. Macintyre's problem: a model companion of a locally axiomatizable class.

The formulation of Macintyre's problem and the corresponding references were given at the beginning of §II.2 before Proposition 4.

A theory T is (*normally*) *autonomous* if any model K of it can be embedded in a (normal) ring F such that $\{F_p\} \models T$.

We denote by $X_1(K)$ the set of all proper ideals in $B(K)$ (and similarly for an arbitrary algebra Ω). If $q \in X_1(K)$, then $\bar{q} = q \cdot K$ is an ideal in K and we write $\{K_q\} = \{K/\bar{q} \mid q \in X_1(K)\}$. A theory T is *totally autonomous* if for any model F of it we have $\{F_q\} \models T$.

Example 12. Any theory T such that $T \vdash \Phi_5$ (all the more if $T \vdash \Phi_4$) is normal and totally autonomous. All the classes of rings mentioned in Example 9 have such local theories.

For a theory T in disjunctive normal form we recall the notation $T' = \{\varphi' \mid \varphi \in T\}$, where φ' was defined before Theorem 12.

Theorem 18. a) *The class $\mathcal{K}^*$ is axiomatizable and, what is more, Horn-axiomatizable. If $\mathcal{K}_1 = \{K \mid \{K_p\} \models T, K \models \Phi_1\}$, where T is an AE-theory (in the absence of the condition $K \models \Phi_1$, and AE-positive theory), then the class $\mathcal{K}_1$ is Horn-axiomatizable, and in fact $\mathcal{K}_1 = \{K \mid K \models T', K \models \Phi_1\}$.*

b) *If $\mathcal{K}^*$ is a Boolean absolute class, then it is model complete.*

c) *If T^* is a normally autonomous theory then $\mathcal{K}$ can be embedded in $\mathcal{K}^*$.*

d) *Suppose that $T \subseteq T^*$. If T^* is a normally autonomous theory and $\mathcal{K}^*$ is a Boolean absolute class, then $\mathcal{K}^*$ is a Horn model companion of $\mathcal{K}$. We have $\mathcal{K}^* \subseteq$ Mod $T' \subseteq \mathcal{K}$. Under the same conditions $\Phi_1 + \Phi_2 + (T^*)'$ is a model companion of T'.*

e) *If T^* is a normally autonomous theory, T is a totally autonomous theory, and $\mathcal{K}^*$ is a Boolean absolute class, then $\mathcal{K}^*$ is a Horn model companion of $\mathcal{K}$ (we presume that the model-embedding of T^* in T is deducible in ZFC).*

Proof. a) We claim that the class $\mathcal{K}_1 = \{K \mid \{K_p\} \models T^*, \; K \models \Phi_1\}$ is Horn-axiomatizable, more precisely $\mathcal{K}_1 = \{K \mid K \models (T^*)', \; K \models \Phi_1\}$. For if $K \in \mathcal{K}_1$, then by Proposition 4 and Theorem 12 we obtain $K \models \varphi'$, where $\varphi \in T^*$. Conversely, by Theorem 12 and Proposition 5 we obtain $\{K_p\} \models \varphi$. All the formulae φ' and the formula Φ_1 are Horn formulae. The class $\mathcal{K}^*$ is distinguished in the class $\mathcal{K}_1$ by the Horn axiom Φ_2. The second assertion of part a) generalizes the first, since a model complete theory is AE-axiomatizable. If $K \in \mathcal{K}_1$, then by Proposition 6 and Theorem 12 we obtain $K \models T'$. If $K \models T'$, then by Theorem 12 and Proposition 5 we obtain $\{K_p\} \models T$.

b) Let ψ be a primitive formula with parameters from K, where $K \in \mathcal{K}^*$, that is, the assertion of existence of a solution of a system of equations and inequalities with coefficients from K, and let L be any extension of K in the class $\mathcal{K}^*$. We denote by ψ_1 the assertion of existence of a solution of the subsystem of this system consisting of all equalities and one inequality or of all equations only. The index l numbers such subsystems in arbitrary order. In [8] it was shown in fact that under the conditions that L is normal and atomless we have

$$(L \models \psi) \Leftrightarrow [\forall l\,(\{p \in X(L) \mid L_p \models \psi_l\} \neq \varnothing),\ \bigcup_l \{p \mid L_p \models \psi_l\} = X(L)].$$

We also verify this equivalence here. By Proposition 4 the set in curly brackets coincides with $[\![\psi_l]\!]_{\mathcal{T}(L)}$ and is an open-and-closed set. The implication from left to right is obvious. For the converse we write $[\![\psi_l]\!]_{\mathcal{T}(L)} = u_l \in B(L)$. We form the Boolean subalgebra in $B(L)$ generated by the finite set $\{u_l\}$. It is finite and thus atomic. In u_1 we choose an atom v_1. If in u_2 we do not have any atom except v_1 (otherwise we choose in u_2 an atom $v_2 \neq v_1$), then we split v_1 in an arbitrary way (in the atomless algebra $B(L)$) into v_1 and v_2 and again obtain $v_1 \leqslant u_1$, $v_2 \leqslant u_2$, and $v_1 \neq v_2$. We use here the fact that the ring L is atomless. Continuing this process for all l we obtain a system of pairwise disjunctive elements $\{v_l\} \subseteq B(L)$, $0 \neq v_l \leqslant u_l$. We supplement it with $w_l \leqslant v_l$ such that $\{\ldots v_l \ldots, w_l \ldots\}$ is a decomposition of 1. By accessibility (Theorem 5) there exist $\bar{k}_l$ such that $u_l \leqslant [\![\psi_l(\bar{k}_l)]\!]$. We paste these $\bar{k}_\alpha$ onto v_l and $\bar{k}_l$ onto w_l, obtaining $\bar{k} \in L$. This $\bar{k}$ is a solution of the system ψ in L. Using this equivalence we obtain

$$(L \models \neg\psi) \Leftrightarrow (\bigvee_l ([\![\psi_l]\!] = 0)) \vee (\bigcup [\![\psi_l]\!] < 1)$$

where in place of $\varnothing$ and $X(L)$ we write 0 and 1, respectively.

We successively rewrite the first disjunctive term as follows: $\bigvee_l([\![\neg\psi_l]\!] = 1)$; by the model completeness of T^* we can find an E-formula ψ_l' (with quantifier-free part of the form $\bigvee_s \psi_{ls}$) equivalent to $\neg\psi_l$ in models for T^*, that is, $\bigvee_l ([\![\psi_l']\!] = 1)$; by accessibility (Theorem 5) we have $\bigvee_l \exists\bar{k}(\bigcup_s [\![\psi_{ls}]\!] = 1)$ that is, $\bigvee_l \exists\bar{k}\exists\bar{e}_{ls} \in B(L)\,(e_{ls} \leqslant [\![\psi_{ls}]\!] \wedge \prod_s (1 - e_{ls}) = 0)$. The last formula can be written in the form

$$\exists\bar{k}\exists\bar{e}_{ls} \bigvee_l \forall e_0 \Big[\prod_s (1 - e_{ls}) = 0 \wedge e_{l_1}\cdot k_1 = e_{l_1}\cdot t_1 \wedge \cdots \wedge (e_0\cdot k_2 = e_0 \cdot t_2) \Rightarrow$$
$$\Rightarrow e_0 \leqslant 1 - e_{l_1}) \wedge \cdots\Big],$$

where $k_1 = t_1$ is one of the equalities and $k_2 \neq t_2$ one of the inequalities occurring in ψ_{l_1}, and e_{ls} and e_0 are special variables running through $B(L)$.

Similarly, for the second disjunctive term we obtain the chain of equivalences $\exists p \in X(L) \bigwedge_l (p \notin [\![\psi_l]\!])$, $\exists p \in X(L)\ (p \notin [\![\bigvee_l \psi_l]\!])$, $[\![\neg\bigvee_l \psi_l]\!] > 0$; by the model completeness of T^* we can find an E-formula ψ with quantifier-free part $\bigvee_s \psi_s$ such that $[\![\psi]\!] > 0$; by accessibility

$$\exists\bar{k}\ ([\![\bigvee_s \psi_s]\!] \geqslant 0),\quad \exists\bar{k}\exists e \bigvee_s\ (e \neq 0 \wedge [\![\psi_s]\!] \geqslant e).$$

The last formula can be written in the form

$$\exists \bar{k} \exists e \bigvee_s \forall e_0 (e_0 \neq 0 \wedge e \cdot k_1 = e \cdot t_1 \wedge \ldots \wedge (e_0 \cdot k_2 = e_0 \cdot t_2 \Rightarrow e_0 \leqslant 1 - e) \wedge \ldots),$$

where $k_1 = t_1$ is one of the equalities and $k_2 \neq t_2$ one of the inequalities occurring in ψ_s, and e and e_0 are special variables running through $B(L)$.

Because the class $\mathcal{K}^*$ is Boolean absolute, and using Proposition 8, part b, we find that $\neg\psi$ carries over from K to L, where $K \subseteq L$, $K, L \in \mathcal{K}^*$. Since Robinson's criterion holds for the axiomatizable class $\mathcal{K}^*$, we find that this class is model complete.

c) If $K \in \mathcal{K}$, then K is contained in $\prod_p K_p$, where $K_p \models T$ (and the ring K_p is indecomposable). By the definition of model companion, K_p is embeddable in F^p, a model of T^*. By the condition that T^* is normally autonomous we can assume that F^p is a normal ring all of whose localizations are models of T^*. We introduce the discrete topology in F^p. Let X_0 be the Cantor set (or any totally disconnected Hausdorff compact space without isolated points). Let $\bar{F}^p = C(X_0, F^p)$ be the set of all continuous functions from X_0 to F^p. The ring $\bar{F}^p$ consists of piecewise-constant F^p-valued functions on X_0 with finitely many values. The ring F^p is included in $\bar{F}^p$. We later show that $\bar{F}^p \in \mathcal{K}^*$.

Since $\mathcal{K}^*$ is a Horn class, it is closed with respect to any products (even filtered ones). Therefore, $\left(\prod_p F^p\right) \in \mathcal{K}^*$ and consequently

$$K \rightarrowtail \prod_p K_p \rightarrowtail \prod_p F^p \rightarrowtail \prod_p \bar{F}^p \in \mathcal{K}^*.$$

So we consider the ring $\bar{F} = C_0(X_0, F)$, where F is a normal ring all of whose localizations are models for T^*. We have $B(\bar{F}) = C(X_0, B(F))$ and the set $\langle x_0, p_0\rangle$, by definition equal to $\{f \in B(\bar{F}) \mid f(x_0) \in p_0\}$, is a prime ideal in $B(\bar{F})$ for any $x_0 \in X$ and $p_0 \in X(F)$. Any point in $X(\bar{F})$ has this form, that is, symbolically $X(\bar{F}) = X_0 \times X(F)$, since for $p \in X(\bar{F})$ there is an $x_0 \in X_0$ such that $p_0 = \{f(x_0) \mid f \in p\}$ does not contain the identity of $B(F)$ (otherwise $\{\{x_0 \in X_0 \mid f(x_0) = 1\} \mid f \in p\}$ is an open covering of X_0 and a finite subcover yields $f_1, \ldots, f_n$ such that $f_1 \vee \ldots \vee f_n \in p$ and $f_1 \vee \ldots \vee f_n \equiv 1$, a contradiction). Such a p_0 is a prime ideal in $B(F)$. Therefore, $p \subseteq \langle x_0, p_0\rangle$, and since p is maximal (everything takes place in Boolean algebras), this is possible only if $p = \langle x_0, p_0\rangle$. Then $\overline{\langle x_0, p_0\rangle} = \{f \in \bar{F} \mid f(x_0) \in \bar{p}_0\}$, where $\bar{p}_0 = p_0 \cdot F$ and $\overline{\langle x_0, p_0\rangle} = \langle x_0, p_0\rangle \cdot \bar{F}$. Therefore, $(\bar{F})_{\langle x_0, p_0\rangle} = \bar{F}/\overline{\langle x_0, p_0\rangle} \simeq F/\bar{p}_0 = F_{p_0}$, where p_0 runs through $X(F)$. By hypothesis we have $F_{p_0} \models T^*$ for all F_{p_0}, so $\{(\bar{F})_{\langle x_0, p_0\rangle}\} \models T^*$. We show that $\bar{F}$ is normal. If $f \in \bar{F}$, then we put $e_0(x) = e_1$, where e_1 is the element of F that corresponds by the normality of F to the element $f(x)$ of F, and such an e_0 satisfies the normality condition for f in $\bar{F}$. Suppose that f is an atom in $B(\bar{F})$. At least one "step" of f, say $f(x_0)$, is different from 0.

This step is taken on an open-and-closed set containing at least two points. Removing one of them together with its open-and-closed neighbourhood, we find that f is not an atom.

d) The first assertion follows at once from b) and c). If $K \in \mathscr{K}^*$, then $K \models \Phi_1 + (T^*)'$, so certainly $K \models T'$. If $K \models T'$, then by Theorem 12 and Proposition 5 we find that $K \in \mathscr{K}$. From this we get the last assertion.

e) We need only supplement d) with the fact that the class $\mathscr{K}^*$ is embeddable in $\mathscr{K}$ (apart from this, in d) the condition we now want to remove was not used at all). This follows from (the general) Theorem 19, and is proved below.

***Theorem* 19.** *Suppose that the theory T is model embeddable in the theory T_1 (and that this is deducible in ZFC). If T is an AE-theory, T_1 is totally autonomous, K is normal (that is, $K \models \Phi_1$), and $\{K_p\} \models T$, then K is embeddable in a ring L such that $L \in \mathscr{K}_1 = \{L \mid \{L_p\} \models T_1\}$.*

To infer Theorem 18, part e from this we put $T = T^*$ and $T_1 = T$. By definition of model companion, T^* is model embeddable in T (and in part d) we discussed the deducibility of this fact in ZFC). It is well-known that T^* is an AE-theory and that K in $\mathscr{K}^*$ is normal.

Remark. 1) In part c of Theorem 18 we used only two properties of the model completeness of T^*: the fact that $\mathscr{K}^*$ is closed with respect to products and that T is model embeddable in T^*. In this way, Theorem 18, part c and Theorem 19 give examples of "embedding theorems for locally-axiomatizable classes". 2) In part d of Theorem 18 we proved the formula $(T')^* = (T^*)'$ under the condition that the notation $(T^*)'$ means that the axioms $\Phi_1 + \Phi_2$ are automatically added.

***Corollary* 1.** *If, under the conditions of part* e *of Theorem* 18, *the class $\mathscr{K}$ is axiomatizable, then $\Phi_1 + \Phi_2 + (T^*)'$ is a Horn model companion of $Th\mathscr{K}$.*

In the next three corollaries we assume that $T \subseteq T^*$.

***Corollary* 2.** *If $\mathscr{K}$ is an axiomatizable subclass of the class of all biregular rings and $\mathscr{K}^*$ is a Boolean regular class, then $\Phi_1 + \Phi_2 + (T^*)'$ is a Horn model companion of $\mathscr{K}$.*

***Corollary* 3.** *If $\mathscr{K}$ is an axiomatizable subclass of the class of all strictly Rickartian rings and $\mathscr{K}^*$ is Boolean prime, then $\Phi_1 + \Phi_2 + (T^*)'$ is a Horn model companion of $Th\mathscr{K}$.*

***Corollary* 4.** *If $\mathscr{K}$ is an axiomatizable subclass of the class of all Abelian regular rings, then $\Phi_1 + \Phi_2 + (T^*)'$ is a Horn model companion of $Th\mathscr{K}$.*

A class $\mathscr{K}$ is *normal* if for any $K \in \mathscr{K}$ there is an $L \in \mathscr{K}$ such that $K \subseteq L$ and $L \models \Phi_1$.

***Corollary* 1 *to Theorem* 19.** *Suppose that the theories T and T_1 satisfy the conditions of Theorem* 19. *If $\mathcal{K}$ is a normal class, then $\mathcal{K}$ is embeddable in $\mathcal{K}_1$.*

We call a ring L a *quasi-sheaf* ring if the sheaf condition is fulfilled relative to a regular Boolean algebra $\mathbb{B} \subseteq B(L)$.

***Corollary* 2 *to Theorem* 19.** *Suppose that in addition to the conditions of Theorem* 19 *we have $\forall x \exists f\,(x \models T \Rightarrow x \subseteq f \wedge f \models T_1 \wedge f \models T_2)$, where T_2 is a Horn theory. Then for the corresponding L we have $L \models T_2$ and L is a quasi-sheaf ring (a sheaf ring if $T_1 \vdash \Phi_5$).*

If we put $T = T_1$, then the ring L corresponding to K has the natural properties of a "sheaf closure of K" and the class $\mathcal{K}$ corresponding to T is closed with respect to "sheaf closure".

Proof of Theorem 19. Suppose that K satisfies the conditions of the theorem. By Proposition 6 we have $[\![T]\!]_{\mathcal{T}(K)} = 1$, so by Theorem 10, part b, we obtain $[\![T]\!]_{B(K)} = 1$. By Theorem 16, part e, the latter means that $[\![T]\!]_{B,K'} = 1$ or, which is the same, $[\![K' \models T]\!]_B = 1$, where $\mathbb{B} = \mathbb{B}(K)$, and the valuation is that defined in Example 8 (that is, a valuation in the language ZF with set of parameters V^B: see Example 1), and the object $K' = \mathcal{F}_1'$ was defined before Theorem 16, part e. We could also refer here to Theorem 16, part a. By hypothesis, the formula

$$\forall x \exists f\,(x \models T \Rightarrow x \subseteq f \wedge f \models T_1)$$

is deducible in ZFC. Hence by accessibility in V^B we obtain

$$[\![K' \subseteq f \wedge f \models T_1]\!]_B = 1,$$

where $f \in V^B$. We write $\hat{f}^B = L$. Again by Theorem 16, part e, we obtain $K \subseteq L$ (in the sense of $k \mapsto p_k$) and it is sufficient to show that $L \in \mathcal{K}_1$ to complete the proof of the theorem.

In fact, $\mathbb{B}$ can be embedded in $B(L)$ by the rule $b \mapsto b\cdot 1 + \neg b\cdot 0$ (the right-hand side is the pasting of 1 and 0 in f). We denote this embedding by h. We write $L_{(p_0)} = L/\overline{h(p_0)} = h(p_0)\cdot L$, where p_0 on the left-hand side is a point of the Stone space $S(\mathbb{B})$ of the Boolean algebra $\mathbb{B}$, and $\overline{h(p_0)}$ on the right-hand side is an ideal in L. We show that $L_{(p_0)}$ coincides with the "fibre" of f at the point p, that is, with the factorization of L by the equivalence relation $(k \sim_{p_0} t) \Leftrightarrow ([\![k = t]\!]_B \notin p_0)$ (see §I.6), and we also show that $([\![f \models \varphi(k_1, \ldots, k_n)]\!]_B = 1) \Leftrightarrow \{L_{(p_0)}\} \models \varphi(|k_1|_{p_0}, \ldots, |k_n|_{p_0})$, where p_0 runs through $S(\mathbb{B})$ and $k_1, \ldots, k_n \in L$. The first assertion means: $\exists b \in p_0\,(h(b)\cdot(k - t) = k - t) \Leftrightarrow [\![k = t]\!]_B \notin p_0$. From left to right: $h(\neg b) = 1 - h(b)$, $(1 - h(b))\cdot(k - t) = 0$ in L, next $\neg b \notin p_0$ and $[\![(1 - h(b))(k - t) = 0]\!]_B = 1$, $\neg b \leqslant [\![k - t = 0]\!]_B$. For the converse, suppose that $b = [\![k = t]\!]_B \notin p_0$, $b \leqslant [\![h(b)\cdot(k - t) = 0]\!]_B$, and

$\neg b \leqslant [\![h(b)\cdot(k-t)=0]\!]_B$, $h(b)\cdot(k-t) = 0$ in L and $\neg b \in p_0$, $h(\neg b)\cdot(k-t) = k-t$. The second assertion can be verified in the form $p_0 \in [\![f \models \varphi(\bar{k})]\!]_B \Leftrightarrow L_{(p_0)} \models \varphi([\bar{k}]_{p_0})$ for the connectives $\wedge$, $\neg$, and $\exists$ by induction on the length of the formula φ (see Theorems 4, 2, part a). Thus $\{L_{(p_0)}\} \models T_1$.

For any $p \in X(L)$ we form $p_0 = h^{-1}(p) \subseteq \mathsf{B}$. Then $p_0 \in S(\mathsf{B})$. We write $a = \overline{h(p_0)}$, an ideal in L. Then $a \subseteq \bar{p}$ and $L_p = L/\bar{p} \simeq (L/a)(\bar{p}/a) = L_{(p_0)}/\overline{p/a}$. We note that $q = p/a$ has the properties: $q \subseteq B(L/a)$ and q is closed with respect to $\vee$ and does not contain $[1]_a$. For if $[1]_a = [e]_a$, where $e \in p$, then $1 - e = e_0\cdot r$, $e_0 \in h(p_0)$, $1 = e + e_0\cdot r = (e \vee e_0)\cdot 1$, $e \vee e_0 \in p$, a contradiction. We add to q all elements $[l]_a$ from $B(L/a)$ that are majorized by some $[e]_a$ from q: the q_1 obtained in this way is a proper ideal in $B(L/a)$ and $\bar{q}_1 = q$. Thus $L_p = L_{(p_0)}/\bar{q}_1$, where $q_1 \in X_1(L_{(p_0)})$ and by hypothesis we obtain $L_p \models T_1$.

II.4. A model companion of a class of localizations. The completeness of the theory of a locally axiomatizable class.

We now consider the question of the transfer of model completeness in the "reverse direction" from the classes $\mathcal{K}^*$ and $\mathcal{K}$ onto their local theories. To this end in the general case it is useful to extend the notion of Boolean primality of a class in the following way (see Proposition 8). Let $\mathcal{K}_1$ and $\mathcal{K}$ be two classes. We say that $\mathcal{K}_1$ is *Boolean prime* for $\mathcal{K}$ if

$$\forall K \in \mathcal{K}\ \forall L \in \mathcal{K}_1\ (K \subseteq L \Rightarrow \exists p_1 \in X(L)\ \exists p \in X(K)$$
$$\forall k \in K\ (p \supseteq p_1 \cap K \wedge \exists e_2 \in p_1\ (e_2\cdot k = k) \Rightarrow \exists e \in p\ (e\cdot k = k)].$$

However, in Theorem 20 it is sufficient to impose a weaker condition on the classes $\mathcal{K}_1$ and $\mathcal{K}$ than the condition "the class $\mathcal{K}_1$ is a model companion of the class $\mathcal{K}$ and the classes $\mathcal{K}_1, \mathcal{K}$ are mutually Boolean prime". We sometimes denote the latter condition by (*). The weak condition takes into account special forms of the classes $\mathcal{K}_1$ and $\mathcal{K}$. We recall that the ring $\bar{L} = C(X_0, L)$ was defined for the ring L in the proof of Theorem 18, part c. A class $\mathcal{K}_1$ is a *weak model companion of a class* $\mathcal{K}$ if: 1) for any two rings $\bar{L}_1$ and $\bar{L}_2$ in $\mathcal{K}$ such that $L_1, L_2 \models T_1$, and any Horn $AEAE$-formula φ with constant parameters from $\bar{L}_1$ (that is, parameters of the form $\lambda(x) \equiv \lambda_0$, where $\lambda_0 \in L_1$ and x runs through X_0), if $\bar{L}_1 \models \varphi$ and $\bar{L}_1 \subseteq \bar{L}_2$, then $\bar{L}_2 \models \varphi$; 2) if $K \models T$, $K \in \mathcal{K}$, then K can be embedded in an L from $\mathcal{K}_1$, and if $L \models T_1$, $\bar{L} \in \mathcal{K}_1$, then $\bar{L}$ can be embedded in a K from $\mathcal{K}$; 3) if $K \models T$, $K \in \mathcal{K}$, and $K \subseteq L$, where $L \in \mathcal{K}_1$, then there exist $p \in X(K)$ and $p_1 \in X(L)$ such that $p \supseteq p_1 \cap K$ and $\forall k \in K \cap \bar{p}_1\ (k \in \bar{p})$; 4) if $L \models T_1$, $\bar{L} \in \mathcal{K}_1$, and $\bar{L} \subseteq K$, where $K \in \mathcal{K}$, then there exist $p \in X(\bar{L})$ and $p_1 \in X(K)$ such that $p \supseteq p_1 \cap \bar{L}$ and $\forall l \in L\ (l \in \bar{p}_1 \Rightarrow l \in \bar{p})$. It is clear that the condition (*) implies the fact that the class $\mathcal{K}_1$ is a weak model companion of the class $\mathcal{K}$. In the next theorem we denote by T_2 the part of T_1 consisting of formulae of the form EAE, that is, $T_2 = (T_1)_{EAE}$.

***Theorem* 20.** *Suppose that*

$$\{K \mid \{K_p\} \models T_1,\ K \models T_1' + \Phi_1 + \Phi_2\} \subseteq \mathcal{K}_1 \subseteq \{K \mid \{K_p\} \models T_1\},$$
$$\mathcal{K} = \{K \mid \{K_p\} \models T\}$$

and that the class $\mathcal{K}_1$ is a weak model companion of the class $\mathcal{K}$. If $T \vdash \Phi_5$ and $T_1 \vdash \Phi_5$, then T_1 is a model companion of T.

(If $T \subseteq T_1$, then condition 4 in the definition of weak model companion can be omitted.)

Proof. We shall prove the model completeness of T_1. Suppose that $F_1 \subseteq F_2$ are two models for T_1 and ψ is a primitive formula over F_1 such that $F_1 \models \neg\psi$. We form the rings $\bar{F}_1 = C(X_0, F_1)$ and $\bar{F}_2 = C(X_0, F_2)$, $\bar{F}_1 \subseteq \bar{F}_2$ (see the proof of Theorem 18, part c). The localizations of $\bar{F}_1$ and $\bar{F}_2$ coincide with the localizations of F_1 and F_2. Since $T_1 \vdash \Phi_5$, we have $\{(\bar{F}_1)_p\} \models T_1$, $\bar{F}_1 \models \Phi_1 \wedge \Phi_2$, and what is more, all localizations of $\bar{F}_1$ (and $\bar{F}_2$) coincide with F_1 (respectively, with F_2). In this situation we can prove by induction on the length of arbitrary formula φ the following generalization of Proposition 6:

$$\{(\bar{F}_1)_p\} \models \varphi(\lambda_1^p, \ldots, \lambda_n^p) \Rightarrow [\![\varphi(\lambda_1, \ldots, \lambda_n)]\!]_{\mathcal{T}(F_1)} = 1,$$

where $\lambda_1, \ldots, \lambda_n$ are constants from $\bar{F}_1$. The proof of Proposition 6 need only be supplemented by the argument that $(\bar{F}_1)_p$ is isomorphic to F_1 by the rule $[f]_{\bar{p}} \mapsto f(x_0)$, where x_0 corresponds to p. Thus, $\bar{F}_1, \bar{F}_2 \in \mathcal{K}_1$. For an A-formula $\neg\psi$ (as well as for any formula with constant parameters) in a normal ring we have $[\![\neg\psi]\!]_{\mathcal{T}(\bar{F}_1)} = 1$. By Theorem 12 we have $\bar{F}_1 \models (\neg\psi)'$. Then by hypothesis, we get $\bar{F}_2 \models (\neg\psi)'$, and using Theorem 12 we find that $[\![\neg\psi]\!]_{\mathcal{T}(\bar{F}_2)} = 1$, that is, $F_2 \simeq (\bar{F}_2) \models \neg\psi$.

Let K be a model for a theory T. Since $T \vdash \Phi_5$ we have $K \in \mathcal{K}$. By hypothesis we get $K \subseteq L \in \mathcal{K}_1$. Because the classes are Boolean prime we can find $p_1 \in X(L)$ and $p \in X(K)$ such that $p \supseteq p_1 \cap K = \{0\}$, that is, $p = p_1 \cap K = \{0\}$. Hence, $K \to L_{p_1}$, $k \mapsto [k]_{\bar{p}_1}$, is an embedding and again by hypothesis we get $L_{p_1} \models T_1$. If $T \subseteq T_1$, then everything has been proved and condition 4 in the definition of weak model companion has not been used.

It remains to show that T_1 can be model-embedded in T. Suppose that $L \models T_1$. We form $\bar{L} \in \mathcal{K}_1$. By hypothesis we have $\bar{L} \subseteq K \in \mathcal{K}$, and we can find $p_1 \in X(K)$, $p \in X(\bar{L})$ such that $p \supseteq q = (p_1 \cap \bar{L})$, and p has the corresponding properties. As we saw in the proof of Theorem 18, part c, p is of the form $\langle x_0, p_0\rangle$, where $p_0 \in X(L)$, that is, $p_0 = \{0\}$. Therefore, $\bar{p} = \{f \in \bar{L} \mid f(x_0) = 0\}$ and we put $L \rightarrowtail K/\bar{p}_1$, $e \mapsto [e]_{\bar{p}_1}$. By hypothesis this is indeed an embedding. Hence we obtain an embedding of L in $K_{p_1} \models T$.

Corollary. *The class $\mathcal{K}_7$ of all Abelian regular rings does not have as a weak (ordinary) model companion any class of the form $\mathcal{K}_1$ (from Theorem 20), where $T_1 \vdash \Phi_5$ (respectively $T_1 \vdash$ "prime").*

Proof. If such a $\mathcal{K}_1$ is a model companion of $\mathcal{K}_7$, then the theory T_1 is a model companion of the theory of fields, which is impossible.

Remark. The conditions $T \vdash \Phi_5$, $T_1 \vdash \Phi_5$ can also be weakened in Theorem 20 and in this corollary.

A primitive formula is *1-primitive* if it does not contain more than one equality. We shall say that *the class of formulae Σ is decidable by the theory T* if, for any formula φ in Σ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Theorem 21. a) *Suppose that the set of all 1-primitive propositions is decidable by the local theory T, and for all rings in the class $\mathcal{K} = \{K \mid \{K_p\} \models T\}$ the set $B(K)$ is infinite. Then the set of all E-propositions is decidable by $Th\mathcal{K}$.*

b) *If $\mathcal{K}^*$ is a Boolean absolute class and the set of all 1-primitive propositions is decidable by the (model complete) theory T^*, then $(Th\mathcal{K}^*) = (\Phi_1 + \Phi_2 + (T^*)')$ is a complete and model complete Horn theory.*

Proof. a) Suppose that $K \in \mathcal{K}$ and ψ is a primitive proposition. (The condition imposed on K guarantees, for example, that K^2 is atomless.) Then all ψ_l, $1 \leqslant l \leqslant L$, formed from ψ in the same way as in the proof of Theorem 18, part b, are 1-primitive propositions. By hypothesis, either $T \vdash \psi_l$ for all l or $T \vdash \neg\psi_{l_0}$ for some l_0. In the first case we have $\forall p\ (K_p \models \bigwedge_l \psi_l)$. We choose exactly L different points $p_1, \ldots, p_L$ in $X(K)$. By the Hausdorff property they have pairwise disjoint neighbourhoods $u_1, \ldots, u_L$. The proposition ψ_1 is fulfilled in p_1 for some $k_1(p_1), \ldots, k_n(p_1)$ and the equalities from ψ_1 are fulfilled in an open-and-closed neighbourhood u'_1 inside u_1. Therefore, $k_1, \ldots, k_n$ satisfy ψ_1 in u'_1. In the same way, $t_1, \ldots, t_n$ satisfy ψ_L in u'_L. We paste together all $k_1, \ldots, k_n$, and so on, and obtain $k_1, \ldots, k_n$ in $u'_1 \cup \ldots \cup u'_n$ satisfying ψ. We extend these $k_1, \ldots, k_n$ to the complement of $u'_1 \cup \ldots \cup u'_n$ in such a way that all the equalities in ψ are fulfilled. Thus, $K \models \psi$. In the second case we have $\forall p\ (K_p \models \neg\psi_{l_0})$, so (without using normality or the atomless property) we obtain $K \models \neg\psi$. We now easily find that the set of all E-formulae is decidable.

b) We at once obtain this assertion by part a and Theorem 18, parts a, b, taking into account that in a model complete theory any formula is equivalent to an E-formula.

Corollary. *Under the conditions of Theorem 21, part b, if the theory T^* is recursively axiomatizable, then $Th\mathcal{K}^*$ is decidable.*

Remark. This theorem also remains true for an extension of the language of rings by an arbitrary set of constants interpreted simultaneously in all rings K in $\mathcal{K}$.

II.5. The transfer of a local theory into a locally axiomatizable class. The theory T is "stronger" than the theory T', in other words, T is the theory of fibres and T' is the theory of global objects. Therefore, assertions of the following type are of some interest: if $T \vdash \psi$, then $T' \vdash \psi'$, where ψ' is obtained from ψ by some syntactical translation (in other words: if ψ holds in fibres, then ψ' holds in global objects).

We denote by $\varphi\neg$ the formula in disjunctive normal form obtained from $\neg\varphi$ by the classical transformation. We denote by $\varphi\neg\neg$ the formula obtained from φ in prenex form by adding $\neg\neg$ before each connective $\exists$.

***Proposition* 9.** *Let K be a normal ring.*

a) *The predicate* $[\![\varphi(k)]\!]_{\mathcal{T}(K)} = 0$ *is expressible in K, that is, there is a syntactical translation* $\varphi(x_1, \ldots, x_n) \mapsto \varphi^0(x_1, \ldots, x_n)$ *such that this predicate is equivalent to* $K \models \varphi^0(\bar{k})$ *(compare with Theorem* 12). *Here* φ *is in disjunctive normal form.*

b) *The predicate* $[\![\varphi\neg\neg(\bar{k})]\!]_{\mathcal{T}(K)} \geqslant e$, $e \in B(K)$, *is expressible in K in the same sense (we denote by* $\varphi(x_1, \ldots, x_n) \mapsto \varphi^+(x_1, \ldots, x_n, e)$ *the corresponding syntactical translation). Here,* φ *is in disjunctive normal form.*

Proof. a) We first describe the syntactical translation $\varphi \mapsto \varphi^0$. If φ is quantifier-free, that is, $\varphi = \bigvee_s \psi_s$, then

$$\varphi^0 = \forall e \exists e_0 \forall t_1 \exists t \bigwedge_s ([e^2 = e \wedge e\cdot t = t\cdot e \wedge e\cdot k_1 = e\cdot t_1 \wedge \ldots$$
$$\ldots \wedge (e_0^2 = e_0 \wedge e_0\cdot t_1 = t_1\cdot e_0 \wedge e_0\cdot k_2 = e_0\cdot t_2 \Rightarrow e_0 \leqslant 1 - e) \wedge \ldots] \Rightarrow e = 0),$$

where $k_1 = t_1$ is one of the equalities, and $k_2 \neq t_2$ one of the inequalities, in ψ_S. Next,

$$(\exists x\varphi)^0 = \forall x \varphi^0, \quad (\forall x\varphi)^0 = \forall e \exists t,k\, (e^2 = e \wedge e\cdot t = t\cdot e \wedge (e \leqslant [\![\varphi(k)]\!]_{\mathcal{T}(K)}]) \Rightarrow$$
$$\Rightarrow e = 0),$$

where the third factor in the premise is replaced by $\varphi'(k, \bar{k}, e)$ according to Theorem 12, part a.

It is clear that $[\![\bigvee_s \psi_s]\!]_{\mathcal{T}(K)} = 0$ is equivalent to $K \models \varphi^0$, and so forth by induction.

b) We describe the syntactical translation $\varphi \mapsto \varphi^+$. If $\varphi(\bar{k})$ is quantifier-free, then $\varphi^+(\bar{k}, e)$ coincides with the $\varphi'(\bar{k}, e)$ of Theorem 12, part a. Next, $(\forall x\varphi)^+ = \forall x\,(\varphi^+)$, and finally,

$$(\exists x\varphi)^+ = \forall e_0 \exists t \exists e_1 \forall t_1 \exists k\, [e_0^2 = e_0 \wedge e_0\cdot t = t\cdot e_0 \wedge e_0 \neq 0 \wedge e_0 \leqslant e \Rightarrow$$
$$\Rightarrow e_1^2 = e_1 \wedge e_1\cdot t_1 = t_1\cdot e_1 \wedge e_1 \neq 0 \wedge e_1 \leqslant e_0 \wedge \varphi^+(k, e_1)].$$

The proof is by induction.

***Lemma* 2.** *If K is a normal ring and* $[\![\varphi(\bar{k})]\!]_{B(K)} = 1$, *then* $[\![\varphi\neg(\bar{k})]\!]_{\mathcal{T}(K)} = 0$, *where* $\bar{k} \subseteq K$.

Proof. If $[\![\varphi\neg(\bar{k})]\!]_{\mathcal{T}(K)} \neq 0$, then by Theorem 10, part b, we obtain $[\![\varphi\neg(\bar{k})]\!]_{B(K)} \neq 0$, that is, $[\![\varphi\neg(k)]\!]_{B(K)} \neq 0$, a contradiction.

***Theorem* 22.** *If* $T \vdash \varphi$, *then* $(T' + \Phi_1) \vdash ((\varphi\neg)^0 \wedge \varphi^+)$.

Proof. We verify the first assertion. Suppose that in an arbitrary ring K we have $K \models (T' + \Phi_1)$. By Theorem 12, part a, we obtain $[\![T]\!]_{\mathcal{T}(K)} = 1$. Hence by Theorem 10, part b, we obtain $[\![T]\!]_{B(K)} = 1$. By hypothesis we obtain $[\![\varphi]\!]_{B(K)} = 1$, and by Lemma 2 above and Proposition 9, part a, we have $[\![\varphi\neg]\!]_{B(K)}=0$, so $K \models (\varphi\neg)^0$.

We verify the second assertion. Suppose that $K \models (T' + \Phi_1)$. Then $[\![T]\!]_{\mathcal{T}(K)} = 1$. We denote by $T\neg\neg$ the Gödel negative translation of any formula in T (the formulae in T and the formula φ are in disjunctive normal form). From the fact that K is normal we obtain $[\![T\neg\neg]\!]_{\mathcal{T}} = 1$. Since $T\neg\neg \vdash \varphi\neg\neg$ we have $[\![\varphi\neg\neg]\!]_{\mathcal{T}} = 1$, where the relevance of normality is the same as in Proposition 9b), and therefore $K \models \varphi^+$.

Remark. A similar theorem is also true for other pairs in place of $\langle \mathcal{T}, B\rangle$. The expressibility of all the above predicates also holds for other languages. The condition that K be normal can be weakened, for example, by replacing it by a condition of the kind $\{[\![k = 0]\!]_{\mathcal{T}(K)} | \ k \in K\} \subseteq B\ (K)$. The language of rings can be extended by any number of constants.

CHAPTER III

A NATURAL TRANSLATION OF CLASSICAL INTO INTUITIONISTIC THEORY FOR ALGEBRAS WITH METRIC

Here we consider applications of Heyting-valued analysis distinct from that in Ch. II. Namely, we consider the problem of passing from the classical validity of a certain statement φ to the intuitionistic validity of φ itself or of a new statement φ' close to φ in meaning and form. This passage allows us to employ the important merits of intuitionistic validity, such as effectiveness, disjunctivity, or extensionality (see the end of §I.5 and [10], [9], [4]). However, even the question of the precise definition of intuitionistic validity is not uniquely solved, and it is not simple when we deal with theories of abstract objects such as sets and algebras. One of the possible definitions was given at the end of §I.5. This is Heyting validity, denoted by $cHa \models (\cdot)$. We emphasize that all considerations on the metamathematical level can be studied within the framework of intuitionistic set theory *HZF*. In this chapter we shall obtain results of the following kind: if $ZFC \vdash (K \models (\varphi \Rightarrow \psi))$, then $cHa \models (K \models (\varphi \Rightarrow \psi))$, where φ and ψ can contain parameters $k_1, ..., k_n$ from K. Here φ are the conditions of a certain theorem, ψ is its conclusion, and K is the object to which the assertion $\varphi \Rightarrow \psi$ refers. Instead of $K \models (\varphi \Rightarrow \psi)$ we can obviously write the relativization $\varphi_K \Rightarrow \psi_K$. Since the question is, at least in the premise, one of deducibility, K should be described by means of a formula. A broad (and the customary) language of description is the language *ZF*, so suppose that K is described by a certain formula $\varkappa$ in ZF, and that the relativization is realized by a variable f satisfying $\varkappa(\cdot)$. Thus we arrive at the following form for the assertion: if $ZFC \vdash \forall f\,(\varkappa(f) \Rightarrow \forall k_1, \ldots, k_n \in f\,(\varphi_f \Rightarrow \psi_f))$, then $cHa \models \forall f\,(\varkappa(f) \Rightarrow \Rightarrow \forall k_1, \ldots, k_n \in f\,(\varphi'_f \Rightarrow \psi'_f))$, where φ' and ψ' are certain transforms of the formulae φ and ψ, respectively. An assertion of this form will be proved in Theorem 23, where we impose the condition on $\varkappa(\cdot)$ that it is a Dedekind formula (for the definition see below), and in connection with Novikov's conjecture ([5], p.127) ψ is any *AE*-formula and φ is an arbitrary formula. It is natural that φ and ψ are written in the language corresponding to the structure of K. In our case we assume that K is a ring and therefore φ and ψ are formulae in the language of rings.

Takeuti and Titani [10], [42] also develop what they call a global intuitionistic analysis. Namely, they prove the Heyting validity of many elementary theorems of mathematical analysis and also the Heyting validity of some theorems of the theory of functions of many complex variables (for example, Weierstrass's theorem). Establishing Heyting validity (which is the subject of study in Chs. III, IV and partially in previous chapters) involves considerable effort even in simple cases, because it is not closed with respect to classical deducibility. Hence, the usual mathematical assertions are, generally speaking, not all Heyting valid (a compensation is the effectiveness that arises once Heyting validity has been established).

An *algebra with metric* is a set

$$K, +, -, \cdot, 0, 1, \|\cdot\|, \quad \text{where} \quad +: K^2 \to K, \; -: K \to K, \; \cdot: K^2 \to \to K, \; 0, 1 \in K, \; \|\cdot\|: K \to \mathbb{R},$$

and $\mathbb{R}$ is described as the set of Dedekind cuts $\lambda = \langle \lambda, \lambda_1 \rangle$ in $\mathbb{Q}$. Here $\mathbb{Q}$ is described as usual in terms of the sequence of natural numbers which, in turn, is defined as the smallest inductive set ω. This definition of $\mathbb{R}$ can be written in the usual way in the language of set theory ZF. In the usual way we define the predicates connected with $\mathbb{R}$ (and other number systems). For example, the predicate $\cdot < \cdot$ on $\mathbb{R}$ is defined by the formula $(\lambda < \mu) \Leftrightarrow \exists r \in \mathbb{Q}\,(r \in \lambda_1 \wedge r \in \mu)$. The functional symbol $\|\cdot\|$ can be read as "metric". In what follows we assume that an algebra with metric is interpreted as a ring with absolute value. In this chapter (as well as in the previous one) the structure of rings is chosen as an example, and we can also consider general algebraic systems. The usual formulae in the language ZF describing the structure of the ordered rings $\mathbb{Q}$, $\mathbb{Q}_{>0}$, $\mathbb{R}$ and the structure of the ring $\mathbb{C}$ (as the set of pairs of Dedekind cuts) will be denoted by $\varkappa_1, \varkappa_2, \varkappa_3$, and $\varkappa_4$.

We denote by $\mathbb{B}$ the algebra $\mathbb{B}(\Omega)$ defined at the end of §I.2 (but with Ω in place of H). We denote by $[\![\cdot]\!]_\Omega$ and $[\![\cdot]\!]_B$ the valuations in the language ZF with sets of parameters V^Ω and V^B, respectively (see Example 1), where of course $V^\Omega \subseteq V^B$.

A formula $\varkappa(\cdot, \ldots, \cdot)$ in the language ZF is called a Dedekind formula if the following three conditions are satisfied:

1) $HZF \vdash \forall f, +, -, \cdot, 0, 1, \|\cdot\| (\varkappa(f, +, -, \cdot, 0, 1, \|\cdot\|) \Rightarrow (f, +, -, \cdot, 0, 1, \|\cdot\|)$ is an algebra with metric), where $\varkappa$ also includes the formula $\varkappa_3$ and $+, -, \cdot, 0, 1, \|\cdot\|$ are ordinary set-theoretic variables whose notation reminds us only of their meaning.

2) $\forall \Omega ([\![\varkappa(f, \ldots, \|\cdot\|)]\!]_\Omega \leqslant ([\![\varkappa(f, \ldots, \|\cdot\|)]\!]_B \wedge (f(k) \wedge f(t) \rightarrow [\![k = t \Leftrightarrow [\![\|k_B - t\| = 0]\!]_B)))$, where f is assumed to be an extensional element in V^Ω.

3) $\forall \Omega ([\![\varkappa(f, \ldots, \|\cdot\|)]\!]_\Omega \leqslant ((f(k) \rightarrow [\![\|k\|_\Omega = \|k\|_B]\!]_B) \wedge (f(k) \wedge f(t) \rightarrow \rightarrow [\![k \underset{\Omega}{+} t \underset{B}{+} t]\!]_B \wedge \ldots))$, where the dots on the right-hand side denote the same condition for the other operations. The subscripts Ω and $\mathbb{B}$ indicate that the terms are computed in V^Ω and V^B, respectively. Condition 1 is trivial and conditions 2 and 3 assert a kind of weak absoluteness of the formula $\varkappa$.

Example 13. 1) Suppose that $\varkappa(f, +, -, \cdot, 0, 1, \|\cdot\|)$ says: f is the set of Dedekind cuts in $\mathbb{Q}$ closed with respect to the usual ring operations in $\mathbb{R}$, together with these operations in f and the "norm" $\|x\| = |x| = \max\{x, -x\}$, that is, $\varkappa(f, \ldots, \|\cdot\|) \Leftrightarrow \forall x \in f (\varkappa_3(x) \wedge \forall x, y \in \in f (x + y, x \cdot y, 0, 1 \in f \wedge \ldots,$ where $\varkappa_3$ also contains the formula $\varkappa_1$. It follows from Proposition 10 below and Theorem 3, part e that this formula is a Dedekind one. It describes in V^Ω and in V^B two families consisting of the rings of all real numbers in $\mathbb{R}_\Omega$ and $\mathbb{R}_B$, respectively. Here $\mathbb{R}_\Omega = \{x \mid \varkappa_3(x)\}$ is the object defined in V^Ω by the formula $\varkappa_3$, and $\mathbb{R}_B$ is the corresponding object in V^B. Of course, it is usually $\mathbb{B}$-globally valid that $\mathbb{R}_\Omega \neq \mathbb{R}_B$. However, by Proposition 10, it is $\mathbb{B}$-globally valid that $[\![\mathbb{R}_\Omega \subseteq \mathbb{R}_B]\!]$. These objects can be explicitly defined, for example, by $\mathbb{R}_\Omega(x) = [\![\varkappa_3(x)]\!]_\Omega$, where $[\![\varkappa_3(x)]\!]_\Omega > 0$, and $\mathbb{R}_B(x) = [\![\varkappa_3(x)]\!]_B$, where $[\![\varkappa_3(x)]\!]_B > 0$.

2) Suppose that $\varkappa$ says: f is the set of pairs of Dedekind cuts in $\mathbb{Q}$ closed with respect to the usual ring operations in $\mathbb{C}$, together with these operations in it, and with the norm in f equal to $\|\langle\lambda, \mu\rangle\| = \lambda^2 + \mu^2$. Again, it follows from Proposition 10 that this formula is a Dedekind one. It describes in V^Ω and in V^B two families consisting of all number rings in $\mathbb{C}_\Omega$ and $\mathbb{C}_B$, respectively.

3) By similar Dedekind formulae we can describe the families of "number" rings in various hypercomplex systems over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

4) In all the cases we have mentioned we can add to $\varkappa$ the formula $\forall x \in f \exists y \in f (x = 0 \vee x \cdot y = 1)$ and so obtain the corresponding families of fields or division rings.

5) In all the cases we have mentioned we can add to $\varkappa$ the condition of real or algebraic closure: $\forall x \in f \exists y \in f (x > 0 \Rightarrow y^2 = x) \wedge \forall a_0, \ldots, a_{2n+1} \in f \exists x \in f (a_0 + \ldots + a_{2n+1} x^{2n+1} = 0)$ or $\forall a_0, \ldots, a_n \in \in f \exists x \in f (a_0 + \ldots + a_n x^n = 0)$. These axiom schemes can be expressed in an equivalent way by a single formula in the language ZF. In this way

practically all the usual classes of "number" rings and fields (and also groups and algebras) can be described by Dedekind formulae, which means that Theorem 23 (below) can be applied to them.

***Proposition* 10.** *We have the following relations*:

a) $1 = [\![\{x \mid \varkappa_1(x) = \check{\mathbb{Q}}]\!]_\Omega = [\![\{x \mid \varkappa_1(x)\} = \check{\mathbb{Q}}]\!]_B$, $1 = [\![\{x \mid \varkappa_2(x)\} = (\mathbb{Q}_{>0})^\vee]\!]_\Omega = [\![\{x \mid \varkappa_2(x)\} = (\mathbb{Q}_{>0})^\vee]\!]_B$, $[\![\varkappa_3(x)]\!]_\Omega \leqslant [\![\varkappa_3(x)]\!]_B$, $[\![\varkappa_4(x)]\!]_\Omega \leqslant [\![\varkappa_4(x)]\!]_B$;

b) $[\![\varkappa_3(x)]\!]_\Omega \leqslant [\![x = \langle\lambda, \lambda_1\rangle]\!]_\Omega$, *where* $\mathscr{D}(\lambda) = \mathscr{D}(\lambda_1) = \{\check{r} \mid r \in \mathbb{Q}\}$ *and* $\lambda(\check{r}) = [\![\exists u, v\ (x = \langle u, v\rangle \wedge \check{r} \in u)]\!]_\Omega \wedge [\![\varkappa_3(x)]\!]_\Omega$, $\lambda_1(\check{r}) = [\![\exists u, v\ (x = \langle u, v\rangle \wedge \check{r} \in v)]\!]_\Omega \wedge [\![\varkappa_3(x)]\!]_\Omega$;

c) $[\![\varkappa_3(x) \wedge \varkappa_3(y)]\!]_\Omega \leqslant ([\![x < y]\!]_\Omega \leftrightarrow [\![x < y]\!]_B)$;

d) $[\![\varkappa_3(x) \wedge \varkappa_3(y)]\!]_\Omega \leqslant [\![x \underset{\Omega}{+} y = x \underset{B}{+} y]\!]_B$ *and the same is true for* $\varkappa_4$ *and the other operations*;

e) $[\![\varkappa_3(x)]\!]_\Omega \leqslant [\![\,\|x\|_\Omega = \|x\|_B]\!]_B$ *and the same is true for* $\varkappa_4$ (*the norms for* $\mathbb{R}$ *and* $\mathbb{C}$ *are defined as above*).

Proof. a) The first relations are obvious. Using them we find that the formula $\varkappa_3$ has the form $\exists u, v\,[u, v \subseteq \check{\mathbb{Q}} \wedge x = \langle u, v\rangle \wedge \exists r, s \in \check{\mathbb{Q}}\,(r \in u \wedge s \in v) \wedge \forall r \in \check{\mathbb{Q}}\,\neg(r \in u \wedge r \in v) \wedge \forall r \in \check{\mathbb{Q}}\,(r \in u \Leftrightarrow \exists s \in u\,(r < s)) \wedge \forall r \in \check{\mathbb{Q}}\,(r \in v \Leftrightarrow \exists s \in v\,(s < r)) \wedge \forall r, s \in \check{\mathbb{Q}}\,(r < s \Rightarrow r \in u \vee s \in v)]$. Then

$$[\![\varkappa_3(x)]\!]_\Omega = \bigvee_{u,v}[[\![u, v \subseteq \check{\mathbb{Q}}]\!]_\Omega \wedge \cdots] = \bigvee_{u,v}[((\bigwedge_{y \in \mathscr{D}(u)}{}_\Omega\, u(y) \underset{\Omega}{\to} [\![y \in \check{\mathbb{Q}}]\!]_\Omega) \wedge [\![v \subseteq \check{\mathbb{Q}}]\!]_\Omega) \wedge$$

$$\wedge [\![x = \langle u, v\rangle]\!]_\Omega \wedge (\bigvee_{r,s \in Q}[\![\check{r} \in u]\!]_\Omega \wedge [\![\check{s} \in v]\!]_\Omega) \wedge$$

$$\wedge(\bigwedge_{r \in Q}([\![\check{r} \in u]\!]_\Omega \wedge [\![\check{r} \in v]\!]_\Omega \to 0)) \wedge (\bigwedge_{r \in Q}([\![\check{r} \in u]\!]_\Omega \leftrightarrow \bigvee_{s \in \mathscr{D}(u)} u(s) \wedge [\![\check{r} < s]\!]_\Omega)) \wedge$$

$$\wedge(\bigwedge_{r \in Q}([\![\check{r} \in v]\!]_\Omega \leftrightarrow \bigvee_{s \in \mathscr{D}(v)} v(s) \wedge [\![s < \check{r}]\!]_\Omega)) \wedge$$

$$\wedge(\bigwedge_{r,s \in Q}([\![\check{r} < \check{s}]\!]_\Omega \to [\![\check{r} \in u]\!]_\Omega \vee [\![\check{s} \in v]\!]_\Omega))].$$

To pass to the valuation in V^B in the fourth, fifth, and sixth factors it is sufficient to show that we can replace u, v by u', $v' \in V^\Omega$ such that $w \leqslant ([\![u = u' \wedge v = v']\!]_\Omega \wedge [\![u = u' \wedge v = v']\!]_B)$, where $w = [\![u, v \subseteq \check{\mathbb{Q}}]\!]_\Omega$ and $\mathscr{D}(u') = \mathscr{D}(v') = \mathbb{Q}$. These elements u' and v' are automatically extensional in V^Ω and in V^B. So we can replace u, v by extensional functions u, v including $\{\check{r} \mid r \in \mathbb{Q}\}$ in their domains (by Theorem 3b)), and we put $u' = u \restriction \{\check{r} \mid r \in \mathbb{Q}\}$ and the same for v'. Then we have $w \leqslant [\![\forall y\,(y \in u \Leftrightarrow y \in u')]\!]_\Omega$ since $w \leqslant (u(y) \to [\![y \in u']\!]_\Omega)$, where $y \in \mathscr{D}(u)$. Hence, we also get the same relation for $[\![\cdot]\!]_B$. Therefore, $w \wedge \bigwedge_r((u'(r) \wedge v'(r)) \to 0) \wedge \bigwedge_r(u'(r) \leftrightarrow \bigvee_{s \in Q} u'(s) \wedge [\![\check{r} < \check{s}]\!]_\Omega) \wedge \bigwedge_r(v'(r) \leftrightarrow \bigvee_{s \in Q} v'(s) \wedge [\![\check{s} < \check{r}]\!]_\Omega)$ carries over to $\mathbb{B}$.

In the same way $[\![\varkappa_4(x)]\!]_\Omega = \bigvee_{y,z}[\![\varkappa_3(y) \wedge \varkappa_3(z)]\!]_\Omega \wedge [\![x = \langle y, z\rangle]\!]_\Omega$ carries over to $\mathbb{B}$.

b) We note that λ and λ_1 are automatically extensional functions in V^{Ω} and in V^B. We employ u' and v' as above and so obtain the inequality

$$[\![\varkappa_3(x)]\!]_\Omega \wedge w \wedge [\![x=\langle u', v'\rangle]\!]_\Omega \wedge \ldots \leqslant ((u'(r) \leftrightarrow \lambda(\check{r})) \wedge v'(\check{r}) \leftrightarrow \lambda_1(\check{r})),$$
where $r \in \mathbb{Q}$.

Then we replace the right-hand side by $[\![u' = \lambda \wedge v' = \lambda_1]\!]_\Omega$, then by $[\![x = \langle\lambda, \lambda_1\rangle]\!]_\Omega$, and finally we sum over all u, v.

c) We define $\cdot<\cdot$ by $((x < y) \leftrightharpoons \exists r \in \check{\mathbb{Q}}\ (r \in P_2\ (x) \wedge r \in P_1\ (y))$, where $P_1(\cdot)$ and $P_2(\cdot)$ are the first and the second members of an ordered pair. We write $w \leftrightharpoons [\![\varkappa_3\ (x) \wedge \varkappa_3\ (y)]\!]_\Omega$. Using b) we obtain

$$w \leqslant [\![x = \langle\lambda, \lambda_1\rangle \wedge y = \langle\mu, \mu_1\rangle]\!]_\Omega.$$

Then we have $w \wedge [\![x < y]\!]_\Omega = w \wedge [\![\exists r \in \check{\mathbb{Q}}\ (r \in \lambda_1 \wedge r \in \mu)]\!]_\Omega =$
$= w \wedge \bigvee_r \lambda_1(\check{r}) \wedge \mu(\check{r}) = w \wedge [\![\exists r \in \check{\mathbb{Q}}\ (r \in \lambda_1, \wedge r \in \mu)]\!]_B = w \wedge [\![x < y]\!]_B.$

d) We define $x + y \leftrightharpoons \langle\{t \in \check{\mathbb{Q}} \mid \exists r \in P_1\ (x)\, \exists s \in P_1\ (y)\, (t < r + s)\}$, $\{t \in \mathbb{Q} \mid \exists r \in P_2\ (x)\ \exists s \in P_2\ (y) (t > r + s)\}\rangle$. Using b) we get $w \leftrightharpoons [\![\varkappa_3\ (x) \wedge \varkappa_3\ (y)]\!]_\Omega \leqslant [\![x = \langle\lambda, \lambda_1\rangle \wedge y = \langle\mu, \mu_1\rangle]\!]_\Omega$. It is easy to verify that $w \leqslant [\![\varkappa_3\ (x + y)]\!]_\Omega \leqslant [\![\varkappa_3\ (x + y)]\!]_B$, and also the global validity of the fact that + is a function. To complete the proof we have to verify the inequality

$$w \leqslant [\![\{t \in \check{\mathbb{Q}} \mid \exists r \in \lambda \exists s \in \mu(t < r + s)\}_\Omega = \{t \in \check{\mathbb{Q}} \mid \exists r \in \lambda \exists s \in \mu(t < r + s)\}_B]\!]_B \wedge \ldots,$$

where the first term can be replaced by the function

$$f(\check{t}) \leftrightharpoons [\![\exists r \in \lambda \exists s \in \mu\ (t < r + s)]\!]_\Omega$$

on $\mathbb{Q}$, in the sense that $[\![f = \{t \in \check{\mathbb{Q}} \mid \exists r \in \lambda \exists s \in \mu\ (t < r + s)\}_\Omega]\!]_\Omega = 1$. But in $\mathbb{B}$ this function is equal to the second term.

e) This follows directly from parts c and d.

Lemma 3. *Let φ be an arbitrary formula in the language of rings transformed classically into the form without $\Rightarrow$ and with $\neg$ only in atomic formulae, and suppose that φ^+ is obtained from φ by replacing all subformulae of the form $\neg\ (k = t)$ by $(0 < \|k - t\|^2)$. Then we have*

$$[\![\varkappa\ (f, \ldots, \|\cdot\|)]\!]_\Omega \wedge f\ (k_1) \wedge \ldots \wedge f\ (k_n) \leqslant ([\![\varphi_f^+\ (k_1, \ldots, k_n)]\!]_\Omega \rightarrow$$
$$\rightarrow [\![\varphi_f^+\ (k_1, \ldots, k_n)]\!]_B).$$

Here φ_f is the relativization of φ by a set-theoretic variable f (the parameters of φ are from $\mathscr{D}(f)$), and the terms and the predicate $\cdot < \cdot$ occurring in φ^+ are interpreted operationally by means of the formulae $\varkappa$ and $\varkappa_3$.

Proof. For $\cdot = \cdot$ we pass to $\mathbb{B}$ with the same terms (operationally computed in Ω) and using Proposition 10 we replace them by terms operationally computed in $\mathbb{B}$. For the connectives we easily apply induction; for example, for $\forall$ we obtain

$$[\![\varkappa]\!]_\Omega \wedge f(k_1) \wedge \cdots \wedge f(k_n) \wedge \bigwedge_{x\in\mathcal{D}(f)} (f(x) \to [\![\varphi f(\bar{k}, x)]\!]_\Omega) \leqslant$$
$$\leqslant \bigwedge_{x\in\mathcal{D}(f)} (f(x) \to [\![\varkappa]\!]_\Omega \wedge f(k_1) \wedge \cdots \wedge f(k_n) \wedge f(x) \wedge [\![\varphi_f(\bar{k}, x)]\!]_\Omega) \leqslant$$
$$\leqslant \bigwedge_{x\in\mathcal{D}(f)} (f(x) \to [\![\varphi_f(\bar{k}, x)]\!]_B) = [\![\forall x \in f\varphi_f(\bar{k}, x))]\!]_B. \square$$

***Lemma* 4.** *Let ψ be an E-formula in the language of rings transformed classically into the form without $\Rightarrow$ and with $\neg$ only in atomic formulae, and suppose that ψ^+ is obtained from ψ by replacing all subformulae of the form $\neg(k = t)$ exactly as in Lemma 3, and all subformulae of the form $k = t$ by the formula $\| k - t \|^2 < \check{\varepsilon}_{k,t}$, where the $\varepsilon_{k,t}$ are arbitrary elements from $\mathbb{Q}_{>0}$. Then we have*

$$[\![\varkappa(f, \ldots \|\cdot\|)]\!]_\Omega \wedge f(k_1) \wedge \cdots \wedge f(k_n) \leqslant ([\![\psi_f^+(k_1, \ldots, k_n)]\!]_B \leftrightarrow$$
$$\leftrightarrow [\![\psi_f^+(k_1, \ldots, k_n)]\!]_\Omega).$$

Proof. The atomic case reduces to the computation of the terms in B and Ω and the absoluteness of $\cdot < \cdot$ (Proposition 10, part c). For the connective $\exists$ the proof is the usual one. $\square$

If ψ is a *AE*-formulae, then we extend the definition of the translation ψ^+ given in Lemma 4: all quantifiers $\forall$ are rewritten without any change and we then bind all variables $\varepsilon_{k,t}$ by universal quantifiers (in the group of initial quantifiers $\forall$). The translation we have just defined for an arbitrary *AE*-formula ψ in the language of rings is called translation in the conclusion and that defined in Lemma 3 of arbitrary formulae φ in the language of rings is called translation in the premise. Now for φ and ψ we define the translation of the formula $\varphi \Rightarrow \psi$ as $\varphi^+ \Rightarrow \psi^+$.

***Theorem* 23.** *Let $\varkappa$ be a Dedekind formula in the language ZF, φ, ψ any formulae in the language of rings, and ψ an AE-formula. If*

$$ZFC \vdash \forall f, \ldots, \|\cdot\| (\varkappa(f, \ldots, \|\cdot\|) \Rightarrow$$
$$\Rightarrow \forall k_1, \ldots, k_n \in f [\varphi_f(k_1, \ldots, k_n) \Rightarrow \psi_f(k_1, \ldots, k_n)]),$$

then we have

$$cHa \models \forall f, \ldots, \|\cdot\| (\varkappa(f, \ldots, \|\cdot\|) \Rightarrow \forall k_1, \ldots, k_n \in$$
$$\in f [\varphi_f^+(k_1, \ldots, k_n) \Rightarrow \psi_f^+(k_1, \ldots, k_n)]).$$

Proof. Let

$f, \ldots, \|\cdot\| \in V^\Omega$, $k_1, \ldots, k_n \in \mathcal{D}(f)$, and

$$u \leftrightharpoons ([\![\varkappa(f, \ldots, \|\cdot\|)]\!]_\Omega \wedge f(k_1) \wedge \cdots \wedge f(k_n) \wedge [\![\varphi_f^+(k_1, \ldots, k_n)]\!]_\Omega.$$

By Lemma 3 we obtain $u \leqslant [\![\varphi_f^+(k_1, \ldots, k_n)]\!]_B$. Since $\varkappa$ is a Dedekind formula we have

$$u \leqslant [\![\varkappa(f, \ldots, \|\cdot\|)]\!]_B \wedge f(k_1) \wedge \cdots \wedge f(k_n) \wedge [\![\varphi_f(k_1, \ldots, k_n)]\!]_B.$$

By hypothesis we obtain $u \leqslant [\![\psi_f(\bar{k})]\!]_B$, and so $u \leqslant [\![\psi_f^+(\bar{k})]\!]_B$. We put

$$\psi_f^+(\bar{k}) = \forall t_1, \ldots, t_m \in f \forall \varepsilon_1, \ldots, \varepsilon_s \in \mathbb{Q}_{>0} \quad \exists l_1, \ldots, l_r \in f\chi(\bar{k}).$$

Then

$$\forall t_1, \ldots, t_m \in \mathscr{D}(f) \forall \varepsilon_1, \ldots, \varepsilon_s \in \mathbb{Q}_{>0} (u \leqslant f(t_1) \wedge \ldots \wedge f(t_m) \to \to [\![\exists l_1, \ldots, l_r \in f\chi(\bar{k}, \bar{t}, \bar{\varepsilon})]\!]_B),$$

that is,

$$u \wedge f(t_1) \wedge \ldots \wedge f(t_m) \leqslant [\![\exists \bar{l} \in f\chi(\bar{k}, \bar{t}, \bar{\varepsilon})]\!]_B.$$

By Lemma 4 we have

$$\wedge f(t_1) \wedge \ldots \wedge f(t_m) \leqslant [\![\exists \bar{l} \in f\chi(\bar{k}, \bar{t}, \check{\varepsilon})]\!]_\Omega,$$

that is,

$$u \leqslant [\![\forall \bar{t} \in f \forall \varepsilon \in \mathbb{Q}_{>0} \exists \bar{l} \in f\chi(\bar{k}, \bar{t}, \check{\varepsilon})]\!]_\Omega.$$

Remark. We denote by $cBa \models \varphi$ the predicate $\forall \mathrm{B} ([\![\varphi]\!]_B = 1)$, where B runs through all complete Boolean algebras and φ is a formula in the language *ZF* with parameters from V. In Theorem 23 the condition $ZFC \vdash (\cdot)$ can obviously be replaced by the condition $cBa \models (\cdot)$. In all that has been said above we can include multi-basis algebraic systems within the notion of algebras with metric, which is really useful in applications, and we can also replace $\mathbb{R}$ and $\mathbb{C}$ generated by the standard $\mathbb{Q}$ by other systems generated by any other standard sets described by formulae.

Example 14. We begin with the Lin–Seidenberg theorem: an injective polynomial map of the complex line $\mathbb{C}$ into the complex plane $\mathbb{C}^2$ has at most one critical point. This theorem can be written as an *AE*-proposition ψ in the language of rings:

$$\psi \rightleftharpoons \forall a_0, \ldots, a_n, b_0, \ldots, b_m \exists z_1, z_2 \forall u, v (z_1 = z_2 \vee f(z_1) \neq f(z_2) \vee g(z_1) \neq \neq g(z_2) \vee f'(u) \neq 0 \vee g'(u) \neq 0 \vee f'(v) \neq 0 \vee g'(v) \neq 0 \vee u = v),$$

where in the classically equivalent way we can rearrange the quantifiers $\exists z_1, z_2$ and $\forall u, v$. Here $f(x) = a_0 + \ldots + a_n x^n$, $g(x) = b_0 + \ldots + b_m x^m$, $\langle f, g\rangle \colon \mathbb{C} \to \mathbb{C}^2$, and $\langle f', g'\rangle$ is the gradient, whose vanishing implies a critical point. Thus, this theorem says: $\mathbb{C} \models \psi$. In accordance with Ch. II, the proposition ψ is valid in all algebraically closed fields K of characteristic 0 or even $p \geqslant p_0$, where p_0 is a fixed constant (this is just the Robinson–Tarski theorem), and also in a broad class of rings K in the sense that $K \models \psi$. For the sake of brevity we consider the case of fields: a certain Dedekind formula $\varkappa_0$ describes the above class of fields (if we restrict ourselves to subsets of $\mathbb{C}$). Therefore Theorem 23 reads:

$$cHa \models \forall f, \ldots, \|\cdot\| [\varkappa_0(f, \ldots, \|\cdot\|) \Rightarrow (\psi)_f^+].$$

Similarly, a quite arbitrary proposition of the form $\varphi \Rightarrow \psi$, where ψ is an *AE*-formula, that is valid in $\mathbb{R}$ or $\mathbb{C}$ (or $\mathbb{Q}$, or in the ring of locally constant functions on the Cantor set, and so on) is also valid by the corresponding completeness theorem, in a suitable class of rings described by a Dedekind formula. Therefore, by applying Theorem 23 we obtain its natural intuitionistic version. For example, this will hold for Artin's theorem. We consider this example in the case of two-basis algebra. It is well known that any splitting field K (of a polynomial h_1 over a field P) has the following property: if K contains one root of any polynomial h that is irreducible over P, then K contains all its roots. This description of the fields P, K is given by a Dedekind formula $\varkappa_1$. Let φ be a natural description of the fact that h is irreducible over P and has a root in K. Since φ contains neither $\Rightarrow$ nor $\neg$, we have $\varphi^+ = \varphi$. We put $\psi \leftrightharpoons \exists x_1, \ldots, x_m \in K$ (h splits over $x_1, \ldots, x_m$). Then $\psi^+ = \forall \varepsilon \in \mathbb{Q}_{>0} \exists x_1, \ldots, x_m (\ldots)$. By applying Theorem 23 we obtain $cHa \models \varphi_{P,K} \Rightarrow (\psi^+)_K$. Finally, in the same way, we can show the Heyting validity of Hilbert's Nullstellensatz. Writing these theorems in various ways and changing the form of the formula $\varkappa$, we obtain their various intuitionistic variants.

CHAPTER IV

HEYTING COMPLETION OF LOCALLY COMPACT TOPOLOGICAL SPACES

In this chapter we consider a further application of Heyting-valued analysis. Here by *global validity* (or *Ω-global validity*) we understand global validity with respect to the valuation in the language ZF with the set of parameters V^Ω (see Example 1). This valuation is denoted by $[\cdot]$ (or $[\cdot]_\Omega$).

Let Ω and Ω_1 be two complete Heyting algebras. We denote by Ω^{Ω_1} the set of all morphisms (in the sense of the cHa-structure $\langle \bigvee, \bigwedge, 0, 1\rangle$) of the algebra Ω_1 into the algebra Ω, and in Ω^{Ω_1} we fix the structure of a complete Heyting algebra with respect to the order $(f \leqslant g) \leftrightharpoons \forall u \in \Omega_1\ (f(u) \leqslant g(u))$. Next, we consider the case when Ω_1 is the topology of a fixed topological space or a ring Y, and in this case we shall write Y^Ω in place of $\Omega^{\mathcal{T}_Y}$. Here Y^Ω is equipped with the canonical structure of an algebra over the original ring of "scalars" Y. From 1977 onwards and in various forms the idea has been realized that the algebra Y^Ω in a "non-standard sense" (that is, in the sense of Ω-global validity) coincides with Y. In particular, important properties of the ring Y carry over to Y^Ω. What is more, homomorphisms into Y^Ω in this non-standard sense coincide with homomorphisms into Y, and important properties of Y-valued homomorphisms carry over to Y^Ω-valued ones. In this chapter and in the appendix we give some realizations of this idea. Namely, it is shown in Theorem 24 that Y^Ω in fact coincides with $\widetilde{Y}$, where $\widetilde{Y}$ is the completion in V^Ω of $\check{Y}$ as a metric or uniform space. Hence, many properties of Y carry over to $\check{Y}$ and then to $\widetilde{Y}$, and therefore to Y^Ω. In the appendix (Theorem 25) it is shown (in the case when Ω is a Boolean algebra) that: a) Y^Ω-valued functionals on a Banach space coincide with Y-valued functionals on it, b) Banach algebras over the ring Y^{Ω} coincide with Banach algebras over the ring Y, in particular their spectra correspond to each other, and c) continuous families of commutative locally compact groups can be identified in a non-standard sense with a single such group, and in particular their groups of characters correspond to each other. Of course, these heuristic considerations are not precise formulations.

In what follows $X = X(\Omega)$ is the Stone space of the algebra Ω. A set $\mathcal{O} \in \mathcal{T}(X)$ is called *Ω-dense* if there is a family $\{u_i\} \subseteq \Omega$ such that $\mathcal{O} = \bigcup_i u_i$ and $\bigvee_{\Omega i} u_i = 1$. Of course, an Ω-dense set is dense (in the topology $\mathcal{T}(X)$) and if, for any element of $\mathcal{T}(X)$, dense implies Ω-dense, then Ω is a Boolean algebra. In the same way we define the sets $\mathcal{O}$ that are Ω-dense at u, where u is any element of Ω (of course $\mathcal{O} \in \mathcal{T}(X)$ and $\mathcal{O} \subseteq u$). The topology $\mathcal{T}(Y)$ on Y is always denoted by $\mathcal{T}$ and its elements by α, β, γ. We recall that u runs through Ω and $\mathcal{O}$ runs through $\mathcal{T}(X)$. We denote by $C_{\Omega}(X, Y)$ the set of all continuous Y-valued functions defined on open sets $\mathcal{O}$, where each $\mathcal{O}$ is dense in an element u of Ω, factorized by being equal on an open and Ω-dense set. We denote by $C_{\Omega, u}(X, Y)$ (or briefly $C_u(X, Y)$) the set of functions in $C_{\Omega}(X, Y)$ defined on sets that are Ω-dense in u. The most important is the space $C_1(X, Y)$. In what follows we assume that the reader is familiar with the notions of the theory of uniform spaces, for example, in Bourbaki, General Topology (1968), Ch. II. Suppose that $[\![\langle \check{Y}, \check{\Sigma}, \check{\mathcal{T}}\rangle$ is a uniform space with basis of neighbourhoods $\check{\Sigma}$ and basis of topology $\check{\mathcal{T}}]\!]_{\Omega} = 1$. Let $\tilde{\check{Y}}$ be the object in V^{Ω} that is (in the sense of global validity) the set of all minimal Cauchy filters in $\check{Y}$. However, it is more convenient to define $\tilde{\check{Y}}$ as the set of all bases of minimal Cauchy filters in the uniform space $\check{Y}$, contained in $\check{\mathcal{T}}$.

Example 15. Let Ω be the topology of some fixed topological space, and let $Y = \mathbb{Q}$ or $Y = \mathbb{R}$. It is easy to see ([1], [10], [42]) that the algebra $(\check{\mathbb{Q}})^{\Lambda\Omega}$ (see the definition in §I.6) is isomorphic to the algebra of locally constant continuous functions of the form $Z \to \mathbb{Q}$, and the same is true for $(\check{\mathbb{R}})^{\Lambda\Omega}$. We have: $[\![\check{\mathbb{R}}$ is the Cauchy completion of the metric space $\check{\mathbb{Q}}]\!]_{\Omega} = 1$. Therefore, we can regard $\tilde{\mathbb{R}}$ as the set of real numbers in V^{Ω} given by the Cauchy definition (namely, by means of sequences), and consequently the Cauchy real (non-standard) numbers in V^{Ω} can be identified with the simple part of the algebra $C(Z, \mathbb{R})$ of all continuous functions. We denote by $\mathbb{R}^d$ the object in V^{Ω} that satisfies the natural definition of the Dedekind completion of the ordered set $\check{\mathbb{Q}}$ (see the proof of Proposition 10, part a. Then $(\mathbb{R}^d)^{\Lambda\Omega}$ can be identified with the whole algebra $C(Y, \mathbb{R})$. The isomorphism has the form $\lambda \mapsto f$ where $[\![\lambda = \langle \lambda, \lambda_1\rangle \in \mathbb{R}^d]\!]_{\Omega} = 1$ and $f(x) \leftrightharpoons \langle U, L\rangle$ is a cut in $\mathbb{Q}$ ($\langle\lambda, \lambda_1\rangle$ clearly denotes a cut in $\check{\mathbb{Q}}$), so that $r \in U \leftrightharpoons x \in [\![\check{r} \in \lambda]\!]_\Omega$ and $r \in L \leftrightharpoons x \in [\![r \in \lambda_1]\!]_\Omega$. Hence, in particular, $(\check{\mathbb{R}})^\wedge$ can be embedded in $(\mathbb{R}^d)^\wedge$ in a canonical way, and usually $\check{\mathbb{R}} \neq \mathbb{R}^d$. There is yet another natural way of completing $\check{\mathbb{Q}}$ which is classically equivalent to the completion via Cauchy sequences. We regard $\check{\mathbb{Q}}$ as a uniform space. We denote by $\mathbb{R}^f$ the object in V^Ω that is the set of all minimal Cauchy filters in the uniform space $\check{\mathbb{Q}}$. A minimal Cauchy filter can be defined by a simple set of intervals in $\mathbb{Q}$, and therefore it is a "constructive" object. Theorem 24 in particular says that $(\mathbb{R}^f)^{\wedge\Omega}$ is isomorphic to $C_1(X, Y)$, where $\mathbb{R}^f$ is just $\widetilde{\check{\mathbb{Q}}}$. The Stone space X contains the absolute $\dot{Z}$ of the space Z, and so there is a continuous surjective map $\varkappa\colon \dot{Z} \to Z$. Hence, $C(Z, \mathbb{R})$ can be embedded in $C_1(X, Y)$ in a canonical way by the map $f \mapsto f \circ \varkappa$, which explains the mutual relations of $\check{\mathbb{R}}, \mathbb{R}^d$, and $\mathbb{R}^f$ in V^Ω.

We now present another description of this object based on sheaves and Theorem 16, part d. We call a *u-morphism* a map $f\colon \Omega_1 \to \Omega$ satisfying the usual conditions of a morphism but with the condition $f(1_{\Omega_1}) = 1_\Omega$ replaced by $f(1_{\Omega_1}) = u$, where u is a fixed element of Ω. We denote the set of all u-morphisms by $\Omega_u^{\Omega_1}$. We form the following presheaf $\mathcal{F}(\cdot)$ on Ω: $\mathcal{F}(u) \leftrightharpoons \Omega_u^{\Omega_1}$ $\forall u \in \Omega$ and $\rho_v^u(f) \leftrightharpoons f(\cdot) \wedge v$. This is a sheaf on Ω. By Theorem 16, part d we form the object $\mathcal{F}'$ such that $\Omega^{\Omega_1} = (\mathcal{F}')^{\wedge\Omega}$. If as above $\Omega_1 \leftrightharpoons \mathcal{T}(Y)$, then the object $\mathcal{F}'$ will be denoted by $\mathcal{F}'_Y$. Theorem 24 says that $\mathcal{F}'_Y = \tilde{Y}$. We write $\mathcal{F}_Y \leftrightharpoons \bigcup \{\mathcal{F}(u) \mid u \in \Omega\}$. In what follows, p, q, r always run through the set $\mathcal{F}_Y$.

Thus, we have introduced external objects $\mathcal{F}_Y$ and $C_\Omega(X, Y)$, Y^Ω and $C_1(X, Y)$ (from the class V), and internal objects $\mathcal{F}'_Y$ and $\tilde{Y}$ (from the class V^Ω). We intend to establish that they are in fact the same: $\mathcal{F}_Y \cong C_\Omega(X, Y)$, $Y^\Omega \cong C_1(X, Y)$, $\mathcal{F}(1) = Y^\Omega$ and $\mathcal{F}'_Y = \tilde{Y}$, $Y^\Omega \cong (\mathcal{F}'_Y)^{\wedge\Omega}$. We make some details more precise before proving these assertions.

Let $\Omega_1 \leftrightharpoons \mathcal{T}(Y)$, where Y is a topological space, and in Y we choose the uniformization of the topology $\mathcal{T}$ of Y by means of a certain basis Σ of open symmetric neighbourhoods in Y.

The space $\mathcal{F}_Y$ (and also $\Omega^{\mathcal{T}}$) defined above will be called the Ω-completion of the topological space Y.

Remark. It can be shown [44] that the object $\mathcal{F}'_Y$ has the following simpler description (compared to that in Theorem 16, part d): $\mathcal{F}'_Y\colon \mathcal{F}_Y \to \Omega$, where for any $p \in \Omega_u^{\mathcal{T}}$ we put $\mathcal{F}'_Y(p) \leftrightharpoons u$. Here $p \in V^\Omega$, since $\mathcal{T}$, on which p is defined, is identified with $\{\check{\alpha} \mid \alpha \in \mathcal{T}\} \subseteq V^\Omega$.

We recall that a Hausdorff space Y is called uniformly locally compact if there is a neighbourhood σ_0 such that $\forall y \in Y$ ($\overline{\sigma_0(y)}$ is compact), where the bar denotes closure, as usual. A uniformly locally compact space is always complete and paracompact. We recall that the topology on Y is defined by the basis of neighbourhoods Σ by taking the basis of the topology to be $\{\sigma(y) \mid y \in Y, \sigma \in \Sigma\}$. In the case when Ω is a Boolean algebra Theorem 24 is in fact close to Theorems 1 and 4 of [11], and is in part contained in the note [14].

In the sets $\mathcal{F}_Y$ and $C_\Omega(X, Y)$ we define the valuations

$$[\![p = q]\!]_1 = \bigwedge \{p(\alpha) \leftrightarrow q(\alpha) \mid \alpha \in \mathcal{T}(Y)\} \quad \text{and}$$
$$[\![f = g]\!]_2 = j(\{x \in O_f \cap O_g \mid f(x) = g(x)\}^\circ),$$

where O_f and O_g are the domains of the corresponding functions. Here the symbol $^\circ$ as usual denotes the interior and $j(O)$ is the smallest open-and-compact set containing O, that is, $j(O) = \bigvee_{i} {}_\Omega u_i$, where $O = \bigcup_i u_i$ (this j is also used implicitly in the definition of Ω-dense sets). Thus we have two valuations: $\langle \mathcal{F}_Y, [\![\cdot = \cdot]\!]_1\rangle$ and $\langle C_\Omega(X, Y), [\![\cdot = \cdot]\!]_2\rangle$.

If we have certain operations defined in Y, then we extend them to $\mathcal{F}_Y$ and $C_\Omega(X, Y)$ in the following canonical way:

$$(p + q)(\alpha) = \bigvee \{p(\beta) \wedge q(\gamma) \mid \beta + \gamma \subseteq \alpha\}$$

and $(f+g)(x) = f(x)+g(x)$, where $x \in O_f \cap O_g$, and similarly for the operations $-$ and $\cdot$. If some K acts on Y, then we define the action of $\mathcal{F}_K$ on $\mathcal{F}_Y$ in a similar way, namely:

$$(k \cdot p)(\alpha) = \bigvee \{k(\beta) \wedge q(\gamma) \mid \beta \cdot \gamma \subseteq \alpha\}.$$

Moreover, Y can be canonically embedded in $\Omega^{\mathcal{T}(Y)}$ and $C_\Omega(X, Y)$ by the formulae $y \mapsto \check{y}$, where $\check{y}(\alpha) = \begin{cases} 1 & \text{for } y \in \alpha \\ 0 & \text{for } y \notin \alpha \end{cases}$ and $y \mapsto f_y$, where $f_y(x) \equiv y$.

In particular, if Y has distinguished elements 0 and 1, then we obtain $\check{0}$ and $\check{1}$. We usually identify y and $\check{y}$. The fact that these operations are well defined and extend the corresponding operations in Y is verified in Theorem 24, part a.

As always, the terms appearing in the valuations $[\![\cdot]\!]_1$ and $[\![\cdot]\!]_2$ are understood operationally. For the valuation $[\![\cdot]\!]_{\Omega, \mathcal{F}'_Y}$ defined in accordance with Example 8, the operations can be defined in two ways. The first is that $+(p, q, r) \rightleftharpoons p(Y) \wedge q(Y) \wedge r(Y) \wedge [\![p + q = r]\!]_\Omega$, that is, $+$ is understood as a non-standard graph. The second can be applied in the case when $+$ is uniformly continuous on Y, and then $+$ is extended from $\check{Y}$ to $\widetilde{\check{Y}}$ by continuity in accordance with Theorem 24, part b, and this extension is taken as the definition of the operation on $\widetilde{Y}$. Both cases give the same result.

***Theorem* 24.** *Let Y be an arbitrary locally compact topological space (with a basis Σ of symmetric open neighbourhoods).*

a) *The valuations* $\langle \mathscr{F}_Y, [\![\cdot]\!]_1\rangle$ *and* $\langle C_\Omega(X, Y), [\![\cdot]\!]_2\rangle$ *are equal and the corresponding bijection* $\cong$ *is stratified, that is,* $\Omega_u^{\mathscr{T}} \cong C_u(X, Y)$. *If Y is a topological ring (topological group), then the valuations remain equal, that is, the bijection* $\cong$ *also preserves the operations in* $\mathscr{F}_Y$ *and* $C_\Omega(X, Y)$. *Here* $\Omega_u^{\mathscr{T}}$ *and* $C_\Omega(X, Y)$ *are algebraic systems of the same kind as Y for any* $u \in \Omega$ *and Y is embedded in them.*

b) *We have* $[\![\mathscr{F}'_Y = \check{Y}]\!]_\Omega = 1$. *The valuations* $[\![\cdot]\!]_1$ *and* $[\![\cdot]\!]_{\Omega, \mathscr{F}'_Y}$ *are equal, and we also have* $Y^\Omega \cong (\mathscr{F}'_Y)^{\wedge\Omega}$.

c) *If Y is a topological ring (topological group), then it is globally valid that "the object* $\dot{\mathscr{F}}_Y$ *is an algebraic system of the same kind as Y". If Y is a conditionally complete lattice (module, with abstract norm, with norm, Banach space), then the same is globally valid for* $\mathscr{F}'_Y$.

Proof. We establish the bijection ψ between $\mathscr{F}_Y$ and $C_\Omega(X, Y)$ which plays the main role in this theorem. We recall that p, q, r run through the set $\mathscr{F}_Y$. Let $p \in \Omega_u^{\mathscr{T}}$ and $x \in u$. We put $\psi(p) = f_p$, where $f_p(x) \leftrightharpoons \lim p^{-1}(\mathscr{T}_x)$ and $\mathscr{T}_x$ is the family of all open-and-compact subsets of X containing a fixed point $x \in X$ (that is, $\mathscr{T}_x$ is the basis of the topology on X at the point x). Such a $p^{-1}(\mathscr{T}_x)$ is the basis of a filter of sets in Y since p is a morphism. We choose an arbitrary open covering of the whole of Y by relatively compact sets $\{\alpha_y \mid y \in Y\}$. Since p is a morphism, the set

$$S_p \leftrightharpoons \bigcup \{p(\alpha_y) \mid y \in Y\}$$

is Ω-dense in $u = p(Y)$. Now let $x \in S_p$. Then $\exists y\, (\alpha_y \in p^{-1}(\mathscr{T}_x))$. It is easy to see that this last condition means that for any $\sigma \in \Sigma$ the basis $p^{-1}(\mathscr{T}_x)$ contains a set of order σ^2. Therefore, $p^{-1}(\mathscr{T}_x)$ is the basis of a Cauchy filter. Since Y is complete, $\lim p^{-1}(\mathscr{T}_x)$ exists. Thus, f_p is defined on the whole of S_p. The set S_p does not depend on the choice of the covering $\{\alpha_y \mid y \in Y\}$: suppose that $\{\beta_y\}$ is another. Then $\bigcup_y p(\alpha_y) = \bigcup_y p(\beta_y)$, since $\bar{\alpha}_y \subseteq \beta_{y_1} \cup \ldots \cup \beta_{y_n}$ and $p(\alpha_y) \leqslant p(\beta_{y_1}) \cup \ldots \ldots \cup p(\beta_{y_n}) \leqslant \bigcup_y p(\beta_y)$. We show that the function $f_p : S_p \to Y$ is continuous. Let $x_0 \in S_p$, $y_0 = f_p(x_0)$, and $\sigma_1 \in \Sigma$. We consider the neighbourhood $\sigma_1(y_0)$. We choose a σ such that $\sigma^4 \subseteq \sigma_1$ and $\sigma(y)$ is relatively compact for all $y \in Y$, which is possible by the condition imposed on Y. Then $\{\sigma(y) \mid y \in Y\}$ is a covering of Y of the same sort as above, and therefore $S_p = \bigcup_y p(\sigma(y))$. For certain y we have $x_0 \in p(\sigma(y))$. This $p(\sigma(y))$ is the required neighbourhood of x_0. If $x_1 \in p(\sigma(y)) \cap S_p$, then $y_1 \leftrightharpoons f_p(x_1) \in \overline{\sigma(y)}$ and $y_0 \in \overline{\sigma(y)}$. Therefore, $y_1, y_0 \in \sigma^2(y)$ and $y_1 \in \sigma^4(y_0)$.

We also denote by f_p the equivalence class in $C_\Omega(X, Y)$ represented by f_p. Thus $\psi\colon \Omega_u^{\mathscr{T}} \to C_u(X, Y)$.

We show that the map ψ is injective. Suppose that $p \neq q$ and let $u \rightleftharpoons p(Y) = q(Y)$, that is, there is an $\alpha \in \mathcal{T}$ such that $p(\alpha) \nleqslant q(\alpha)$ (otherwise we interchange p and q). Also suppose that f_p coincides with f_q on a set O which is Ω-dense in u. If, for all β such that $\overline{\beta} \subseteq \alpha$, we have $p(\beta) \cap S_p \cap S_q \cap O \subseteq q(\alpha)$, then $p(\beta) \subseteq q(\alpha)$. For we can easily prove that $j(O_1 \cap O_2) = j(O_1) \cap j(O_2)$, and since $\bigcup \{\beta \mid \overline{\beta} \subseteq \alpha\} = \alpha$ and p is a morphism we have $\bigvee_\beta p(\beta) = p(\bigcup \beta) = p(\alpha)$ and $p(\alpha) \leqslant q(\alpha)$, a contradiction. Therefore, we can find β such that $\overline{\beta} \subseteq \alpha$, $x \in (p(\beta) \cap S_p \cap S_q \cap O)$ and $x \notin q(\alpha)$, that is, $x \in p(\beta) \backslash q(\alpha)$, $x \in (S_p \cap S_q \cap O) = O$. Hence $f_p(x) \in \overline{\beta} \subseteq \alpha$ and $f_q(x) \in \alpha$ by assumption. By the definition of limit and taking into account the fact that q is monotone, we obtain $x \in q(\alpha)$, which gives a contradiction. We show that ψ is surjective. Let $f\colon O \to Y$ be a continuous map, where the set O is open and Ω-dense in u. We put $p(\alpha) = j(f^{-1}(\alpha)) \in \Omega$, $p\colon \mathcal{T} \to \Omega$. This p is a morphism, since j is a cHa-morphism of the form $\mathcal{T}(X) \to \Omega$, that is, $p \in \Omega_u^{\mathcal{T}}$. We form f_p and S_p. If $x \in S_p \cap O$, then $f(x) = f_p(x)$, since $f(x) = \lim p^{-1}(\mathcal{T}_x)$. For if $f(x) \in \alpha$, then $x \in p(\alpha)$, $\alpha \in p^{-1}(\mathcal{T}_x)$. Thus f and f_p coincide on an Ω-dense set $S_p \cap O$, that is, $\psi(p) = f$. We note a useful formula $p(\alpha) \equiv j(f_p^{-1}(\alpha))$, where here and in what follows $f_p = \psi(p)$. Thus $\psi\colon \Omega_u^{\mathcal{T}} \leftrightarrow C_u(X, Y)$ for all $u \in \Omega$. The bijection ψ preserves the valuations $[\![\cdot]\!]_1$ and $[\![\cdot]\!]_2$, that is, $\bigwedge_\alpha (p(\alpha) \leftrightarrow q(\alpha)) = j(\{x \mid f_p(x) = f_q(x)\}^\circ$, which by definition also implies the equality of these two valuations. For if $f_p(x) = f_q(x)$ and $x \in p(\beta)$, where $\overline{\beta} \subseteq \alpha$, then $f_p(x) \in \overline{\beta}$, $f_q(x) \in \alpha$, $x \in q(\alpha)$, that is, $\{x \mid f_p(x) = f_q(x)\}^\circ \wedge p(\beta) \subseteq q(\alpha)$, $[\![f_p = f_q]\!]_2 \wedge p(\beta) \leqslant q(\alpha)$, $[\![f_p = f_q]\!]_2 \leqslant (p(\alpha) \leftrightarrow q(\alpha))$. Conversely, if $f_p(x) \neq f_q(x)$ and $x \in \bigcap_\alpha (p(\alpha) \leftrightarrow q(\alpha))$, then there exist $\alpha, \beta \in \mathcal{T}$ such that $\alpha \cap \beta = \varnothing$, $f_p(x) \in \alpha$, $f_q(x) \in \beta$. Then

$$x \in p(\alpha) \wedge q(\beta) \wedge (q(\beta) \to p(\beta)) \leqslant p(\alpha) \wedge p(\beta) = \varnothing,$$

a contradiction. We obtain

$$[\![p = q]\!]_1 \leqslant \bigcap_\alpha (p(\alpha) \leftrightarrow q(\alpha)) \subseteq \{x \mid f_p(x) = f_q(x)\},$$

which gives the required equality.

Thus the map ψ is the required bijection between $\mathcal{F}_Y$ and $C_\Omega(X, Y)$.

Let Y be an algebraic system as above (or any other with positive axioms). The sum of the functions p and q (as well as f_p and f_q) is computed by first passing to $p(\cdot) \wedge u$ and $q(\cdot) \wedge u$, where $u = p(Y) \wedge q(Y)$ (respectively, to $f_p \restriction O$ and $f_q \restriction O$, where $O = \mathcal{D}(f_p) \cap \mathcal{D}(f_q)$), and then by addition. It is clear that O is Ω-dense in u. Therefore, it is sufficient to verify that the operations are well-defined and ψ is an isomorphism for $p, q \in \Omega_u$. We take such p and q. We show that the sum $p+q$ belongs to $\Omega_u^{\mathcal{T}}$ (and the same will hold for all operations). For we have $(p+q)(\varnothing) = 0$, $(p+q)(Y) = p(Y) \wedge q(Y) = u$, $(p+q)(\alpha_1 \cap \alpha_2) = \bigvee \{p(\beta) \wedge p(\gamma)\}$, where

$p(\beta) \wedge q(\gamma) \leqslant (p+q)(\alpha_1) \wedge (p+q)(\alpha_2)$. Conversely,

$(p+q)(\alpha_1) \wedge (p+q)(\alpha_2) = \bigvee \{(p(\beta_1) \wedge q(\gamma_1) \wedge p(\beta_2) \wedge q(\gamma_2) \mid \beta_1 + \gamma_1 \subseteq$
$\subseteq \alpha_1, \beta_2 + \gamma_2 \subseteq \alpha_2\} \leqslant \bigvee \{p(\beta_1 \cap \beta_2) \wedge q(\gamma_1 \cap \gamma_2) \mid \beta_1 \cap \beta_2 + \gamma_1 \cap \gamma_2 \subseteq$
$\subseteq \alpha_1 \cap \alpha_2\} \leqslant (p+q)(\alpha_1 \cap \alpha_2)$.

From this we find that any function $p+q$ is monotone, and in particular $[\bigvee_i (p+q)(\alpha_i) \leqslant (p+q)(\bigcup_i \alpha_i)]$. If we can verify the reverse inequality we shall have proved that $(p+q)$ is a u-morphism. To this end (and mainly for the proof of the next part of the theorem) we need the following lemma.

***Lemma* 5.** *Let Y be a uniform locally compact space and suppose that p: $\mathcal{T} \to \Omega$ is a function such that $p(\emptyset) = 0$, $p(Y) = u$ (where $u \in \Omega$), and $p(\alpha \cap \beta) = p(\alpha) \wedge p(\beta)$. Then the following three properties of p are equivalent. The first: $p(\bigcup_i \alpha_i) = \bigvee_i p(\alpha_i)$ ("full additivity"); the second: $p(\alpha) = \bigvee \{p(\beta) \mid \overline{\beta} \subseteq \alpha, \ \overline{\beta}$ is compact$\}$ ("compact regularity"); the third: $p(\alpha) = \bigvee \{p(\beta) \mid \exists \sigma \in \Sigma \ (\sigma(\beta) \subseteq \alpha)\}$ and $\forall \sigma \in \Sigma \ (\bigvee \{p(\alpha) \mid \alpha^2 \subseteq \subseteq \sigma\} = p(Y))$ ("uniform regularity"). The property of uniform regularity does not depend on the choice of the basis of neighbourhoods Σ.*

Proof. a) We first verify the second assertion of the lemma:

$$\bigvee \{p(\beta) \mid \exists \sigma \in \Sigma \ (\sigma(\beta) \subseteq \alpha)\} = \bigvee \{p(\beta) \mid \exists \sigma_1 \in \Sigma \ (\sigma_1(\beta) \subseteq \alpha)\}$$

which follows from the definition of the basis of the filter. In the same way we verify the second property. We turn to the first assertion and we number the three properties consecutively by 1, 2, and 3. Obviously, $1 \Rightarrow 2$ and it is easy to see that $2 \Rightarrow 3$. The implication $3 \Rightarrow 1$ is non-trivial. We choose a new basis of neighbourhoods $\Sigma_1 = \{\sigma \cap \sigma_0 \mid \sigma \in \Sigma\}$, where σ_0 is such that $\overline{\sigma_0(y)}$ is compact for all $y \in Y$. By what has already been proved, it also has the property of uniform regularity. In what follows the variables of the form σ, σ_0, ..., run through Σ_1. Suppose that $\sigma(\beta) \subseteq \alpha$. We choose σ_1 such that $\sigma_1^2(\beta) \subseteq \alpha$. Since

$$p(Y) = \bigvee \{p(\alpha) \mid \alpha^2 \subseteq \sigma_1\} \leqslant \bigvee \{p(\sigma_1(y)) \mid y \in Y\},$$

we have

$$(\beta) \leqslant \bigvee \{p(\beta) \wedge p(\sigma_1(y)) \mid y \in Y\} \leqslant \bigvee \{p(\sigma_1(y)) \mid y \in \sigma_1(\beta)\}.$$

Suppose that $y \in \sigma_1(\beta)$. Then $\sigma_1^2(y) \subseteq \alpha$, $\overline{\sigma_1(y)} \subseteq \alpha$, and $\overline{\sigma_1(y)}$ is compact. Therefore, $p(\beta) \leqslant \bigvee \{p(\sigma_1(y)) \mid \overline{\sigma_1(y)} \subseteq \alpha, \ \overline{\sigma_1(y)}$ is compact$\} \leqslant$
$\leqslant \bigvee \{p(\gamma) \mid \overline{\gamma} \subseteq \alpha, \ \overline{\gamma}$ is compact$\}$, $p(\alpha) \leqslant \bigvee \{p(\gamma) \mid \overline{\gamma} \subseteq \alpha, \overline{\gamma}$ is compact$\}$. We have thus proved that $3 \Rightarrow 2$. Suppose that $\overline{\beta} \subseteq \alpha_1 \cup \alpha_2$ and $\overline{\beta}$ is compact. We show that $p(\beta) \leqslant p(\alpha_1) \vee p(\alpha_2)$ and hence by compact regularity, which we have already proved, we find that the function p is

finitely additive. For any $y \in \overline{\beta}$ we can find σ_y such that $(\sigma_y^2(y) \subseteq \alpha_1) \vee \vee (\sigma_y^2(y) \subseteq \alpha_2)$. We obtain a covering of $\overline{\beta}$ by such σ_y's, choose a finite subcover $\sigma_{y_1}(y_1), \ldots, \sigma_{y_n}(y_n)$, and put $\sigma = \sigma_{y_1} \cap \ldots \cap \sigma_{y_n}$. We obtain $\forall y \in \beta\ (\sigma(y) \subseteq \alpha_1 \vee \sigma(y) \subseteq \alpha_2)$, since any y is contained in say $\sigma_{y_1}(y_1)$, which is contained in α_1 or α_2, and if, for example, $\sigma_{y_1}(y_1) \subseteq \alpha_1$, then $\sigma(y) \subseteq \alpha_1$. We put $\beta_1 \leftrightharpoons \{y \in \beta \mid \sigma(y) \subseteq \alpha_1\}$ and $\beta_2 \leftrightharpoons \{y \in \beta \mid \sigma(y) \subseteq \alpha_2\}$. Then $\beta = \beta_1 \cup \beta_2$ and $\sigma(\beta_1) \subseteq \alpha_1$, $\sigma(\beta_2) \subseteq \alpha_2$. We choose σ_1 such that $\sigma_1^2 \subseteq \sigma$ and obtain $\sigma_1^2(\beta_1) \subseteq \alpha_1$, $\sigma_1^2(\beta_2) \subseteq \alpha_2$, and $\sigma_1(\beta) = \sigma_1(\beta_1) \cup \sigma_1(\beta_2)$. We have already proved the useful relation $p(\beta) \leqslant \bigvee \{p(\sigma_1(y)) \mid y \in \sigma_1(\beta)\}$ (for arbitrary β). We extend it as follows:

$$p(\beta) \leqslant (\bigvee \{p(\sigma_1(y)) \mid \sigma_1(y) \subseteq \alpha_1) \vee (\bigvee \{p(\sigma_1(y)) \mid \sigma_1(y) \subseteq \alpha_2\}) \leqslant \leqslant p(\alpha_1) \vee p(\alpha_2).$$

It remains to obtain the full additivity of the function p from its compact regularity and finite additivity. Suppose that $\alpha = \bigcup_i \alpha_i$, $\overline{\beta} \subseteq \alpha$, and $\overline{\beta}$ is compact. Then $\beta \subseteq \alpha_1 \cup \ldots \cup \alpha_n$ and $p(\beta) \leqslant p(\alpha_1) \vee \ldots \vee p(\alpha_n) \leqslant \leqslant \bigvee_i p(\alpha_i)$. Hence $p(\alpha) \leqslant \bigvee_i p(\alpha_i)$ and Lemma 5 is proved.

We continue with the proof of part a of Theorem 24. We verify the compact regularity of the function $p+q$. Since $p(\alpha_1) = \bigvee \{p(\beta) \mid \overline{\beta} \subseteq \alpha_1$, $\overline{\beta}$ is compact$\}$ and $q(\alpha_2) = \bigvee \{q(\gamma) \mid \gamma \subseteq \alpha_2$, $\overline{\gamma}$ is compact$\}$, we have $(p+q)(\alpha) = \bigvee \{p(\alpha_1) \wedge q(\alpha_2) \mid \alpha_1 + \alpha_2 \subseteq \alpha\} \leqslant \bigvee \{p(\beta) \wedge q(\gamma) \mid \overline{\beta} + \overline{\gamma} \subseteq \alpha$, $\overline{\beta}$ and $\overline{\gamma}$ are compact$\} \leqslant \bigwedge \{p(\beta) \wedge q(\gamma) \mid \overline{\beta + \gamma} \subseteq \alpha$, $\overline{\beta + \gamma}$ is compact$\} \leqslant \bigvee \{(p+q)(\beta+\gamma) \mid \overline{\beta+\gamma} \subseteq \alpha$, $\overline{\beta+\gamma}$ is compact$\} \leqslant \leqslant \bigvee \{(p+q)(\delta) \mid \overline{\delta} \subseteq \alpha$, $\overline{\delta}$ is compact$\}$. By Lemma 5 (its simple part) we find that $(p+q) \in \Omega_u^{\mathcal{F}}$.

Finally, we show that the bijection ψ preserves the operations in the sheaves $\mathcal{F}_Y$ and $C_\Omega(X, Y)$, namely, that $\psi(p+q) = f_{p+q}$ and $f_p + f_q = = \psi(p) + \psi(q)$(and similarly for other operations). For f_{p+q} and $f_p + f_q$ are defined on a set $\mathcal{O}$ which is Ω-dense in $u = p(Y) = q(Y)$. We compare them at a point $x \in \mathcal{O}$, that is, we compare $z = \lim (p+q)^{-1}(\mathcal{T}_x)$ and $y_1 + y_2$, where $y_1 = \lim p^{-1}(\mathcal{T}_x)$, $y_2 = \lim q^{-1}(\mathcal{T}_x)$. Let α be any neighbourhood of the point $(y_1 + y_2) \in Y$. We choose a neighbourhood $\overline{\delta}$ such that $(y_1+y_2) \in \delta$ and $\overline{\delta} \subseteq \alpha$. We choose α_1, α_2 such that $y_1 \in \alpha_1$, $y_2 \in \alpha_2$, and $\alpha_1 + \alpha_2 \subseteq \delta$. Then $x \in p(\alpha_1)$, $x \in q(\alpha_2)$, $x \in (p+q)(\alpha_1 + \alpha_2)$, and $z \in \overline{\alpha_1 + \alpha_2} \subset \overline{\delta} \subseteq \alpha$. Hence $y_1 + y_2 = z$.

We shall demonstrate that Y is embedded in $\mathcal{F}_Y$ and in fact in the set of global elements of Y^Ω. If $y_1 \neq y_2$, then obviously $\check{y}_1 \neq \check{y}_2$. In addition $(y_1 + y_2)^\vee = \check{y}_1 + \check{y}_2$ (and similarly for all operations in Y). For using the fact that the operation + is continuous we obtain $(y_1 + y_2)^\vee(\alpha) = = \bigvee \{\check{y}_1(\beta) \wedge \check{y}_2(\gamma) \mid \beta + \gamma \subseteq \alpha\}$.

b) The global validity of the proposition

$$\forall \sigma, \sigma_1 \in \check{\Sigma} \exists \sigma_0 \in \check{\Sigma}\ (\sigma_0 \subseteq \sigma \cap \sigma_1) \wedge (\check{Y})^2 \subseteq \sigma \wedge \sigma = \sigma^{-1} \wedge \exists\, \sigma_2 \in \check{\Sigma}\ (\sigma_2^2 \subseteq \sigma))$$

means that $\langle \check{Y}, \check{\Sigma} \rangle$ is a uniform space with the basis $\check{\Sigma}$ of the filter of neighbourhoods. In the same way we verify the global validity of the fact that $\check{\mathcal{T}}$ is one of the bases of the topology in the uniform space $\check{Y}$.

We shall use the definition of the object $\mathcal{F}_Y$ given in the remark on p.95. We recall that $b \in \check{Y} \rightleftharpoons b$ is "a basis of the minimal Cauchy filter in the uniform space $\langle \check{Y}, \check{\Sigma} \rangle$ contained in the basis of the topology $\check{\mathcal{T}}$". Instead of the expression in quotation marks we shall say briefly: "b is a basis in $\check{\mathcal{T}}$ of the minimal Càuchy filter". It remains to show the global validity of the following proposition: "$b \in \mathcal{F}'_Y \Leftrightarrow b$ is a basis in $\check{\mathcal{T}}$ of the minimal Cauchy filter".

We recall that the last notion means the following: 1) $b \subseteq \check{\mathcal{T}}$, 2) $\forall \alpha,\ \beta \in b\ (\alpha \cap \beta \in b)$, 3) $\forall \alpha, \beta \in \check{\mathcal{T}}\ (\alpha \subseteq \beta \wedge \alpha \in b: \Rightarrow): \beta \in b)$, 4) $\exists \alpha \in \check{\mathcal{T}}\ (\alpha \in b)$, 5) $\check{\varnothing} \notin b$, 6) $\forall \sigma \in \check{\Sigma} \exists \alpha \in \check{\mathcal{T}}\ (\alpha \in b \wedge \alpha \times \alpha \subseteq \sigma)$, 7) $\forall \alpha \in b \exists \beta \in b \exists \sigma \in \Sigma\ (\sigma(\beta) \subseteq \alpha)$.

From left to right: let $p \in \Omega_u^{\mathcal{T}}$, and we have to show that $u \leqslant [\![p$ is a basis in $\check{\mathcal{T}}$ of the minimal Cauchy filter$]\!]$. We have 1) $\bigwedge_{\alpha \in \mathcal{T}} (p(\alpha) \to [\![\check{\alpha} \in \check{\mathcal{T}}]\!]) = 1$, 2) $\bigwedge_{\alpha, \beta \in \mathcal{T}} (p(\alpha) \wedge p(\beta) \to [\![(\check{\alpha} \cap \check{\beta}) \in p]\!]) = 1$, since $[\![\check{\alpha} \cap \check{\beta} = (\alpha \cap \beta)^{\vee}]\!] = 1$, 3) $\bigwedge_{\alpha, \beta \in \mathcal{T}} ([\![\check{\alpha} \subseteq \check{\beta}]\!] \wedge [\![\check{\alpha} \in p]\!] \to [\![\check{\beta} \in p]\!]) = 1$, since $[\![\check{\alpha} \in p]\!] = p(\alpha)$, 4) $\bigvee_{\alpha} [\![\check{\alpha} \in p]\!] = \bigvee_{\alpha} p(\alpha) = u$,
5) $[\![\check{\varnothing} \in p]\!] = p(\varnothing) = 0$, $[\![\check{\varnothing} \notin p]\!] = 1$, 6) $\bigwedge_{\sigma} \bigvee_{\alpha} p(\alpha) \wedge [\![\check{\alpha} \times \check{\alpha} \subseteq \check{\sigma}]\!] = u$, since $\check{\alpha} \times \check{\alpha} = (\alpha \times \alpha)^{\vee}$ and $\{p(\alpha) \mid \alpha \times \alpha \subseteq \sigma\} = p(Y) = u$,
7) $\bigwedge_{\alpha} (p(\alpha) \to \bigvee_{\beta, \sigma} (p(\beta) \wedge [\![\check{\sigma}(\check{\beta}) \subseteq \check{\alpha}]\!])) = \bigwedge_{\alpha} (p(\alpha) \to \bigvee_{\sigma(\beta) \subseteq \alpha} p(\beta)) = 1$, since $[\![\check{\sigma}(\check{\beta}) = (\sigma(\beta))^{\vee}]\!] = 1$ and chiefly $p(\alpha) = \bigvee \{p(\beta) \mid \exists \sigma\ (\sigma(\beta) \subseteq \alpha)\}$.

We now verify the above equivalence from right to left. Let $[\![b$ be a basis in $\check{\mathcal{T}}$ of the minimal Cauchy filter$]\!] \rightleftharpoons u \in \Omega$. We put $p(\alpha) \rightleftharpoons [\![\check{\alpha} \in b]\!] \wedge u$ and show that $[\![p = b]\!] \geqslant u$ and $p \in \Omega_u^{\mathcal{T}}$, which proves the required implication. We have $u \wedge [\![z \in b]\!] \leqslant [\![z \in \check{\mathcal{T}} \wedge z \in b]\!] \leqslant \bigvee_{\alpha} [\![z = \check{\alpha} \wedge z \in b]\!] \leqslant \bigvee_{\alpha} [\![z = \check{\alpha}]\!] \wedge p(\alpha) \leqslant \bigvee_{\alpha} [\![z = \check{\alpha} \wedge \check{\alpha} \in p]\!] \leqslant [\![z \in p]\!]$, since $[\![\check{\alpha} \in p]\!] = \bigvee_{\beta} [\![\check{\beta} \in b]\!] \wedge u \wedge [\![\check{\alpha} = \check{\beta}]\!] = p(\alpha)$. Conversely, $1 = \bigwedge_{\alpha} (p(\alpha) \to [\![\check{\alpha} \in b]\!])$, since $p(\alpha) \leqslant [\![\check{\alpha} \in b]\!]$. We verify the second assertion: $p(\varnothing) = [\![\check{\varnothing} \in b]\!] \wedge u = 0$, $p(Y) = [\![\check{Y} \in b]\!] \wedge u \leqslant u$ and $u \leqslant \bigvee_{\alpha} [\![\check{\alpha} \in b]\!] \wedge \wedge ([\![\check{\alpha} \subseteq Y \wedge \check{\alpha} \in b]\!] \to [\![\check{Y} \in b]\!]) \leqslant [\![\check{Y} \in b]\!]$, that is $p(Y) = u$;

$p(\alpha \cap \beta) = [\![(\alpha \cap \beta)^{\vee} \in b]\!] \wedge u = [\![\check{\alpha} \in b]\!] \wedge [\![\check{\beta} \in b]\!] \wedge u = p(\alpha) \wedge p(\beta)$;

$p(\alpha) = [\![\check{\alpha} \in b]\!] \wedge u \leqslant [\![\check{\alpha} \in b]\!] \wedge ([\![\check{\alpha} \in b]\!] \to \bigvee_{\beta, \sigma} [\![\check{\beta} \in b]\!] \wedge [\![\check{\sigma}(\check{\beta}) \subseteq \check{\alpha}]\!]) \leqslant$

$\leqslant \bigvee \{p(\beta) \mid \exists\sigma\, (\sigma(\beta) \subseteq \alpha)\}$, that is, $p(\alpha) = \bigvee \{p(\beta) \mid \exists\sigma\, (\sigma(\beta) \subseteq \alpha)\}$,

and $\bigvee \{p(\alpha) \mid \alpha^2 \subseteq \sigma\} = \bigvee \{[\![\check{\alpha} \in b]\!] \wedge u \mid \alpha^2 \subseteq \sigma\} \geqslant [\![\exists \alpha \in \in \mathcal{T} (\alpha \in b \wedge \alpha^2 \subseteq \check{\sigma})]\!] \geqslant u$, that is, p has the property of uniform regularity. By Lemma 5 we find that p is fully additive.

c) This can be proved by a direct computation of valuations. □

Remark. There are known theorems on the transfer of properties of Y (as well as of any structure with Y as the domain) to $\check{Y}$, and then from $\check{Y}$ to $\widetilde{Y}$ (by continuity, and only this step is non-trivial), and finally to $\Omega^{\mathcal{T}} \simeq C_1(X, Y)$, as has been done in the preceding chapters. The construction similar to $\widetilde{Y}$ can be performed for locally compact spaces, metric spaces, and certain classes of topological spaces of a similar kind. It can also be done with arbitrary Ω_1 in place of $\mathcal{T}_Y$. It is also possible to associate in a canonical way with every p an Ω-valued measure μ_p with p equal to the mean of this measure. We can define integration of functions with respect to μ_p in such a way that $\int f d\mu_p = f(p)$. In the Boolean-valued case, integration and various integral representations with Boolean-valued measures are considered in particular in [11] and [15].

Appendix

EVALUATION IN BOOLEAN ALGEBRAS

In the papers [11] and [15] there are many thorough computations of valuations in Boolean algebras in the context of the questions of functional analysis, and the theory of integration and integral representations, and there are also general theorems on Boolean valuations. Following these papers, we present here some examples of specific arguments characteristic of Boolean-valued analysis. The details omitted are contained in these papers. The results presented below partially carry over to arbitrary complete Heyting algebras and to classes of such algebras, for example Stone algebras. The mathematical content of the appendix as well as its general plan are described at the beginning of Ch. IV.

If the valuation under consideration has its range in a fixed complete Boolean algebra $\mathbb{B}$, then there arises a quite specific situation. First of all, it is concerned with the following: if $ZFC \vdash \varphi$, then $[\![\varphi]\!]_B = 1$, where $[\![\cdot]\!]_B$ is the valuation in the language ZF with set of parameters V^B (see Example 1 and Theorem 8, part a). We can consider the following thesis: "if an assertion φ of traditional mathematics is valid, then $ZFC \vdash \varphi$". Therefore, "all objects and assertions of traditional mathematics exist in V^B and are globally valid in V^B, respectively". What is more, they even exist in the stronger sense that follows from the fact that the valuation is accessible in the language of V^B; see Theorem 4. As a result it is considerably easier to work with valuations in Boolean algebras, and in particular with the valuation in the language ZF. In addition, it is often convenient to use in arguments (see below) the facts that the valuation $[\![\varphi]\!]_B$ is Boolean in the language ZF, $f(g)$ is Boolean, where $f \in V^B$, and the Stone space $X(\mathbb{B})$ is Hausdorff (extremally disconnected). The theory of valuations in Boolean algebras is sometimes called Boolean-valued analysis, but before we turn to it let us say a few words on Robinson's non-standard analysis.

It follows at once from Theorem 3, part d that

$$([\![\varphi(\check{x}_1, \ldots, \check{x}_n)]\!]_{Z_2} = 1) \Leftrightarrow ([\![\varphi(\check{x}_1, \ldots, \check{x}_n)]\!]_\Omega = 1,$$

where φ is any bounded formula, Ω is an arbitrary Heyting algebra, and $Z_2 \subseteq \Omega$, $V^{Z_2} \subseteq V^\Omega$. We call all objects of the form $\check{x}$ in V^Ω standard objects ("sets"), that is, we identify the class V with the class $\{\check{x} \mid x \in V\}$, the class of standard objects. The objects from $V^\Omega \setminus V^{Z_2}$ are called non-standard or fuzzy objects ("sets"). The above equivalence says that global validity on the class of standard objects coincides with global validity on the class of all objects (for bounded formulae with standard parameters). Essentially new possibilities arise in the case when this equivalence is true for all formulae with standard parameters. This is true if Ω is a particularly simple complete Boolean algebra, namely if Ω is a discrete complete Boolean algebra, that is, Ω is the lattice of all subsets of a fixed set I (we denote this by $\mathcal{P}(I)$). The theory of $\mathcal{P}(I)$-valuations is sometimes called Robinson's non-standard analysis. Of course, Robinson's non-standard analysis can be presented without mentioning any valuations formally. However by Łos's theorem, validity in an ultraproduct $(\prod_{\alpha \in I} K_\alpha)/D \models \varphi([k_1], \ldots, [k_n])$ is equivalent to the fact that $[\![\varphi(k_1, \ldots, k_n)]\!] = 1$, where

$$[\![\varphi]\!] \leftrightharpoons \{\alpha \in I \mid K_\alpha \models \varphi(k_1(\alpha), \ldots, k_n(\alpha))\}.$$

Therefore, $\mathcal{P}(I)$-valuations enter Robinson's non-standard analysis in a substantial way. One might think that many of the achievements of Robinson's analysis are also possible in Heyting-valued analysis. We recall the axiomatics of Robinson's non-standard analysis given in Nelson's paper [16]. This is also interesting because the corresponding system of notions can be directly transferred to Boolean-valued Heyting-valued analyses, for which we can also give the corresponding axiomatics. We describe Nelson's axiomatics in a quite informal way. Let M be the "world of objects (sets)" and S (a part of the world M) the "world of standard objects (sets)". The language (which we denote by ZFS) used to describe the properties of the world M is the usual language ZF supplemented by a one-place predicate $st(x)$, "x is standard", and it is clear that $(M \models st\,(x)) \Leftarrow (x \in S)$. Next, $x^\wedge = \{y \in M \mid M \models (y \in x)\}$ is called an external standard set if $x \in S$, and an external set if $x \in M$, and a set z that cannot be represented in the form $z = x^\wedge$ for any $x \in M$ is called strictly external. Nelson writes $^*x$ instead of $x^\wedge$ (the same notation is generally used in Robinson's non-standard analysis). Any formula of the language ZF is called internal. The axioms are all the usual axioms of the theory ZFC formulated for internal formulae together with three new axioms (the intended role of the axioms is that they are valid in M):

1) $\varphi \Leftrightarrow \varphi^{st}$ for all internal formulae φ containing only standard parameters, where φ^{st} denotes the relativization of φ by the predicate $st(\cdot)$ (in this way the universes M and S cannot be distinguished by internal formulae with standard parameters). In what follows the notation $\forall^{stfin}$ denotes the relativization of the quantifier $\forall z$ by the predicate "$st\,(z) \wedge z$ is finite".

2) $\forall^{stfin} z \exists x \forall y \in z \varphi\,(x, y) \Leftrightarrow \exists^{st} x \forall^{st} y \varphi\,(x, y)$ for all internal formulae φ with any parameters (in this way everything that is true for all standard finite sets is also true for all standard sets).

3) $\forall^{st} x \exists^{st} y \forall^{st} z\,(z \in y \Leftrightarrow z \in x \wedge \varphi\,(z))$, where φ is any formula with parameters (a delicate connection between standard and non-standard sets).

We turn to Boolean-valued analysis and consider as examples some questions of duality.

a) Everywhere in what follows B is an arbitrary fixed complete Boolean algebra and $[\![\cdot]\!]_B$ is a valuation in the language ZF with range in B and family of parameters V^B (see Example 1).

Let $\mathscr{X}$ be a real or complex Banach space (we denote the field of scalars by K), $\mathscr{X}^*$ the adjoint Banach space over K, and D the unit ball (with boundary) in $\mathscr{X}^*$. The operation $(\cdot)^\sim$ was defined in Theorem 24. Then $[\![\tilde{\mathscr{X}}$ is a Banach space over $\tilde{K}]\!] = 1$. We denote by $\mathscr{D}$ the object in V^B such that $[\![\mathscr{D}$ is the unit ball (with boundary) in $(\tilde{\mathscr{X}})^*]\!] = 1$. If $h \in D$ (that is, $h\colon \mathscr{X} \to K$), then we extend it by uniform continuity to $\check{h}$ (where $[\![\check{h}\colon \check{\mathscr{X}} \to K$ is a functional with norm $\leqslant 1]\!] = 1$) defined on $\tilde{\mathscr{X}}$. This extension will be denoted by $\tilde{h}$. (We can extend any uniformly continuous function in exactly the same way.) We put $\psi(h) = \tilde{h}$. Of course, we have $\tilde{h} \in (\mathscr{D}^\wedge)$. We put $\underline{\psi} = \{\langle \check{h}, \psi\,(h)\rangle \mid h \in D\}_-$. Of course, we have $[\![\underline{\psi}\colon \check{D} \to \mathscr{D}]\!] = 1$.

In D we take the uniform structure with subbase of the filter of neighbourhoods of the form

$$\{\{\langle h_1, h_2\rangle \in D^2 \mid |\, h_1\,(x) - h_2\,(x)\,| < \varepsilon\} \mid x \in \mathscr{X},\ \ \varepsilon \in \mathbb{Q}_{>0}\},$$

which, by definition, induces in D a topology called *weakly dual.*[1] We recall that it is the weakest topology in D such that all functions $h(x): D \to K$ (where h is variable and $x \in \mathscr{X}$ is fixed) are continuous. This topology is convenient because the ball D is compact in it, and consequently we can apply Theorem 24 to the uniform topological space D. In particular, $\check{D}$ and $\tilde{D}$ are uniform spaces (in V^B). In the same way we can define a uniform structure in $\mathscr{D}$ inducing in $\mathscr{D}$ a topology which by analogy we can call *globally weakly dual.* It has the properties of the weakly dual topology mentioned above. It is easy to see that a subbase of the filter of neighbourhoods of this topology is

$$\{\{\langle h_1, h_2\rangle \in \mathscr{D}^2 \,\|\, h_1(x) - h_2(x) \,| < \varepsilon\} \mid x \in \check{\mathscr{X}},\ \ \varepsilon \in (\mathbb{Q}_{>0})^{\vee}\}.$$

***Lemma* 6.** *We have:* $[\![\psi\!\!\!\!\_: \check{D} \to \mathscr{D}$ *(as well as* $(\psi\!\!\!\!\_)^{-1}$*) is a uniformly continuous injection (of uniform structures with weakly dual topologies) and a local isomorphism* $]\!] = 1$ *and* $[\![$*the image of* $\psi\!\!\!\!\_$ *is dense in* $\mathscr{D}$ $]\!] = 1$.

Proof. The computation of the first valuation is not complicated (we note that $(\psi\!\!\!\!\_)^{-1} = h \upharpoonright \mathscr{X}$). To compute the second it is sufficient to prove the assertion:

$$\forall Y \in \mathscr{P}^{\mathrm{fin}}(\check{\mathscr{X}})\ \forall h_1 \in \mathscr{D} \forall \varepsilon \in (\mathbb{Q}_{>0})^{\vee}\, \exists h_2 \in$$
$$\in \check{D} \forall x \in Y\ (\,|\,\psi\!\!\!\!\_(h_2)(x) - h_1(x)\,| < \varepsilon),$$

where $\mathscr{P}^{\mathrm{fin}}(Z)$ is the set of all finite subsets of Z. Since the equality $\mathscr{P}^{\mathrm{fin}}(\check{\mathscr{X}}) = (\mathscr{P}^{\mathrm{fin}}(\mathscr{X}))^{\vee}$ is globally valid, we have to verify that

$$\forall Y \in \mathscr{P}^{\mathrm{fin}}(\mathscr{X})\ \forall h_1 \in \mathscr{X}^{\wedge} \forall \varepsilon \in$$
$$\in \mathbb{Q}_{>0} \bigvee \{ \bigwedge_{x \in Y} [\![\,|\,(h_2(\check{x})) - h_1(\check{x})\,| < \check{\varepsilon}]\!] \mid h_2 \in D\} = 1.$$

Let $Y = \{x_1, \ldots, x_n\}$. Thus, we verify that

$$\bigvee_{h_2} \bigwedge_{i=1}^{n} [\![\,|\,(h_2(\check{x}_i)) - h_1(\check{x}_i)\,| < \check{\varepsilon}]\!] = 1.$$

It is sufficient to consider the case when $\|h_1\| < 1$.

We first consider the case when $Y = \{x_1, \ldots, x_n\}$ is a linearly independent system in $\mathscr{X}$. We denote by L the linear subspace of $\mathscr{X}$ spanned by Y. Then $[\![\tilde{L}$ is spanned by the basis $\check{Y} = \{\check{x}_1, \ldots, \check{x}_n\}$ over $\tilde{K}$ in $\tilde{\mathscr{X}}]\!] = 1$. We take the norm in $\tilde{L}$ induced from $\tilde{\mathscr{X}}$. We restrict h_1 to $\tilde{L}$ and obtain h'_1, which satisfies $\|h'_1\| < 1$. We put $\lambda_1 = h'_1(\check{x}_1), \ldots, \lambda_n = h'_1(\check{x}_n)$, $\lambda = \langle \lambda_1, \ldots, \lambda_n\rangle \in \tilde{K}^n$. We denote by $\check{f} = \langle \check{f}_1, \ldots, \check{f}_n\rangle$ the dual basis to the basis $\langle \check{x}_1, \ldots, \check{x}_n\rangle$ in $\tilde{L}$ and by g an isomorphism of $(\tilde{K})^n$ and $(\tilde{L})^*$ of the form $\mu \mapsto \mu_1 \check{f}_1 + \ldots + \mu_n \check{f}_n$, which exists since any two norms on a

[1]This is often called the weak* topology. (Translator)

finite-dimensional space are equivalent. The function g_1: $(\check{K})^n \to \check{\mathbb{R}}$ given by $g_1(\mu) = \| \mu_1\check{f}_1 + \ldots + \mu_n\check{f}_n \|$ is continuous. Therefore,

$$[\![\exists\delta \in \check{\mathbb{Q}}_{>0}\forall\mu \in \check{K}^n (| \mu - \lambda | < \delta \Rightarrow g_1(\mu) < 1)]\!] = 1,$$

where $|\cdot|$ is any of the usual norms in $(\tilde{K})^n$ (for example, $\max_{1\leqslant i\leqslant n} | \mu_i |$).

Hence, we have $[\![\exists\mu \in (\check{K})^n (| \mu - \lambda| < \check{\varepsilon} \wedge \| \mu_1\check{f}_1 + \ldots + \mu_n\check{f}_n \| < 1$ and $\bigvee_{\mu_1\in K}\ldots\bigvee_{\mu_n\in K} [\![| \check{\mu} - \lambda | < \check{\varepsilon} \wedge \| (\check{\mu}, \check{f}) \| < 1]\!] = 1$. The argument that follows is for one term with valuation equal to b_μ, $b_\mu > 0$. This μ in K^n defines a functional by $(\mu, f) : L \to K$. Then $[\![\| (\check{\mu}, \check{f}) < 1]\!] \geqslant b_\mu$, that is, $\|(\mu, f)\| \leqslant 1$. By the Hahn-Banach theorem we can extend (μ, f) to $\mathscr{X}$ without increasing the norm, and so obtain the required functional h_2 on D. Next $b_\mu \leqslant \bigwedge_i [\![| \check{\mu}_i - \lambda_i | < \varepsilon]\!]$, where we have $\check{\mu}_i = h_2(\check{x}_i)$ and $\lambda_i = h_1'(\check{x}_i) = h_1(\check{x}_i)$, since $\check{x}_i \in \check{L}$. By summing over all b_μ we obtain the required result.

If Y is a linearly dependent system, then we choose a maximal linearly independent subsystem of it and apply the previous argument, with simple valuations for the vectors not included in the basis.

By Lemma 6 we can extend $\underline{\psi}$ to $\widetilde{D}$, that is, $[\![\underline{\psi}: \underline{D} \to \mathfrak{D}]\!] = 1$. For $p \in D^B$ we put $\psi^\wedge(p) = s$, where $[\![\underline{\psi}(p) = s]\!] = 1$ and $s \in \mathfrak{D}^\wedge$. Thus $\psi^\wedge$: $D^B \to \mathfrak{D}^\wedge$. If we consider the extension of $\underline{\psi}$ as a graph, then $\psi^\wedge$ can be obtained as $(\underline{\psi})^\wedge$. Hence, we have the following theorem.

***Theorem* 25a).** *The map $\psi^\wedge$ is an isomorphism of $C_1(X, D)$ and $\mathfrak{D}^\wedge$, where X is the Stone space of the algebra* B. *(In the spirit of Theorem* 9 *the properties of $\mathfrak{D}$ in V^B carry over to $C_1(X, Y)$.)*

Remark. 1) It is clear from the proof that a similar assertion holds for fields other than K, and also for other weak topologies.

2) If in Theorem 24 Y is a space of functions (as it is, for example, in Theorem 25, part a, then an f in $C_1(X, Y)$ can be expressed more descriptively in the form $f(p, x) \rightleftharpoons f(p)(x)$, where p runs through $\mathfrak{D}(f)$, an open Ω-dense set in X, and x runs through $\mathfrak{D}(f(p))$. Such a class of functions "of two variables" is usually well described in ordinary terms. In this way, a question about functions of two or more variables can be reduced to the same question in V^B for functions of a smaller number of variables or of one variable (this has applications, for example, in complex analysis). In the situation of Theorem 25, part a, $C_1(X, D)$ coincides with the set $D(X, \mathscr{X}, K)$ of all continuous functions $f(p, x)$: $O_f \times \mathscr{X} \to K$, where O_f is a dense open set in X and $\forall p$ ($f(p, x)$ is linear and homogeneous and $|f(p, x)| \leqslant \|x\|$) is factorized as before.

b) Let A be a Banach algebra (with identity e) over $\mathbb{C}$ with continuous involution $* : A \to A$ satisfying the condition $h(x^*) = h(x)^-$, where $h \in D$ (see below) and $^-$ is complex conjugation (such an algebra is called self-conjugate). Here D denotes the set of all complex homomorphisms of A into $\mathbb{C}$ (except the one which is identically zero), that is, $\varphi(\lambda_1 x + \lambda_2 y) = \lambda_1 \varphi(x) + \lambda_2 \varphi(y)$, $\varphi(xy) = \varphi(x)\varphi(y)$. Hence it follows that $\varphi(e) = 1$ and φ is continuous. The function $h(x)$, where h is variable (running through D) and $x \in A$ is fixed, will be denoted by $\hat{x}$, that is, $\hat{x}$: $D \to \mathbb{C}$. In what follows we shall need the property: $\forall x \in A$ (x is invertible in $A \Leftrightarrow \forall h \in D$ $(h(x) \neq 0)$), which is true if, for example, the algebra is commutative (see for example W. Rudin, Functional Analysis, p.265). Therefore, for simplicity, we assume that A is a commutative algebra. We put

$$f_{x_1, \ldots, x_n}(c_1, \ldots, c_n) = \sum_{i=1}^{n} (c_i e - x_i)(c_i e - x_i)^*: \ \mathbb{C}^n \to A$$

(with variables $c = \langle c_1, \ldots, c_n \rangle$ and parameters $Y = \{x_1, \ldots, x_n\}$), and this function is continuous. It is clear that ($f_x(c)$ is not invertible in A) $\Leftrightarrow$ $\Leftrightarrow \exists h \in \bigwedge (c_1 = h(x_1) \wedge \ldots \wedge c_n = h(x_n))$. The set $G(A)$ of all invertible elements of A is open and the set $F(A)$ of all non-invertible elements in A is closed. We write $S(A, x_1, \ldots, x_n) = f_x^{-1}(F(A))$, and this set is closed in $\mathbb{C}^n$.

The set D with the weakest topology under which all functions $\{\hat{x} \mid x \in A\}$ are continuous is a compact set, and is sometimes called the *spectrum of the algebra A*. We can apply Theorem 24 to it. We denote by $\mathscr{D}$ the object in V^B such that $[\![\mathscr{D}$ is the spectrum of the Banach algebra $\tilde{A}]\!] = 1$. We note that the same description of the algebra $\tilde{A}$ that was given for the algebra A is globally valid. If $h \in D$, then we extend $\check{h}$, where $[\![\check{h}$: $\check{A} \to \check{\mathbb{C}}$ is a homomorphism$]\!] = 1$, onto $\tilde{A}$ by uniform continuity, and denote this extension by $\tilde{h}$. Of course, $\tilde{h} \in \mathscr{D}^\wedge$. We put $\psi(h) = \tilde{h}$ and $\underline{\psi} = \{\langle \check{h}, \tilde{h} \rangle \mid h \in D\}$. Of course, $[\![\underline{\psi}$: $\check{D} \to \mathscr{D}]\!] = 1$. Finally, we put $\underline{\psi}^\wedge = (\underline{\psi})^\wedge$, where $\underline{\psi}$ is understood as the graph of the map $\underline{\psi}$.

***Lemma* 7.** *If A is a Banach algebra as above, then $(G(A))^\sim = G(\tilde{A})$, $(F(A))^\sim = F(\tilde{A})$, and*

$$S(\tilde{A}, \check{x}_1, \ldots, \check{x}_n) = (S(A, x_1, \ldots, x_n))^\sim = (S(A, x_1, \ldots, x_n))^{\vee -},$$

where $^-$ denotes the closure, and

$$\langle c_1, \ldots, c_n \rangle \in (S(A, x_1, \ldots, x_n))^\vee \Leftrightarrow \exists h \in \check{D}$$
$$(c_1 = h(\check{x}_1) \wedge \ldots \wedge c_n = h(\check{x}_n)).$$

***Lemma* 8.** *If A is a Banach algebra as above, then the following assertion is globally valid "$\underline{\psi}$ is a uniformly continuous injection (as well as the map inverse to $\underline{\psi}$), and the range of $\underline{\psi}$ is dense in $\mathscr{D}$".*

Proof. The first assertion is simple. We verify the second. Let $Y = \{x_1, \ldots, x_n\}$, $\varepsilon \in \mathbb{Q}_{>0}$, and $h_1 \in \mathcal{D}^\wedge$. We have to establish that $\bigvee \{\bigwedge_{i=1}^{n} [\![| h_2(\check{x}_i) - h_1(\check{x}_i) | < \check{\varepsilon} \mid h_2 \in D\}]\!] = 1$. We write $c_1 = h_1(\check{x}_1), \ldots$ $\ldots, c_n = h_1(\check{x}_n)$, and $c = \langle c_1, \ldots, c_n \rangle \in \widetilde{\mathbb{C}}^n$. Then $c \in S(\check{A}, \check{x}_1, \ldots, \check{x}_n)$ (it is sufficient to compute h_1 on $f_x(c)$). By Lemma 7 there is a $c' = \langle c_1, \ldots, c_n \rangle \in S(A, x_1, \ldots, x_n)^\vee$ such that $| c' - c | < \varepsilon$. Hence

$$\bigvee \{[\![c'_1 = \lambda_1^\vee]\!] \wedge \ldots \wedge [\![c'_n = \lambda_n^\vee]\!] \mid \langle \lambda_1, \ldots, \lambda_n \rangle \in S(A, x_1, \ldots, x_n)\} = 1,$$

and we denote the components by $b_{\lambda_1, \ldots, \lambda_n}$. There is an $h_2 \in D$ such that $\lambda_1 = h_2(x_1), \ldots, \lambda_n = h_2(x_n)$. The valuation of the fact that $| \check{\lambda}_i - c_i | < \check{\varepsilon}$, where $\lambda_i^\vee = \widetilde{h}_2(\check{x}_i)$, is equal to $b_{\lambda_1, \ldots, \lambda_n}$, and we sum over all $b_{\lambda_1, \ldots, \lambda_n}$.

By thus extending $\underline{\psi}$ to $\widetilde{D}$ by uniform continuity we obtain the following theorem.

***Theorem* 25b).** *If A is a commutative self-adjoint Banach algebra with continuous involution, then $\psi^\wedge$ is an isomorphism of $C_1(X, D)$ and $\mathcal{D}^\wedge$, where X is the Stone space of the algebra $\mathbb{B}$. As usual, the properties of $\mathfrak{D}$ in V^B carry over to $C_1(X < D)$.*

c) Let G be a commutative locally compact group that is also uniformly locally compact. Then we can apply Theorem 24. We denote by G^+ its character group, that is, the continuous homomorphisms of G into the unit sphere S (the operations in G^+ are defined pointwise, and the topology is given by the basis at the identity consisting of all

$$O(c, \Delta) = \{\chi \in G^+ \mid \chi(c) \subseteq \Delta\},$$

where c runs through all compact sets in G and Δ runs through all neighbourhoods of the identity in S; see for example L.S. Pontryagin, Topological groups, Ch. V, Princeton 1958).

It is globally valid that $\widetilde{G}$ is a group of the same kind as G. If $\chi \in G^+$, then $[\![\check{\chi}: \check{G} \to \widetilde{S}$ is a character $]\!] = 1$, and we can extend it by continuity to $\widetilde{G}$ (as always, we denote the extension by $\widetilde{\chi}$). Of course, we have $[\![\widetilde{\chi} \in (\widetilde{G})^+]\!] = 1$. We put $\psi(\chi) = \tilde{\chi}$ and $\underline{\psi} = \{\langle \check{\chi}, \tilde{\chi} \rangle \mid \chi \in G^+\}_{-}$, and finally $\psi^\wedge = (\underline{\psi})^\wedge$, where $\underline{\psi}$ is understood as a graph. It is clear that $[\![\underline{\psi}: (G^+)^\vee \to (\widetilde{G})^+$ is a monomorphism $]\!] = 1$. We show that $[\![\underline{\psi}$ is continuous $]\!] = 1$. Any element of the basis at the identity of the topology on $\widetilde{G}$ is given by a compact set c' in $\widetilde{G}$ and $\varepsilon \in (\mathbb{Q}_{>0})^\vee$. There is a compact set c in G such that $\widetilde{c} \supseteq c$. Therefore, it is sufficient to consider the neighbourhood of the identity in $(\widetilde{G})^+$ given by the compact set $\widetilde{c}$ and $\check{\varepsilon} > 0$. In G^+ we take a neighbourhood of the identity of the form $O = O(c, \varepsilon)$. Then $[\check{O}$ is a neighbourhood in $(G^+)^\vee$, $\underline{\psi}(O) \subseteq O(\check{c}, \check{\varepsilon})] = 1$. It is easy to see that $(\underline{\psi})^{-1}$ is also continuous.

We extend $\underline{\psi}$ to $(G^+)^\sim$ by continuity and so obtain a monomorphism which is a homeomorphism onto its range, $\underline{\psi} : (G^+)^\sim \to (\widetilde{G})^+$.

We denote by Γ the image of $\underline{\psi}$ in $(\widetilde{G})^+$. Then Γ is a locally compact subgroup of $(\widetilde{G})^+$. We define an isomorphism i between $\widetilde{G}$ and Γ^+ by $g \mapsto \chi_g \mapsto \chi_g \upharpoonright \Gamma$, where χ_g is the character of $(\widetilde{G})^+$ that corresponds to evaluating g at a point. Such an i is an epimorphism, since any character on Γ can be extended to $(\widetilde{G})^+$ (using Pontryagin duality).

To show that i is a monomorphism it is sufficient to verify that its kernel $\{g \in \widetilde{G} \mid \forall \chi \in \Gamma\ (\chi(g) = \check{e})\} = \{g \in \widetilde{G} \mid \forall \chi \in (G^+)^\vee\ (\underline{\psi}(\chi)(g) = \check{e})\}$ is trivial, that is, to verify that $(\bigwedge \{[\![\widetilde{\chi}(g) = \check{1}]\!] \mid \chi \in G^+\} = 1) \Rightarrow ([\![g = \check{e}]\!] = 1)$, and $\forall \chi \in G^+ ([\![\widetilde{\chi}(g) = \check{e}]\!] = 1) \Rightarrow [\![g = \check{e}]\!]$. We have the relation $\check{1}(w) = g(\chi^{-1}(w))$, where w is an arbitrary Borel set on the sphere S and $\check{1}(\cdot)$ and $g(\cdot)$ are the measures on S corresponding to $\check{1}$ in $\widetilde{S}$ and g in $\widetilde{G}$ by the formula $g(w) \rightleftharpoons [f_g^{-1}(w)]$, where $[\cdot]$ denotes the factorization by thin sets. We consider the weakest topology on G such that all χ in G^+ are continuous. It is Hausdorff and contained in the original topology in G. Therefore, they coincide on any compact subspace of G, and the measures $\check{e}(\cdot)$ and $g(\cdot)$ on G also coincide on the elements of this new topology. Therefore, they coincide on the closed sets in this new (weak) topology. If O is an open relatively compact set in G, then there is a set u open in the weak topology such that $O = \bar{O} \cap u$ and $\bar{u}$ is closed in the weak topology. Therefore, $g(O) = g(\bar{O}) \wedge g(u) = \check{e}(\bar{O}) \wedge \check{e}(u) = \check{e}(O)$. Hence the measures $g(\cdot)$ and $\check{e}(\cdot)$ coincide, that is, $[\![g = \check{e}]\!] = 1$.

Thus, i is an algebraic isomorphism of $\widetilde{G}$ and Γ^+. It is easy to see that i is also a homeomorphism. By Pontryagin duality we have an isomorphism $i_1 : \Gamma \leftrightarrow (\widetilde{G})^+$. We put $\varphi = i_1 \circ \underline{\psi} : (G^+)^\sim \to \Gamma \to (\widetilde{G})^+$ and, as always, $\varphi^\wedge = (\underline{\varphi})^\wedge$, where φ on the right-hand side is understood as a graph. From this we obtain the following theorem.

Theorem 25c). *If G is a commutative locally compact group, then $\varphi^\wedge$ is an isomorphism of $C_1(X, G^+)$ and $((\widetilde{G})^+)^\wedge$. (In the usual sense, the properties of $(\widetilde{G})^+$ in V^B carry over to $C_1(X, G^+)$.)*

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