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Undecidable hypotheses in Edward Nelson's internal set theory

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Preface

The modern theory of nonstandard (= infinitesimal) methods had its origins in Abraham Robinson's investigations in the early 1960s. Robinson established the possibility of completely rigorous arguments with infinitesimal and infinitely large numbers. The first approach to nonstandard analysis was model-theoretic or "constructive", see Robinson [31], [32]. Lindstrøm's large article [21], part I of Albeverio et al [1], and many other sources, among them Davies [4], Kanovei [14], Lutz and Gose [23], Lyubetskii [24], Uspenskii [35], present this approach in all the necessary detail. Universal constructions of ultrapowers and ultralimits (iterated ultrapowers) gave many fruitful nonstandard extensions of various mathematical structures. Meanwhile many properties of different extensions were found to be similar. This was the reason for searching for an appropriate axiomatization.

Several axiomatic systems were proposed: Nelson [28], [29], Kawai [17], [18], Hrbáček [10], [11], Henson and Keisler [9], Vopenka [36] and some others (see [20] for a survey). Edward Nelson's internal set theory (briefly, IST) appears to be the most fruitful among them. IST extends the usual set theory

ZFC by adding a new unary predicate of standardness and three intuitively acceptable and easy to use new axioms governing its action.

Today **IST** is certainly accepted as a base for the nonstandard mathematics. The following investigations are in any event connected with **IST**: Diener and Stroyan [5], Diener and Diener [6], Shubin and Zvonkin [34], the monographs of Robert [30] and van den Berg [2], and some other textbooks and applied works, including Gordon [7] and [15], [16] of the author, where some previous variants of the theorems of this paper were presented. Lutz and Gose [23], Kusraev and Kutateladze [20] considered **IST** among some other nonstandard systems.

It is the applied side of **IST** that has been the usual topic of nonstandard investigations concerning the **IST**. Purely logical questions, as a rule, were avoided. Meanwhile problems concerning the boundaries of the area of the provable and the area of the undecidable are among the most important in logic, especially for theories of set theoretic type, see Jech [13]. Being a conservative extension of **ZFC**, **IST** takes over a lot of famous undecidabilities from **ZFC** (the continuum hypothesis, the Souslin hypothesis, and so on). Hence only those undecidable sentences may be of real interest which are much more connected with the spirit of the nonstandard mathematics, and those which discover this spirit.

The aim of this article is to prove the undecidability of some sentences, or *hypotheses*, in **IST**. All of them are in fact the extensions of some **ZFC** axiom or theorem to the case of an external (that is, containing the predicate st) core formula. (All **ZFC** axioms and theorems hold in **IST** only in the case when the core formula is internal—that is, without the st.) Among those hypotheses are the following four:

(a) $$\forall^{\mathrm{st}} x \in X\ \exists! y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x));$$

(b) $$\forall^{\mathrm{st}} x \in X\ \exists!^{\mathrm{st}} y\ \Phi(x, y) \to \exists^{\mathrm{st}} \tilde{y}\ \forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x));$$

(c) $$\forall^{\mathrm{st}} x \in X\ \exists y \in Y\ \Phi(x, y) \to \exists \tilde{y} : X \to Y\ \forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x));$$

(d) $$\forall^{\mathrm{st}} x\ \ \exists!^{\mathrm{st}} y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\mathrm{st}} x\ \Phi(x, \tilde{y}(x))$$

(X, Y are arbitrary standard sets). We show that (a), (b), (c), (d) are undecidable in **IST**. Moreover each of them holds for all standard X, Y and all Φ in Nelson's ultralimit model and fails for $X = Y = \mathbb{N}$ and some special Φ in another model of **IST**. (We allow Φ to be an *external* formula.) We recall that (c) is the well-known *extension principle*. We study the question about how complicated Φ may be in non-provable examples.

Three more results. Theory **IST** + (b) is strong enough to prove the consistency of **ZFC** and **IST**. It is possible to express in **IST** the truth of all internal formulae with standard parameters by some external formula (this fails if nonstandard parameters are allowed). The following *collection* axiom

$$\forall X\ \exists Y\ \forall x \in X\ [\exists y\ \Phi(x, y) \to \exists y \in Y \Phi(x, y)]$$

holds in **IST** for all formulae Φ (internal as well as external).

Also we study the *bounded set theory* **BST**. This is the modification of **IST** which guarantees that all sets are members of standard sets. **BST** is equiconsistent with **IST** and **ZFC** and is sufficiently strong to make (a)–(d) decidable.

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§1. Introduction. Hypotheses and results

1.1. Internal set theory.

The **IST** language contains the equation $=$ and two predicates: the membership relation $\in$ and the standardness st; st x means "x is a standard set".

A formula of the **IST** language (shortly, *st-$\in$-formula*) that does not contain the symbol st is called *internal*, a formula containing st is called *external*. Internal fomulae are just $\in$-formulae or those of the **ZFC** language.

Two simple abbreviations are often used:

$$\exists^{\mathrm{st}} z\ \varphi(z) \ \text{ means } \ \exists z\,[\mathrm{st}\ z\ \&\ \varphi(z)];$$
$$\forall^{\mathrm{st}} z\ \varphi(z) \ \text{ means } \ \forall z\,[\mathrm{st}\ z \rightarrow \varphi(z)].$$

Note that st x is equivalent to $\exists^{\mathrm{st}} z\ (z = x)$. Thus the only necessary use of the standardness predicate is its use in quantifiers $\exists^{\mathrm{st}}$, $\forall^{\mathrm{st}}$. The quantifiers $\exists^{\mathrm{st}}$, $\forall^{\mathrm{st}}$ are called *external*, while the usual $\exists$ and $\forall$ are called *internal*.

Internal set theory contains all the axioms of Zermelo–Fraenkel set theory **ZFC** (with choice) formulated in the $\in$-language and three additional principles or axioms. These are the *idealization* **I**, the *standardization* **S**, the *transfer* **T**:

$$\mathbf{I}:\ \forall^{\mathrm{stfin}} A\ \exists x\ \forall a \in A\ \ \Phi(x,\ a) \leftrightarrow \exists x\ \ \forall^{\mathrm{st}} a\ \ \Phi(x,\ a)$$

for any internal formula Φ;

$$\mathbf{S}:\ \exists^{\mathrm{st}} Y\ \ \forall^{\mathrm{st}} x\,[x \in Y \leftrightarrow x \in X\ \&\ \Phi(x)]$$

for any st-$\in$-formula Φ and any standard set X;

$$\mathbf{T}:\ \exists x\ \ \Phi(x) \leftrightarrow \exists^{\mathrm{st}} x\ \ \Phi(x)$$

for any internal formula Φ with standard parameters.

Of course none of **I**, **S**, **T** is a single axiom. Principles of such a kind are called *axiom schemes*. Their strength depends on our choice of a *core formula* Φ.

$\forall^{\mathrm{stfin}} A$ is an abbreviation for "for all standard finite A". The slightly vague phrase "with standard parameters" means that every free variable occurring in Φ except x may be replaced only by a *standard* set when **T** acts within the universe of internal sets (see below). From a logical point of view we imply that the list of external quantifiers $\forall^{\mathrm{st}} v_1 \ldots \forall^{\mathrm{st}} v_n$ (for all the free variables $v_1, \ldots, v_n$ of Φ except x) is written down outside **T**.

We note that core formulae $\Phi(x, a)$ in **I** and $\Phi(x)$ in **S** can also contain other free variables which may be replaced by arbitrary sets, standard or nonstandard. Briefly, Φ is a formula with *arbirtary* parameters in **I** and **S**. In general a *parameter* is a set that replaces a free variable in a formula.

Thus the main special feature of **IST** (among other nonstandard systems) is that it takes into consideration only the following two types of sets: standard and internal. Unlike **IST**, many other theories, especially those introduced by Kawaï, Hrbáček, Henson and Keisler, Vopenka, admit the third kind: *external* sets. One can say that **IST** is an *internal* axiomatization of the nonstandard mathematics while some other theories gave an axiomatization of an *external* world containing a nonstandard model. The last approach makes the logical structure significantly more complicated, so it prevents one obtaining a large amount of natural applications. Maybe this is the reason why, unlike **IST**, the known external systems are not of common use.

1.2. The universe of internal set theory.
A set-theoretic universe (that is, a collection of sets governed by the axioms of a theory that we consider) is usually denoted by some form of the letter V. We choose the double-lined form $\mathbb{V}$ to denote the universe of all sets governed by the axioms of **IST** (or of **ZFC** if arguing in **ZFC**). Historically, sets contained in $\mathbb{V}$ are called *internal*. So

$$\mathbb{V} = \text{all sets} = \text{all internal sets}.$$

The standardness predicate st works in $\mathbb{V}$. Thus we may define the universe of standard sets

$$\mathbb{S} = \{x : \mathrm{st}\, x\} = \{x \in \mathbb{V} : x \text{ is standard}\}.$$

One can easily prove that $\mathbb{S}$ is not a member of $\mathbb{V}$, though $\mathbb{S} \subseteq \mathbb{V}$. Collections of such a kind are called *external*. They are considered in **IST** in the same manner as proper classes in **ZFC**.

Every "individual" set of classical mathematics is standard; this is guaranteed by transfer. Thus $\mathbb{N}$ (integers), $\mathbb{R}$ (reals), and so on, are standard members of $\mathbb{V}$. The reader familiar with "superstructural" investigations must keep in mind that the sets $\mathbb{N}$, $\mathbb{R}$ and in general all infinite standard sets contain both standard and nonstandard elements. For example,

$\mathbb{N}$ corresponds to ${}^*\mathbb{N}$ in a superstructure, while the collection

$$^\sigma\mathbb{N} = \{n \in \mathbb{N} : \text{st } n\} = \{\text{standard integers}\}$$

is analogous to superstructural $\mathbb{N}$. By the way, ${}^\sigma\mathbb{N}$ and generally ${}^\sigma X$ for all standard infinite sets X are external collections.

Finiteness is realized in **IST** in the usual sense, that is, the number of elements of a set which we say is finite must be equal to some integer n, not necessarily standard. So it corresponds to hyperfiniteness in a superstructure.

A rigorous "superstructuralist" may realize the universe $\mathbb{V}$ as a model in the usual set-theoretic Zermelo – Fraenkel world. In fact one can construct $\mathbb{V}$ by Nelson's [28] adequate ultralimits rather than by direct construction of a superstructure due to Lindstrøm [21]. From this point of view one must distinguish between two meanings of the notion "external". Firstly, "external" can be taken to mean "defined in $\mathbb{V}$ by an external formula" (or, equivalently, by some st-$\in$-formula). Secondly, "external" is any $X \subseteq \mathbb{V}$ that is not a member of $\mathbb{V}$. According to what has been said above, externality has the first meaning here. The word "outer" is a good substitute for the second meaning. Thus its opposite "inner" means internal + external, that is, definable in $\mathbb{V}$ by a st-$\in$-formula. Finally, "outer" means undefinable in $\mathbb{V}$.

1.3. General approach to problems.

Certainly all the axioms and theorems of **ZFC** remain provable in **IST** because **IST** is an extension of **ZFC**. However let us look at this proposition more carefully. We note that **ZFC** contains, among others, two axiom schemes, namely, *separation*

$$\textbf{Sep}: \ \exists Y \ \forall x \in X \ [x \in Y \leftrightarrow x \in X \ \& \ \Phi(x)],$$

and *replacement*

$$\textbf{Repl}: \ \forall x \in X \ \exists! \ y \, \Phi(x, \ y) \rightarrow \exists \tilde{y} \ \forall x \in X \ \Phi(x, \ \tilde{y}(x)),$$

where Φ is an arbitrary $\in$-formula and X is an arbitrary set. Every variable with $\tilde{}$ over it designates a function, and an occurrence of a term of type $\tilde{y}(x)$ means that $\tilde{y}$ is defined at x. Thus **Repl** $\tilde{y}$ is a function with domain (including) X.

Let us emphasize that **Sep** and **Repl** are included in **IST** only in the case of internal core formulae Φ.

What can one say about external formulae Φ?

The main aim of our investigation is to make clear the status of "external" forms of separation, replacement, choice and collection in **IST**. Exact formulations will be given below after some preliminaries that we need in order to select the forms of real interest and to discard trivialities. We shall investigate, together with **Sep** and **Repl**, two more axiom schemes, *choice* **Chc** and *collection* **Coll**, and a principle of another kind, *uniqueness* **Uniq**. We consider all of these schemes as *hypotheses in* **IST** because it is not quite clear whether one should accept or reject them.

1.4. Separation.
We wish to select "reasonable" forms of the separation axiom with external formula Φ. First of all let us get rid of trivial variations, trivially true or trivially false in **IST**.

We note that the true (= provable) form is **Sep** with

(1) $X = \{1, 2, ..., n\}$, $n \in \mathbb{N}$ finitely large (= standard)

and with arbitrary Φ, internal or external. One can easily derive **Sep** in this case with the help of the external induction theorem, see Nelson [28] or 2.6 below.

The false (= disprovable) form is **Sep** with

(2) $X = \{1, 2, ..., n\}$, $n \in \mathbb{N}$ **infinitely large (=nonstandard)**

and the formula st x as Φ. (The collection ${}^{\sigma}\mathbb{N}$ of all standard integers is not an internal set in **IST**.)

To recognize the crucial difference between these two cases, let us look at elements of a set of the kind (1) and of the kind (2). Any set of the first kind contains only standard elements but not all standard integers ($n+1 \notin X$). Any set of the second kind contains all standard integers together with some part of nonstandard integers (those less than or equal to n).

The really interesting and non-trivial case may be the "intermediate" collection X, that is, the collection of all standard integers. (This collection is not an internal set in **IST**, of course.) The required property of a set Y in **Sep** takes the form

$$\forall^{\mathrm{st}} x\, [x \in Y \leftrightarrow x \in \mathbb{N} \,\&\, \Phi(x)].$$

Natural generalization to an arbitrary standard set X or to the whole universe $\mathbb{V}$ leads to the following three forms of **Sep**:

$$\begin{array}{llll} \mathbf{Sep}_1: & (\mathrm{st}\ X) & \exists^{\mathrm{st}} Y & \forall^{\mathrm{st}} x\, [x \in Y \leftrightarrow x \in X \,\&\, \Phi(x)]; \\ \mathbf{Sep}_2: & (\mathrm{st}\ X) & \exists Y & \forall^{\mathrm{st}} x\, [x \in Y \leftrightarrow x \in X \,\&\, \Phi(x)]; \\ \mathbf{Sep}_3: & & \exists Y & \forall^{\mathrm{st}} x\, [x \in Y \leftrightarrow \Phi(x)]. \end{array}$$

We write st X in brackets instead of the quantifier $\forall^{\mathrm{st}} X$.

Some comments. $\mathbf{Sep}_1$ is just the standardization axiom. Hence $\mathbf{Sep}_1$ and of course $\mathbf{Sep}_2$ are true in **IST**. Thus only $\mathbf{Sep}_3$ is really new. Note that there is (in **IST**) a set Y such that $\mathbb{S} \subseteq Y$, see 2.9, hence $\mathbf{Sep}_3$ does not produce an immediate contradiction. Further we cannot require the standardness of Y in $\mathbf{Sep}_3$ (let Φ be $x = x$). Also we cannot extend $\mathbf{Sep}_1$ to nonstandard sets X. It is easy to check that $\mathbf{Sep}_2$ for nonstandard sets X is equivalent to $\mathbf{Sep}_3$.

1.5. Replacement.
The approach to the selection of external forms which is taken above is applicable to replacement as well. Of course, **Sep** is **Repl** restricted by the assumption that values of the variable y may be 0 or 1 only. Generalizing to

all standard values, we obtain three variants of **Repl** similar to the corresponding forms of **Sep**. They are as follows:

$$\begin{aligned}
&\mathbf{Repl}_1\text{:}\quad (\mathrm{st}\ X)\quad \forall^{\mathrm{st}}x \in X\ \ \exists!^{\mathrm{st}}y\ \Phi(x,\ y) \to \exists^{\mathrm{st}}\tilde{y}\ \ \forall^{\mathrm{st}}x \in X\Phi(x,\ \tilde{y}(x));\\
&\mathbf{Repl}_2\text{:}\quad (\mathrm{st}\ X)\quad \forall^{\mathrm{st}}x \in X\ \ \exists!^{\mathrm{st}}y\ \Phi(x,\ y) \to\\
&\qquad\qquad\qquad \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x \in X\ [\Phi(x,\ \tilde{y}(x))\ \&\ \mathrm{st}\ \tilde{y}(x)];\\
&\mathbf{Repl}_3\text{:}\qquad\qquad\quad \forall^{\mathrm{st}}x\ \ \exists!^{\mathrm{st}}y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x\ [\Phi(x,\ \tilde{y}(x))\ \&\ \mathrm{st}\ \tilde{y}(x)].
\end{aligned}$$

Generalization to all nonstandard values leads to two more forms:

$$\begin{aligned}
&\mathbf{Repl}_4\text{:}\quad (\mathrm{st}\ X)\quad \forall^{\mathrm{st}}x \in X\ \ \exists!y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x \in X\ \ \Phi(x,\ \tilde{y}(x));\\
&\mathbf{Repl}_5\text{:}\qquad\qquad\quad \forall^{\mathrm{st}}x\qquad \exists!y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x\ \Phi(x,\ \tilde{y}(x)).
\end{aligned}$$

Here Φ is an arbitrary st-$\in$-formula, internal or external, X is an arbitrary standard set, $\exists!^{\mathrm{st}}y$ means: "there is a unique standard y such that ... (and maybe many nonstandard such sets y exist too)".

Of course, "$\tilde{y}$ is a function and $x \in$ domain of $\tilde{y}$" is assumed on the right-hand sides, thus $\tilde{y}$ is defined at all standard x in $\mathbf{Repl}_{3,5}$ and at all standard $x \in X$ in $\mathbf{Repl}_{1,2,4}$ (at least). In fact the additional requirement that $\tilde{y}$ is defined at *all* $x \in X$ should not really strengthen $\mathbf{Repl}_{1,2,4}$. Finally, the additional term st $\tilde{y}(x)$ is not necessary in $\mathbf{Repl}_1$ because the value $\tilde{y}(x)$ is standard provided $\tilde{y}$ and $x \in$ dom $\tilde{y}$ are standard.

The following relations are easily provable in **IST**:

$$\begin{tikzcd}
 & \mathbf{Repl}_1 \arrow[dr] & & \\
\mathbf{Repl}_5 \arrow[r] \arrow[dr] & \mathbf{Repl}_4 \arrow[r] & \mathbf{Repl}_2 & \\
 & \mathbf{Repl}_3 \arrow[rr, leftrightarrow] & & \mathbf{Sep}_3
\end{tikzcd}$$

1.6. Choice.

Within **ZFC** the choice scheme

$$\mathbf{Chc}\text{:}\ \forall x \in X\ \ \exists y\ \ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall x \in X\ \Phi(x,\ \tilde{y}(x))$$

follows from the usual axiom of choice **AC** by replacement and separation schemes. However, such a reasoning is impossible in **IST** if Φ is external. Hence it would be interesting to consider the following analogues of the five forms of **Repl** given above, obtained by deleting the uniqueness on the left-hand sides.

$$\begin{aligned}
&\mathbf{Chc}_1\text{:}\ (\mathrm{st}\ X)\ \ \forall^{\mathrm{st}}x \in X\ \ \exists^{\mathrm{st}}y\ \Phi(x,\ y) \to \exists^{\mathrm{st}}\tilde{y}\ \ \forall^{\mathrm{st}}x \in X\ \Phi(x,\ \tilde{y}(x));\\
&\mathbf{Chc}_2\text{:}\ (\mathrm{st}\ X)\ \ \forall^{\mathrm{st}}x \in X\ \ \exists^{\mathrm{st}}y\ \Phi(x,\ y) \to\\
&\qquad\qquad\qquad \to \exists\tilde{y}\ \forall^{\mathrm{st}}x \in X\ [\Phi(x,\ \tilde{y}(x))\ \&\ \mathrm{st}\ \tilde{y}(x)];\\
&\mathbf{Chc}_3\text{:}\qquad\qquad\ \forall^{\mathrm{st}}x\ \exists^{\mathrm{st}}y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x\ [\Phi(x,\ \tilde{y}(x))\ \&\ \mathrm{st}\ \tilde{y}(x)];\\
&\mathbf{Chc}_4\text{:}\ (\mathrm{st}\ X)\ \ \forall^{\mathrm{st}}x \in X\ \ \exists y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x \in X\ \ \Phi(x,\ \tilde{y}(x));\\
&\mathbf{Chc}_5\text{:}\qquad\qquad\ \forall^{\mathrm{st}}x\ \exists y\ \Phi(x,\ y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x\ \Phi(x,\ \tilde{y}(x)).
\end{aligned}$$

Evidently $\mathbf{Chc}_i \to \mathbf{Repl}_i$, $i = 1, 2, 3, 4, 5$, and the chart as in 1.5 is valid for these five variants of choice:

$$\begin{tikzcd} & \mathbf{Chc}_1 \arrow[dr] & \\ \mathbf{Chc}_5 \arrow[r] \arrow[dr] & \mathbf{Chc}_4 \arrow[r] & \mathbf{Chc}_2 \\ & \mathbf{Chc}_3 & \end{tikzcd}$$

In fact $\mathbf{Chc}_1 \leftrightarrow \mathbf{Repl}_1$ in **IST**, see 2.7 below.

1.7. Replacement and choice with bounded range.
It should be natural to modify $\mathbf{Chc}_i$ and $\mathbf{Repl}_i$ by restricting the variable y to some standard set Y. Modified forms of such a kind will have the letter **B** (bounded) in front of the usual notation. For example,

$$\mathbf{BChc}_1\text{:}\ (\mathrm{st}\ X,\ Y)\ \forall^{\mathrm{st}} x \in X\ \exists^{\mathrm{st}} y \in Y\ \Phi(x,\ y) \to \\ \to \exists^{\mathrm{st}} \tilde{y}\ \forall^{\mathrm{st}} x \in X\ [\tilde{y}\ (x) \in Y \,\&\, \Phi(x,\ \tilde{y}\ (x))],$$

where the variable y is bounded by a previously fixed standard set Y. The same reformulation of "y-bounded" forms can be applied to all the hypotheses $\mathbf{Repl}_i$, $\mathbf{Chc}_i$. "Global" hypotheses are rather senseless in the "y-bounded" form. So let us look at "local" forms. Nelson [28] proved $\mathbf{BChc}_1$ in **IST**, see 2.3 below; thus $\mathbf{BChc}_2$, $\mathbf{BRepl}_1$, $\mathbf{BRepl}_2$ also hold in **IST**. Only the case $i = 4$ remains, that is, the *extension principle* of Lutz and Gose [23], Diener and Stroyan [5] and others:

$$\mathbf{BChc}_4\text{:}\quad (\mathrm{st}\ X,\ Y)\ \forall^{\mathrm{st}} x \in X\ \exists y \in Y\ \Phi(x,\ y) \to \\ \to \exists \tilde{y}\ \forall^{\mathrm{st}} x \in X\ [\tilde{y}\ (x) \in Y \,\&\, \Phi(x,\ \tilde{y}(x))],$$

and $\mathbf{BRepl}_4$ with $\exists!y \in Y$ on the left-hand side.

It is easy to see that $\mathbf{Repl}_4 \to \mathbf{BRepl}_4$ and $\mathbf{Chc}_4 \to \mathbf{BChc}_4$.

1.8. Collection.
The scheme of collection

$$\mathbf{Coll}\text{:}\ \forall X\ \exists Y\ \forall x \in X\ [\exists y\ \Phi(x,\ y) \to \exists y \in Y\ \Phi(x,\ y)]$$

(there are several equivalent forms, see Chang and Keisler [3], Makkai [25]; we choose the most convenient for our use) is an easy consequence of replacement in **ZFC**. Hence **Coll** remains true in **IST** for internal core formulae Φ. Conversely, **Repl** is an easy consequence of **Coll**+**Sep** in **ZFC** without **Repl**. Collection is often used implicitly in set-theoretic arguments. It becomes important and explicitly involved in some modified theories (as the Kelley–Morse theory of classes or the Kripke–Platek theory of admissible sets).

Now we want to "standardize" some (or all) variables X, Y, x, y as we have done above in 1.3–1.5. As matter of fact, **Coll** is provable in **IST** for any st-$\in$-formula Φ (see §2 below). We do not take into consideration some weaker forms. In fact every variant of **Coll** with $\exists Y$ and at least one of the quantifiers $\forall^{st}X$, $\forall^{st}x$, $\exists^{st}y$ is weaker than the ground form and may be omitted.

Certainly every variant of type $\forall X\ \exists^{st}Y$ is false (disprovable) in **IST** (we choose Φ to be $x = y$). Hence the unique reasonable combination for X and Y (except the basic form, of course) is $\forall^{st}X\ \exists^{st}Y$. Let us consider its subvariants. The first of them is $\forall^{st}x\ \exists^{st}y$, that is,

$$\mathbf{Coll}_1\colon\ \forall^{st}X\ \ \exists^{st}Y\ \ \forall^{st}x \in X\ [\exists^{st}y\ \Phi(x,\ y) \to \exists^{st}y \in Y\ \Phi(x,\ y)].$$

It appears that $\mathbf{Coll}_1$ is equivalent to $\mathbf{Repl}_1$ and $\mathbf{Chc}_1$ in **IST** (see 2.7). Subvariants $\forall x\ \exists y$ and $\forall^{st}x\ \exists y$ are false in **IST** (with the formula $\forall^{st}z\ (y \notin z)$. The last subvariant $\forall x\ \exists^{st}y$, that is,

$$\mathbf{Coll}_2\colon\ \forall^{st}X\ \ \exists^{st}Y\ \ \forall x \in X\ \ [\exists^{st}y\ \ \Phi(x,\ y) \to \exists^{st}y \in Y\ \ \Phi(x,\ y)],$$

is also disprovable. Indeed, let H be a finite (nonstandard) set containing all standard sets (see Nelson [28] or 2.9 below for the existence of such a set in **IST**). Now we take a surjection $h : \mathbb{N}$ onto H and consider the formula

$$x \in \mathbb{N}\ \&\ [(\mathrm{st}\ h\ (x)\ \&\ y = h\ (x))\ \text{ or }\ (\neg\ \mathrm{st}\ h\ (x)\ \&\ y = 0)]$$

as $\Phi(x, y)$. We clearly violate $\mathbf{Coll}_2$ by taking $X = \mathbb{N}$.

The evident necessity of some nonstandard set as the parameter in Φ in our argument forces us to a special form of $\mathbf{Coll}_2$, namely $\mathbf{Coll}_2(\mathrm{st}\ \Phi)$. By $\mathbf{Coll}_2(\mathrm{st}\ \Phi)$ we denote $\mathbf{Coll}_2$ restricted by the assumption that only *standard* parameters are allowed in the core formula Φ.

1.9. The uniqueness property.

The hypothesis we consider next is of a slightly different nature from those discussed above. In general the property of uniqueness for some class $\mathbb{K}$ is as follows: any set definable by a formula with parameters from $\mathbb{K}$ belongs to $\mathbb{K}$. Let $\mathbb{K}$ be the class $\mathbb{S}$ of all standard sets. Following Nelson [28], we write

$$\mathbf{Uniq}\colon\ (\mathrm{st}\ \Phi)\ \ \exists! x\ \Phi\ (x) \to \forall x\ [\Phi\ (x) \to \mathrm{st}\ x].$$

The uniqueness hypothesis says that for any st-$\in$-formula $\Phi(x)$ with standard parameters, if only one x exists such that $\Phi(x)$ holds, then this unique x must be standard.

Clearly **Uniq** restricted by the assumption that Φ is internal is an easy consequence of the transfer principle **T**. The real problem appears for external formulae. Note that **T** itself does not generalize to external formulae.

1.10. Comments.

After the selection made above the following list of hypotheses is formed for further study:

$$(3)\qquad \begin{cases} \mathbf{Sep}_3,\ \ \mathbf{Repl}_i\ \text{ and }\ \mathbf{Chc}_i,\ \ i = 1, 2, 3, 4, 5,\ \ \mathbf{BRepl}_4,\ \ \mathbf{BChc}_4, \\ \mathbf{Coll},\ \ \mathbf{Coll}_1,\ \ \mathbf{Coll}_2\,(\mathrm{st}\ \Phi),\ \ \mathbf{Uniq}. \end{cases}$$

The main question is: whether they are true or false in **IST**. Only three answers are possible for any hypothesis of the list:

a) The hypothesis we consider is true, that is, provable in **IST** for an arbitrary internal or external core formula Φ. Only **Coll** is involved in this case.

b) The hypothesis is false, that is, disprovable in **IST** for some core formula. In fact this case has already been eliminated.

c) The undecidability case. Firstly, the hypothesis (for all Φ) is consistent with **IST** (in other words, one cannot disprove it for any Φ). Secondly, the negation of some example (that is, for some formula Φ) is consistent with **IST** too (in other words one cannot prove this example).

Some "fine structure" investigations are natural for the third case. We may look for a type of "simple" core formulae for which one can prove some hypothesis of the list in **IST**, and for a slightly more "complicated" core formula which generates an undecidable example of the hypothesis. Of course, a certain notion of complexity of st-$\in$-formulae must be given.

The second part of the introduction contains formulations of our main results. We begin with the consistency (1.11) and the independence (1.12), then turn to the collection hypothesis and truth definitions (1.13). We next present some "fine structure" results (1.14), "hierarchical" investigations, and some comments on the special role of bounded formulae (1.15). Finally we introduce (1.16) bounded set theory **BST** as a modification of **IST** which allows only those internal sets that are members of standard sets. **BST** makes decidable all the hypotheses we consider. Many related problems remain open; some of them are included in the exposition.

1.11. Consistency.

Let Cons T be the statement saying that the theory T is consistent. By **ZFCI** we denote the theory **ZFC** plus the existence of a strongly inaccessible cardinal.

***Theorem* 1A** [Cons **ZFCI**]. *The union of all the hypotheses of the list* (3) *from* 1.10 *is consistent with* **IST**.

***Theorem* 1B** [Cons **ZFC**]. *The union of* **Uniq**, $\mathbf{Sep}_3$, $\mathbf{Repl}_i$ *and* $\mathbf{Chc}_i$, $i = 2, 3, 4, 5$ (*but not* 1!), $\mathbf{BChc}_4$, $\mathbf{BRepl}_4$ *is consistent with* **IST**.

Theorem 1A is proved by the inner model method. In fact we show (in **IST**) that $\mathbb{V}_\varkappa$ is a model for Theorem 1A provided $\varkappa$ is a standard strongly inaccessible cardinal. ($\mathbb{V}_\varkappa$ is the $\varkappa$th level of von Neumann set hierarchy.) The proof of Theorem 1B is almost similar, but an additional logical trick is involved.

Is it possible to prove Theorem 1A with only Cons **ZFC** assumed? The answer is "no" because the following hypotheses are "transcendentally" strong over **IST**:

$$(4) \qquad \mathbf{Repl}_1,\ \mathbf{Chc}_1,\ \mathbf{Coll}_1,\ \mathbf{Coll}_2\ (\mathrm{st}\ \Phi)$$

(the first three of them are mutually equivalent in **IST**, see 2.7). The following theorem shows the effect we have in mind.

***Theorem* 2.** *Let* **H** *be a hypothesis from the list* (4). *Then* **H** $\rightarrow$ Cons **ZFC** *in* **IST**.

Thus "**IST** + any hypothesis of the list (4)" is not equiconsistent with **IST** by the second incompleteness theorem. However Theorem 2 is not really surprising. It is easy to recognize that the hypotheses (4) are just axioms of infinity (in some sense) for the class $\mathbb{S}$ of standard sets in **IST**. No other hypothesis (3) possesses this property.

The inaccessibility assumption is perhaps too strong for proving Theorem 1A.

Problem 1. Does Theorem 1A remain true when only Cons **KMC** is assumed? **KMC** is the Kelley–Morse impredicative theory of classes, see Kelley [19], Jansana [12], Chang and Keisler [3]. Is **IST** + (4) equiconsistent with **KMC**?

As a matter of fact, **KMC** is interpretable in **IST** + $\mathbf{Repl}_1$ + $\mathbf{Sep}_3$. In the interpretation we have constructed $\mathbf{Sep}_3$ ensures comprehension (or class formation) while $\mathbf{Repl}_1$ guarantees the replacement axiom.

We note that $\mathbf{Chc}_1$, $\mathbf{Chc}_5$, $\mathbf{Coll}_2(\mathrm{st}\ \Phi)$, **Uniq** are maximally strong in the list (3).

Problem 2. Prove that the four hypotheses just mentioned are mutually independent over **IST**.

Problem 3. Show that $\mathbf{Chc}_i$ is not provable in **IST** + $\mathbf{Repl}_i$ for $i = 2, 3, 4, 5$. (It fails for $i = 1$.)

1.12. Independence.
This word means the consistency of the negation.

***Theorem* 3** [Cons **ZFC**]. *All the hypotheses of the list* (3) *except maybe* **Uniq** *are independent of* **IST**.

Problem 4. Prove that **Uniq** is independent of **IST**.

The independence of $\mathbf{BChc}_4$ is especially important. Theorem 3 shows that in general one cannot freely use the extension principle for arbitrary core formulae. Nevertheless, Nelson [28] has shown that $\mathbf{BChc}_4$ holds in **IST** for all ext-bounded (see below) core formulae. In fact this result justifies all the known applications of $\mathbf{BChc}_4$.

To prove Theorem 3 we use the adequate-like ultralimit construction of Nelson [28]. However, our ultralimit differs in an essential way from Nelson's. Namely we use only definable functions to create the ultralimit, as well as a special choice of the ground **ZFC** model.

1.13. Collection and truth definitions.
The main difference between the collection hypothesis **Coll** on the one hand and **Sep**, **Repl** and **Chc** on the other is the provability of the first:

***Theorem* 4** [**IST**]. **Coll** *holds for every internal or external core formula.*

One can easily infer the following:

Corollary. *There is no* st-$\in$-*formula* $\tau(x)$ *with only one free variable* x *such that for any internal formula* $\Phi(x_1, \ldots, x_n)$ *the following is provable in* **IST**:

$$\forall x_1 \ldots \forall x_n \, [\Phi(x_1, \ldots, x_n) \leftrightarrow \tau(\ulcorner \Phi(x_1, \ldots, x_n) \urcorner)].$$

In other words, the truth of internal formulae (with parameters allowed) cannot be expressed in **IST** *by a* st-$\in$-*formula.*

By $\ulcorner \Phi \urcorner$ we denote the finite sequence of (coded) logical symbols and sets (used as parameters) by which Φ is written down.

In fact the nonstandardness of parameters plays a key role in the corollary. Otherwise we obtain the opposite result:

***Theorem* 5.** *There is an external formula* $\tau(x)$ *such that for each internal formula* $\Phi(x_1, \ldots, x_n)$ *the following holds in* IST:

$$\forall^{\mathrm{st}} x_1 \ldots \forall^{\mathrm{st}} x_n \; [\Phi(x_1, \ldots, x_n) \leftrightarrow \tau(\ulcorner \Phi(x_1, \ldots, x_n) \urcorner].$$

Thus the truth of internal formulae with standard parameters can be expressed in **IST**. The result remains true for the wider class of *bounded* parameters.

Theorem 5 is involved in the proof of Theorem 3. In addition, it throws some light on the question about which models of **ZFC** can be extended to a model of **IST**. Indeed, if M is a model of **ZFC** and the standard part of a model ${}^*M$ of **IST**, then the set

$$T_M = \{\ulcorner \Phi \urcorner : \Phi \text{ is an } \in\text{-sentence (without parameters) true in } M\}$$

belongs to M according to Theorem 5 (and the standardization). So "$T_M \in M$" is a necessary condition for M to be extendable to a model of **IST**. We recall that being of the form $\mathbb{V}_\varkappa$ with strongly inaccessible $\varkappa$ is a sufficient condition.

Problem 5. Find reasonable necessary and sufficient conditions for an $\in$-model of **ZFC** to be extendable to a model of **IST**. We suppose that being the "set part" of a model of **KMC** (or maybe some piece of **KMC**) can serve as the condition we look for.

1.14. Fine structure results.
A rather interesting problem concerning Theorem 3 is to find an extremely simple core formula giving the unprovable example. But what should be the

measure of simplicity? Several different approaches to this problem, by taking into a consideration the number and the positions of quantifiers in a formula, are generally possible. A criterion of choosing the best must be coordinated with the **IST** axioms. We choose the definition grounded mainly on the *external* quantifiers $\exists^{st}$, $\forall^{st}$.

Firstly we define the class or *ext-prenex* formulae, that is, formulae of type

$$Q_1^{st} x_1 Q_2^{st} x_2 \dots Q_n^{st} x_n \Psi\,(x_1, x_2, \dots, x_n),$$

where Ψ is an internal formula and each Q is $\exists$ or $\forall$. This class of formulae splits into the hierarchy of classes Σ_n^{st} and Π_n^{st}, defined as usual:

$$(\Sigma_n^{st})\ \exists^{st} x_1\ \forall^{st} x_2\ \exists^{st} x_3 \dots \forall\ (\exists)^{st} x_n\ \ \Psi(x_1, x_2, x_3, \dots, x_n),$$

$$(\Pi_n^{st})\ \forall^{st} x_1\ \exists^{st} x_2\ \forall^{st} x_3 \dots \exists\ (\forall)^{st} x_n \Psi(x_1, x_2, x_3, \dots, x_n),$$

Ψ is an internal formula. (We learned the notation Σ_n^{st}, Π_n^{st} from van den Berg [2].) The simplest cases of non-ext-prenex formulae are $\exists\Pi_2^{st}$ and $\forall\Sigma_2^{st}$, that is, of the form

$$\exists x \forall^{st} y \exists^{st} z\ \Psi \quad \text{and} \quad \forall x\ \exists^{st} y\ \forall^{st} z\ \Psi$$

respectively with an internal Ψ.

This piece of hierarchy is rich enough for the following two "fine structure" theorems to be formulated. These theorems almost cover the case of core formulae with standard parameters.

***Theorem* 6** [**IST**]. (a) *The hypotheses* **Uniq** *and* $\mathbf{Sep}_3$, $\mathbf{Repl}_{1,2,3,4,5}$, $\mathbf{Chc}_{1,2}$, $\mathbf{Coll}_1$, $\mathbf{Coll}_2(\mathrm{st}\ \Phi)$, $\mathbf{BChc}_4$, $\mathbf{BRepl}_4$ *are true for ext-prenex core formulae* Φ *with standard parameters.*

(b) $\mathbf{Chc}_4$ *holds for* Σ_2^{st} *core formulae with standard parameters.*

(c) $\mathbf{Chc}_5$ *holds for* Π_1^{st} *formulae (with arbitrary parameters).*

(There is nothing reasonable for $\mathbf{Chc}_3$ in this series.)

***Theorem* 7** [**Cons ZFC**]. (a) *Any hypothesis from the list* $\mathbf{Sep}_3$, $\mathbf{Repl}_{1,2,3,4,5}$, $\mathbf{Chc}_{1,2,3}$, $\mathbf{Coll}_1$, $\mathbf{Coll}_2(\mathrm{st}\ \Phi)$ *is non-provable in* **IST** *for some "parameter-free"* $\exists\Pi_2^{st}$ *core formula and for some "parameter-free"* $\forall\Sigma_2^{st}$ *core formula.*

(b) $\mathbf{Chc}_4$ *and* $\mathbf{Chc}_5$ *are non-provable in* **IST** *for some "parameter-free"* Π_2^{st} *core formula.*

"*Parameter-free*" formulae are those that do not contain any parameter (see 1.1).

An essential gap remains between the results of Theorems 6 and 7 as regards the hypotheses $\mathbf{Chc}_3$ and $\mathbf{Chc}_5$. Namely, assuming that Theorem 6 is the best possible, the required counterexamples might be given by some internal formula Φ (for $\mathbf{Chc}_3$) and some Σ_1^{st} formula Φ (for $\mathbf{Chc}_5$), both having no parameters, but this is somewhat stronger than Theorem 7 guarantees.

Problem 6. Show that **Chc**$_3$ is non-provable in **IST** for some internal core formula without parameters.

It is not hard to realize that the model of **ZFC** chosen to be the ground model for an **IST**-model where **Chc**$_3$ fails in the manner just mentioned must possess some special properties—for example, the non-existence of a definable well-order.

Certainly Theorem 7 is a more precise form of Theorem 3 with the exception of the hypotheses **BRepl**$_4$ and **BChc**$_4$ of replacement and choice with bounded range. The case of those two hypotheses is covered by the following theorem:

***Theorem* 8** [Cons **ZFC**]. **BChc**$_4$ *is non-provable in* **IST** *for some* Π_2^{st} *core formula.* **BRepl**$_4$ *is non-provable in* **IST** *for some core formulae of types* $\exists\Pi_2^{st}$ *and* $\forall\Sigma_2^{st}$.

Unfortunately all the non-provable examples of **BRepl**$_4$ and **BChc**$_4$ we know need a nonstandard parameter in the core formula.

Problem 7. Show that **BRepl**$_4$ and **BChc**$_4$ are non-provable for a "parameter-free" core formula. (All non-provable instances of **BRepl**$_4$ and **BChc**$_4$ that we know need a nonstandard parameter.)

Problem 8. Does there exist a "parameter-free" st-$\in$-formula $\Phi(k, n)$ such that following is consistent with **IST**: Φ defines a $1-1$ map of ${}^{\sigma}\mathbb{N}$ (= all standard integers) onto a cofinal part of $\mathbb{N}$?

One more open question is connected with the uniqueness property (Problem 3). The following theorem gives a partial answer.

***Theorem* 9** [**IST**]. *For any* st-$\in$-*formula* $\Phi(x)$ *with standard parameters, if there is a unique* x *such that* $\Phi(x)$ *holds, then this unique* x *belongs to some standard set.*

Elements of standard sets will be called *bounded* sets below.

We note that Nelson [28], [29] has shown that **Uniq** is provable for Σ_2^{st}-formulae Φ and **Chc**$_4$ is provable for Σ_2^{st}-formulae of special kind (the external quantifier $\exists^{st}$ must be bounded by some standard set).

Problem 9. Study the case of core formulae with nonstandard parameters.

At least one part of Theorem 7 does not remain true when nonstandard parameters are allowed: **Repl**$_1$ is not provable in **IST** for some internal formula Φ with nonstandard parameters.

1.15. Hierarchy.

The hierarchy theorem for a given Σ/Π hierarchy of formulae claims that every class of the hierarchy contains a formula that is not equivalent (in some sense) to any formula of the dual class at the same level. For the hierarchy

Σ_n^{st}/Π_n^{st} the question is whether there is a Σ_n^{st} formula that is not equivalent (in **IST**) to any Π_n^{st} formula, and conversely.

The natural answer "yes" is easy for $n = 1$. Indeed the formula st x of class Σ_1^{st} (for st $x \leftrightarrow \exists^{st} z\ (x = z)$) is not equivalent to any Π_1^{st} formula. The following theorem ensures the answer "yes" for $n = 2$ too; its proof is essentially more complicated.

***Theorem* 10.** *The Σ_2^{st} formula*

$$\Phi(X) =_{\text{def}} \exists^{st} a\ \forall^{st} b\ (\langle a, b\rangle \in X)$$

is not equivalent in **IST** *to any Π_2^{st} formula $\Psi(X)$.*

The author has no results for $n \geqslant 3$.

Problem 10. Prove the hierarchy theorem for $n \geqslant 3$.

Problem 11. Define a reasonable hierarchy involving all external formulae (not necessarily ext-prenex).

One can significantly simplify some questions concerning the hierarchy by considering those st-$\in$-formulae that contain the standardness predicate st only through the *bounded external quantifiers* $\exists^{st} z \in Z$ and $\forall^{st} z \in Z$, where Z is a standard set. Formulae of such a kind are called *ext-bounded* below.

Nelson [28], [29] has shown that his class of formulae is, logically speaking, rather simple. Indeed, every ext-bounded formula is equivalent to some ext-bounded Σ_2^{st} formula as well as to some ext-bounded Π_2^{st} formula. Hence the hierarchy of ext-bounded formulae contains only four classes: internal, Σ_1^{st}, Π_1^{st} and those that are Σ_2^{st} as well as Π_2^{st} up to the equivalence in **IST**. According to Theorem 10 at least one more level is adjoined by non-ext-bounded formulae.

Thus a large part of Theorem 6 is automatically applicable to ext-bounded formulae paying no regard to the number of quantifiers. Let us recall a result that does not follow directly from Theorem 6: **Chc**$_4$ is true in **IST** for any ext-bounded core formula. This is the saturation theorem of Nelson [29].

Nevertheless one can arrange matters so that every formula will be ext-bounded. The way is to construct something like a type-theoretic superstructure over the **ZFC**/**IST** pair. This theory, the *super*-**IST**, is organized so that every set is placed into some level n, $n \in \mathbb{N}$. The level 0 is the usual **IST** universe of (standard and nonstandard) sets, while every (internal) collection of sets of level n is a set of level $n+1$. For any n the set $\mathbb{V}_n$ of all sets of level n is a set of level $n+1$. Moreover, $\mathbb{V}_n$ is standard according to transfer. Finally, every variable has its own level; therefore every quantifier is bounded by an appropriate standard set $\mathbb{V}_n$. See Nelson [28] for the details. (Our exposition is somewhat different from Nelson's.) Super-**IST** is much stronger than **IST** itself, of course.

1.16. Bounded set theory.
Theorems 1 and 3 show that internal set theory **IST** is strongly incomplete as regards the hypotheses we discuss. We claim that a reason for the incompleteness is connected with the vague behaviour of some very large sets that do not belong to any standard set, or more exactly with insufficient regulation of their behaviour by the **IST** axioms.

To justify this claim we modify **IST** in order to exclude "bad" sets. The modified theory is the *bounded set theory* **BST**, see [16]. We define **BST** as the extension of **ZFC** by transfer **T**, standardization **S**, the weakened form of idealization (*bounded idealization*)

$$\mathbf{BI}\text{: (st } A_0, \text{ int } \Phi) \quad \forall^{\text{stfin}} A \subseteq A_0\ \exists x\ \forall a \in A\ \ \Phi(x, a) \leftrightarrow \\ \leftrightarrow \exists x\ \forall^{\text{st}} a \in A_0\ \ \Phi(x, a)$$

and the *bounded sets axiom*

$$\mathbf{B}\text{: } \forall x\ \exists^{\text{st}} X\ \ (x \in X).$$

Certainly **B** contradicts the full idealization **I**, therefore **I** really must be weakened.

The **BST** has an inner model in **IST**. We say that a set x is *bounded* if and only if $x \in X$ for some standard set X. Thus **B** claims that every set is bounded. (We chose the name "bounded" in [16] bearing in mind the notion of bounded quantifier, which is deeply rooted in logic.) Let us define

$$\mathbb{B} = \{x \in \mathbb{V}\colon x \text{ is bounded}\} = \{x \in \mathbb{V}\colon \exists^{\text{st}} X\ (x \in X)\}$$

(the class of all bounded sets in **IST**). Thus $\mathbb{S} \subseteq \mathbb{B} \subseteq \mathbb{V}$; in fact both inclusions are strict in **IST**.

***Theorem* 11** [**IST**]. $\mathbb{B}$ *is a model of* **BST**.

Thus the theories **ZFC**, **BST**, **IST** are equiconsistent.

Certainly the global forms of the hypotheses we consider, that is, $\mathbf{Sep}_3$, $\mathbf{Repl}_{3,5}$, and $\mathbf{Chc}_{3,5}$, are senseless in **BST** since **BST** does not allow a set containing all the standard sets. Local forms have the affirmative solution:

***Theorem* 12** [**BST**]. *The hypotheses*

$$\mathbf{Repl}_{1,2,4},\ \mathbf{Chc}_{1,2,4},\ \mathbf{Coll},\ \mathbf{Coll}_1,\ \mathbf{Coll}_2\ (\text{st } \Phi)\ \ \textit{and}\ \ \mathbf{Uniq}$$

hold for any internal or external core formula Φ.

One can add the "y-bounded" forms $\mathbf{BRepl}_{1,2,4}$ and $\mathbf{BChc}_{1,2,4}$ to the last theorem because each of them is a consequence of the corresponding $\mathbf{Repl}_i$ or $\mathbf{Chc}_i$ in **BST** as well as in **IST**.

The following theorem serves as a key technical tool in proofs of the two preceding theorems, as well as discovering one more difference between **IST** and **BST** (compare with Theorem 10!).

***Theorem* 13.** *Given a "parameter-free"* st-$\in$-*formula* $\Phi(x_1, \ldots, x_n)$ *with only* $x_1, \ldots, x_n$ *as free variables, there is a* Σ_2^{st} *formula* $\sigma(x_1, \ldots, x_n)$ *of the same kind such that*

$$\forall x_1 \ldots \forall x_n\, [\Phi(x_1, \ldots, x_n) \leftrightarrow \sigma(x_1, \ldots, x_n)]$$

is provable in **BST**.

Theorem 13 together with Theorems 11 and partially 12 is included in Kanovei [16].

Bounded sets resemble standard sets in some respects. For example, Theorem 5 remains true for bounded parameters as well.

Problem 12. Prove Theorems 6 and 10 for core formulae with bounded parameters.

Problem 13. Find a reasonable hypothesis similar to those we consider that is undecidable in **BST**.

1.17. The guide for exposition of the proofs.
The main aim of our paper is to prove Theorems 1A, B to 13.

Section 2 contains some preliminary results, that is, several more or less well-known facts which are extensively used throughout the text. Among them we present two theorems of Nelson [28] concerning uniqueness and $\mathbf{Chc}_4$ for Σ_2^{st} core formulae.

Section 3 is devoted to the key technical result, namely Theorem 3.1, which allows us to bound external quantifiers of ext-prenex formulae, hence serves as a cornerstone in our proofs of Theorems 1A, 1B, 6, 9, 12. As the first application of that theorem, we present the proof of Theorem 6 in §3.

A second application will be the proof of Theorems 1A and 1B in §4, arranged by a careful investigation of inner models of type $\mathbb{V}_\varkappa$. A third application, that is, Theorem 12, is included in §5, where we also present proofs of Theorems 11 and 13; all of them are connected with our theory **BST**.

Section 6 presents the proof of the hierarchy theorem for second level (Theorem 10).

Investigations on the truth definability of internal formulae (Theorem 5 and Theorem 2 as an application) are placed in §7.

The main result of §8 is Theorem 4 about the full collection **Coll** in **IST**. Section 8 also contains proofs of Theorem 9 and the corollary mentioned in 1.13.

Theorem 3 (the independence theorem) will be proved in §9, together with Theorems 7 and 8.

Finally the last §10 explains how the idea of "externalization" might lead to some new and (so the author hopes) interesting problems.

§2. Basic internal set theory

A special feature as well as the power of **IST** is that the postulates upon which Nelson constructed **IST** were not connected with **ZFC** or any other standard theory. Rather their action may be applied to any theory of set-theoretic nature. Nevertheless the ground theory has to be strong enough to discover all the possibilities of nonstandard methods.

The theory **BIST** (that is, basic **IST**) is just sufficiently strong to prove the most useful (for our aims) classical theorems of nonstandard mathematics and is sufficiently weak to be the common part of the mutually contradicting **IST** and **BST**. It contains:

1) all the axioms of **ZFC**;
2) transfer **T** and standardization **S** (usual forms, see 1.1);
3) bounded idealization **BI** as in 1.16.

Note that all the results of this section (except maybe 2.7) are more or less known from the works of Nelson [28], [29], and some others. However, we present them with proofs instead of making references, since the original forms do not cover all the cases we need. The additional reason is that the author has tried to obtain a self-contained exposition.

All the following theorems are proved in **BIST** except additional assertions in 2.9–2.12, where the full idealization is assumed. Theorems are grouped according to what additional principle (that is, **BI**, S or T) plays the key role in the proof. We begin with transfer.

2.1. ***Theorem*** [**BIST**]. *Let $\Phi(x)$ be an internal formula with standard parameters. If there is a unique x such that $\Phi(x)$ holds, then this unique x is standard.*

The proof is evident: use transfer. □

Let us recall a model-theoretic definition. A model M is called an *elementary extension* of a submodel $M' \subseteq M$ (and M' is called an *elementary submodel* of M) if every statement (of some fixed language) true in M' remains true in M. Thus the following theorem claims that the universe $\mathbb{V}$ of all (internal) sets is an elementary extension of the class $\mathbb{S}$ of all standard sets with respect to internal formulae with standard parameters.

2.2. ***Theorem*** [**BIST**]. *Let Φ be an internal statement with standard parameters. Let Φ^{st} be the result of replacing every quantifier $\exists$, $\forall$ in Φ by $\exists^{st}$, $\forall^{st}$ respectively. Then $\Phi \leftrightarrow \Phi^{st}$.*

The proof is carried out by induction on the complexity of Φ. Transfer is used through the induction step $\exists$. □

Of course, what Φ^{st} says is the truth of Φ within $\mathbb{S}$.

Now let us turn to standardization.

2.3. *Theorem* **[BIST].** *Let $\Phi(x, y)$ be a* st-$\in$-*formula (internal or external). For any pair of standard X, Y the following holds:*

$$\forall^{\mathrm{st}} x \in X\ \exists^{\mathrm{st}} y \in Y\ \Phi(x, y) \leftrightarrow \exists^{\mathrm{st}} \tilde{y}\colon X \to Y\ \forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x)).$$

Proof. The direction $\leftarrow$. If a function $\tilde{y}$ and a set $x \in \operatorname{dom} \tilde{y}$ are standard, then the value $\tilde{y}(x)$ is standard too by 2.1.

The direction $\to$. Using S, we obtain a standard W such that

$$\forall^{\mathrm{st}} x \in X\ \forall^{\mathrm{st}} y \in Y\ [\langle x, y\rangle \in W \leftrightarrow \Phi(x, y)].$$

Then $\forall^{\mathrm{st}} x \in X\ \exists^{\mathrm{st}} y \in Y\ (\langle x, y\rangle \in W)$ by the left-hand side, hence $\forall x \in X\ \exists y \in Y\ (\langle x, y\rangle \in W)$ by transfer. The usual axiom of choice gives a function $\tilde{y} : X \to Y$ such that $\langle x, \tilde{y}(x)\rangle \in W$ for each $x \in X$. We may assume that $\tilde{y}$ is standard (apply transfer again). This ends the proof. $\square$

Let us recall that *ext-bounded* formulae are those in which the standardness predicate st occurs only through the bounded external quantifiers $\exists^{\mathrm{st}} z \in Z$, $\forall^{\mathrm{st}} z \in Z$, Z is a standard set. Theorem 2.3 is involved in Nelson's reduction algorithm. This powerful syntactic tool is fairly strong to convert any ext-bounded formula to an ext-bounded formula of type Σ_2^{st} or Π_2^{st}, that is, of the form

$$\exists^{\mathrm{st}} a \in A\ \forall^{\mathrm{st}} b \in B \Phi \ \text{or}\ \forall^{\mathrm{st}} a \in A\ \exists^{\mathrm{st}} b \in B \Phi$$

respectively, where Φ is internal and A, B are standard.

2.4. *Theorem* **[BIST].** *Let $\Phi(x_1, \ldots, x_n)$ be an ext-bounded formula with standard parameters and only $x_1, \ldots, x_n$ free. There is an* ext-*bounded Σ_2^{st} formula $\Psi(x_1, \ldots, x_n)$ (also having only standard parameters) such that the following holds:*

$$\forall x_1 \ldots \forall x_n\ [\Phi(x_1, \ldots, x_n) \leftrightarrow \Psi(x_1, \ldots, x_n)].$$

To be more precise, we claim the following. Let $\Phi(\mathbf{x})$ be a "parameter-free" st-$\in$-formula with the list $\mathbf{x} = x_1, \ldots, x_n$ of free variables. Let $\mathrm{Q}_1^{\mathrm{st}} z_1, \ldots, \mathrm{Q}_k^{\mathrm{st}} z_k$ be the list of all external quantifiers contained in Φ. Let $\Phi'(\mathbf{x}, Z_1, \ldots, Z_k)$ denote the formula obtained by replacing each $\mathrm{Q}_i^{\mathrm{st}} z_i$ by $\mathrm{Q}_i^{\mathrm{st}} z_i \in Z_i$. There is an internal "parameter-free" formula $\varphi(\mathbf{x}, a, b)$ such that

$$\forall^{\mathrm{st}} Z_1 \ldots \forall^{\mathrm{st}} Z_k\ \exists^{\mathrm{st}} A\ \exists^{\mathrm{st}} B\ \forall \mathbf{x}\ [\Phi'(\mathbf{x}, Z_1, \ldots, Z_k) \leftrightarrow$$
$$\leftrightarrow \exists^{\mathrm{st}} a \in A\ \forall^{\mathrm{st}} b \in B\ \varphi(\mathbf{x}, a, b)].$$

Proof. The proof is carried out by induction on the complexity (the number of logical signs) of formulae. Assume that Φ is composed by $\neg$, &, $\exists$ and $\exists^{\mathrm{st}}$ only. (It is evident that other logical functors can be expressed using the four mentioned.)

The step & is evident.

The step $\neg$. We need an ext-bounded Σ_2^{st} formula $\Phi(x)$ that is equivalent to a Π_2^{st} formula $\forall^{st} a \in A\ \exists^{st} b \in B\ \varphi(x, a, b)$ taken as $\Phi(x)$ (where φ is internal, A and B are standard). We denote by $\Psi(x)$ the formula

$$\exists^{st} f \in F \quad \forall^{st} a \in A \ \ \varphi(x, a, f(a)), \quad \text{where} \quad F = {}^A B = \{f\colon A \to B\}.$$

Clearly F is standard by 2.1. The required equivalence $\Phi(x) \leftrightarrow \Psi(x)$ is guaranteed by Theorem 2.3.

The step $\exists^{st}$ is evident since one can collapse two quantifiers of the form $\exists^{st}$ into one $\exists^{st}$ using the pair function.

Finally the step $\exists$. We search for an ext-bounded Σ_2^{st} formula $\Psi(x)$ that is equivalent to the following formula $\Phi(x)$:

$$\exists y\ \exists^{st} a \in A\ \ \forall^{st} b \in B\ \ \varphi(x, y, a, b),$$

φ is internal, A and B are standard. The following formula is as required:

$$\Psi(x) =_{\text{def}} \exists^{st} a \in A\ \ \forall^{st} B' \in P\ \ \exists y\ \ \forall b \in B'\ \ \varphi(x, y, a, b),$$

where $P = \{B' \subseteq B : B' \text{ is finite}\}$ is standard together with B. To see the equivalence $\Phi \leftrightarrow \Psi$, apply **BI**. $\square$

2.5. *Theorem* (external transfinite induction) **[BIST]**. *Let $\Phi(\alpha)$ be a* st-$\in$-*formula and suppose that $\Phi(\alpha)$ holds for some standard ordinal α. Then there is a least standard α such that $\Phi(\alpha)$ holds.*

Proof. Let a standard $\alpha_0 \in \mathrm{Ord}$ be such that $\Phi(\alpha_0)$ holds. (Ord is the class of all ordinals, standard together with nonstandard.) By standardization there is a standard set $A \subseteq \mathrm{Ord}$ such that

$$\forall^{st} \alpha \leqslant \alpha_0\ [\alpha \in A \leftrightarrow \Phi(\alpha)].$$

Then A is non-empty, since $\alpha_0 \in A$. We take the least member α of A; α is standard by 2.1—therefore $\Phi(\alpha)$ holds. $\square$

2.6. *Corollary* [**BIST**]. *Let $\varphi(n)$ be a* st-$\in$-*formula such that*

$$\varphi(0)\ \&\ \forall^{st} n \in \mathbb{N}\, [\varphi(n) \to \varphi(n+1)].$$

Then $\varphi(n)$ holds for all standard integers n.

Proof. Apply 2.5 to the formula $\exists k \leqslant n\ \ \neg\varphi(k)$. $\square$

2.7. *Theorem* [**BIST**]. $\mathbf{Repl_1} \leftrightarrow \mathbf{Chc_1} \leftrightarrow \mathbf{Coll_1}$.

Proof. Firstly we recall the definitions.

$$\begin{aligned}
\mathbf{Repl_1}&: (\mathrm{st}\ X)\ \ \forall^{st} x \in X\ \ \exists!^{st} y\ \ \Phi(x, y) \to \exists^{st} \tilde{y}\ \ \forall^{st} x \in X\ \ \Phi(x, \tilde{y}(x));\\
\mathbf{Chc_1}&: (\mathrm{st}\ X)\ \ \forall^{st} x \in X\ \ \exists^{st} y\ \ \Phi(x, y) \to \exists^{st} \tilde{y}\ \ \forall^{st} x \in X\ \ \Phi(x, \tilde{y}(x));\\
\mathbf{Coll_1}&: \qquad\quad \forall^{st} X\ \ \exists^{st} Y\ \ \forall^{st} x \in X\ \ [\exists^{st} y\ \ \Phi(x, y) \to \exists^{st} y \in Y\ \ \Phi(x, y)].
\end{aligned}$$

Evidently $\mathbf{Chc_1} \to \mathbf{Repl_1}$. Further, $\mathbf{Chc_1}$ easily follows from $\mathbf{Coll_1}$, using Theorem 2.3.

The claim $\mathbf{Repl}_1 \to \mathbf{Coll}_1$ is slightly more complicated. By **ZFC** replacement we can build up the von Neumann hierarchy of classes $\mathbb{V}_\alpha$, $\alpha \in \mathrm{Ord}$. We recall that

$$\mathbb{V}_0 = \varnothing,\ \mathbb{V}_{\alpha+1} = \mathscr{P}(\mathbb{V}_\alpha) = \{X\colon X \subseteq \mathbb{V}\},\ \mathbb{V}_\lambda = \bigcup_{\alpha<\lambda}\mathbb{V}_\alpha$$

for all limit ordinals λ, see Jech [13]. Each set x is a member of some $\mathbb{V}_\alpha$ by the replacement. We denote by $\alpha(x)$ the least $\alpha \in \mathrm{Ord}$ such that $x \in \mathbb{V}_\alpha$. It follows from Theorem 2.1 that $\mathrm{st}\ x \to \mathrm{st}\ \alpha(x)$.

Now we consider a standard X and a st-$\in$-formula $\Phi(x, y)$ and try to derive $\mathbf{Coll}_1$ from $\mathbf{Repl}_1$. For any x, we denote by γ_x the least ordinal among the ordinals $\alpha(y)$, where y is standard and $\Phi(x, y)$ holds, if such standard sets y exist; otherwise let $\gamma_x = 0$. (The definition is correct by Theorem 2.5.) The equality $\gamma = \gamma_x$ can be expressed by a certain st-$\in$-formula. Using $\mathbf{Repl}_1$, we obtain a standard function f such that $\gamma_x = f(x)$ for all standard $x \in X$. The set

$$Y = \bigcup_{x\in X,\, f(x)\in \mathrm{Ord}}\mathbb{V}_{f(x)}$$

is standard (Theorem 2.1 is applied again), hence $\mathbf{Coll}_1$ really holds. $\square$

2.8. ***Theorem*** [**BIST**]. *If X is standard finite, then all $x \in X$ and all $Y \subseteq X$ are standard.*

Proof. The number n of elements of X is a standard integer by Theorem 2.1. Now apply Corollary 2.6 to the formula

$$\forall^{\mathrm{st}}X\ [X \text{ has } \leqslant n \text{ members } \to \forall x \in X\ (\mathrm{st}\ x)\ \&\ \forall Y \subseteq X\ (\mathrm{st}\ Y)].$$

This ends the proof. $\square$

Now we turn to several consequences of idealization. All of them will be presented in two forms: the first form is based on **BIST** only, while the second form needs the full idealization **I** (this is displayed by the "full **I**" in brackets), hence it is proved in **IST**.

2.9. ***Theorem*** [**BIST**]. *Let X be a standard set. Then there is a finite set H such that ${}^\sigma X \subseteq H$.*

(full **I**) *there is a finite set H such that $\mathbb{S} \subseteq H$.*

We recall that $\mathbb{S}$ is the class of all standard sets, while ${}^\sigma X$ is the (external) collection of all standard $x \in X$; thus ${}^\sigma X = X \cap \mathbb{S}$.

Proof. We apply the equivalence

$$\forall^{\mathrm{stfin}}X' \subseteq X\ \exists^{\mathrm{fin}}H\ \forall x \in X'\ (x \in H) \leftrightarrow \exists^{\mathrm{fin}}H\ \forall^{\mathrm{st}}x \in X\ (x \in H)$$

(**BI** for the formula "$x \in H$ & H is finite"). The left-hand side evidently holds (let $H = X'$). So the right-hand side holds as well.

The full **I** case goes similarly. $\square$

The following theorem extends idealization to a wider class of formulae; sometimes this is useful as a technical tool.

2.10. ***Theorem*** **[BIST].** **BI** *holds for all ext-bounded* Π_1^{st} *formulae* Φ.
(full **I**) **I** *holds for all* Π_1^{st} *formulae* Φ.

Proof (the full **I** variant). All we need to prove is

$$\forall^{\mathrm{stfin}} A \ \exists x \ \forall a \in A \ \forall^{\mathrm{st}} z \ \varphi(x, a, z) \leftrightarrow \exists x \ \forall^{\mathrm{st}} a \ \forall^{\mathrm{st}} z \ \varphi(x, a, z)$$

for an internal φ. The implication $\leftarrow$ follows from Theorem 2.8. Let us prove the opposite direction. Changing the positions of a and z and applying **I** to the block $\exists x \ \forall^{\mathrm{st}} z$, we convert the left-hand side to the form

$$\forall^{\mathrm{stfin}} A \ \forall^{\mathrm{stfin}} Z \ \exists x \ \forall a \in A \ \forall z \in Z \ \varphi(x, a, z),$$

and then to $\forall^{\mathrm{stfin}} W \ \exists x \ \forall w \in W \ \psi(x, w)$, where $\psi(x, w)$ is the internal formula $\exists a \ \exists z \ [w = \langle a, z\rangle \ \& \ \varphi(x, a, z)]$.

Hence $\exists x \ \forall^{\mathrm{st}} w \ \psi(x, w)$ holds by **I**. We turn back again to φ and obtain the right-hand side. □

We finish this section with two theorems related more closely to the hypotheses we are studying.

2.11. ***Theorem*** (the uniqueness theorem) **[BIST].** *Let* $\Phi(x)$ *be an* ext-*bounded* Σ_2^{st} *formula with standard parameters. If there is unique* x *such that* $\Phi(x)$ *holds, then this unique* x *is standard.*
(full **I**) *The same is true for all* Σ_2^{st} *formulae, not necessarily* ext-*bounded.*

Proof. Let $\Phi(x)$ be the formula $\exists^{\mathrm{st}} a \in A \ \forall^{\mathrm{st}} b \in B \ \varphi(x, a, b)$, where φ is internal, A and B are standard. For some standard $a \in A$ the only set x satisfying $\forall^{\mathrm{st}} b \in B \ \varphi(x, a, b)$ is the x fixed above. Hence

$$\forall \xi \ [\forall^{\mathrm{st}} b \in B \ \varphi(\xi, a, b) \rightarrow \xi = x],$$

that is, $\forall \xi \ \exists^{\mathrm{st}} b \in B \ [\varphi(\xi, a, b) \rightarrow \xi = x]$. Using **BI**, we obtain a standard finite set $B' \subseteq B$ such that

$$\forall \xi \ [\forall b \in B' \ \varphi(\xi, a, b) \rightarrow \xi = x].$$

We note that all elements of the set B' are standard by Theorem 2.8. Thus x is the unique set satisfying the internal formula $\forall b \in B' \ \varphi(x, a, b)$ which has only standard parameters. So x is standard (Theorem 2.1 is applied).
The full **I** case proceeds similarly. □

2.12. ***Theorem*** **[BIST].** $\mathbf{Chc}_4$ *holds for all* ext-*bounded* Σ_2^{st} *core formulae.* $\mathbf{Chc}_5$ *holds for all* ext-*bounded* Π_1^{st} *core formulae.*
(full **I**) $\mathbf{Chc}_4$ *holds for all* Σ_2^{st} *core formulae with bounded external quantifier* $\exists^{\mathrm{st}}$. $\mathbf{Chc}_5$ *holds for all* Π_1^{st} *core formulae.*

Proof. We begin with

$$\mathbf{Chc}_5\colon \forall^{\mathrm{st}} x\ \exists y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\mathrm{st}} x\ \Phi(x, \tilde{y}(x)),$$

where $\Phi(x, y)$ is a Π_1^{st} formula $\forall^{\mathrm{st}} b \in B\ \varphi(x, y, b)$, B is standard, φ is internal. The left-hand side converts to $\forall^{\mathrm{st}} x\ \exists y\ \forall^{\mathrm{st}} b \in B\ \varphi(x, y, b)$, hence

$$\forall^{\mathrm{stfin}} X'\ \forall x \in X'\ \exists y\ \forall^{\mathrm{st}} b \in B\ \varphi(x, y, b)$$

by Theorem 2.8. Now the key point. If X' is a standard finite set, then it follows from 2.6 by induction on the number of elements of X' that

$$\forall x \in X'\ \exists y\ \psi(x, y) \to \exists \tilde{y}\ \forall x \in X'\ \psi(x, \tilde{y}(x))$$

for all st-$\in$-formulae ψ. Thus

$$\forall^{\mathrm{stfin}} X'\ \exists \tilde{y}\ \forall x \in X'\ [\forall^{\mathrm{st}} b \in B\ \varphi(x, \tilde{y}(x), b)].$$

Finally we apply Theorem 2.10 to the ext-bounded formula [...] and obtain the right-hand side of $\mathbf{Chc}_5$.

Now we consider

$$\mathbf{Chc}_4\colon \forall^{\mathrm{st}} x \in X\ \exists y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x)),$$

where Φ is $\exists^{\mathrm{st}} a \in A\ \varphi(x, y, a)$, A is a standard set, φ is an ext-bounded Π_1^{st} formula. Changing the places of y and a on the left-hand side and using 2.3, we obtain a standard function $\tilde{a} : X \to A$ such that $\forall^{\mathrm{st}} x \in X\ \exists y\ \varphi(x, y, \tilde{a}(x))$ holds.

Finally we apply the $\mathbf{Chc}_5$ just derived to the formula φ and find a function $\tilde{y}$ such that $\forall^{\mathrm{st}} x \in X\ \varphi(x, \tilde{y}(x), \tilde{a}(x))$. The right-hand side of $\mathbf{Chc}_4$ holds, since $\tilde{a}(x)$ is standard provided x and $\tilde{a}$ are standard.

The "full I" case does not differ essentially. □

§3. External quantifiers limitation theorem

This section is devoted to a theorem which says that one can bound external quantifiers of an ext-prenex formula by standard sets. So ext-prenex formulae are transformable into ext-bounded and then into Σ_2^{st} by Theorem 2.4. This extends the results of Theorems 2.11, 2.12 to more general cases than those theorems directly provide. As the first application we prove Theorems 6 and 10 at the end of this section. Two additional applications will appear in §§4 and 5.

To begin with we define

$$\mathscr{P}^n(X) = \mathscr{P}(\mathscr{P}(\mathscr{P}(\ldots \mathscr{P}(X) \ldots))) \ (n \text{ times } \mathscr{P});\ \mathscr{P}(X) = \{Y\colon Y \subseteq X\}.$$

If $\theta = \operatorname{card} X$ (the cardinality of X), then let $\exp^n(\theta) = \operatorname{card} \mathscr{P}^n(X)$; so $\exp^0(\theta) = \theta$ and $\exp^1(\theta) = 2^\theta$.

We recall that *bounded* sets are those that are members of standard sets. We define the *order* of a bounded set x to be the least (standard) cardinal $\varkappa$

such that x is a member of some standard set X of cardinality $\varkappa$. For example, all integers (standard and nonstandard) are bounded of order $\aleph_0$.

3.1. ***Theorem*** **[BIST].** *Let $\varphi(x_1, ..., x_m, z_1, ..., z_n)$ be an internal formula with only $x_1, ..., x_m, z_1, ..., z_n$ free and with bounded parameters, and let $\mathbf{Q}_1, ..., \mathbf{Q}_n$ be a string of quantifiers $\forall$, $\exists$. Let θ' be* max *of orders of nonstandard parameters of φ, $\theta = \max\{\theta', \operatorname{card} X\}$, and $\lambda = \exp^n\theta$.*

Then for every standard X there are standard sets $Z_1, ..., Z_n$, each of cardinality $\leqslant\lambda$, such that for all k, $0 \leqslant k < n$, and all standard $z_i \in Z_i$, $1 \leqslant i \leqslant k$, and all (not necessarily standard) $x_1, ..., x_m \in X$ the following holds:

$$\mathbf{Q}_{k+1}^{\mathrm{st}} z_{k+1} \ldots \mathbf{Q}_n^{\mathrm{st}} z_n \varphi(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n, x_1, \ldots, x_m) \leftrightarrow$$
$$\leftrightarrow \mathbf{Q}_{k+1}^{\mathrm{st}} z_{k+1} \in Z_{k+1} \ldots \mathbf{Q}_n^{\mathrm{st}} z_n \in Z_n \varphi(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n, x_1, \ldots, x_m).$$

Proof. Let us write $\mathbf{x}$ instead of $x_1, ..., x_m$; $\mathbf{x} \in X^m$ We may suppose that φ contains no nonstandard parameters, for if not, we replace each nonstandard parameter (hence bounded of order θ) by a free variable ranging over a set of cardinality $\leqslant\theta$, add these new variables to the list $\mathbf{x}$, and add corresponding sets of cardinality $\leqslant\theta$ to X. Thus assume that φ contains only standard parameters.

Now we define for arbitrary $z_1, ..., z_n$

$$Y[z_1, \ldots, z_n] = \{\mathbf{x} \in X^m \colon \varphi(z_1, \ldots, z_n, \mathbf{x})\},$$

and then for all k, $1 \leqslant k < n$, and for all $z_1, ..., z_n$ define

$$Y[z_1, \ldots, z_k] = \{Y[z_1, \ldots, z_k, z_{k+1}] \colon z_{k+1} \in \mathbb{V}\}.$$

The definition is correct: each $Y[z_1, ..., z_k]$ is a "legal" set (in the universe $\mathbb{V}$), since $Y[z_1, ..., z_k] \subseteq \mathscr{P}^{n-k}(X^m)$.

Finally we put $Y[\,] = \{Y[z_1] : z_1 \in \mathbb{V}\}$ for $k = 0$; $Y[\,] \subseteq \mathscr{P}^n(X^m)$.

***Assertion* 1.** *There are standard sets $Z_1, ..., Z_n$, each of cardinality $\leqslant\lambda$, such that the following holds:*

(5) *for all k, $0 \leqslant k < n$, all $z_i \in Z_i$, $1 \leqslant i \leqslant k$,*

$$\forall z'_{k+1}\, \exists z_{k+1} \in Z_{k+1}\ (Y[z_1, \ldots, z_k, z_{k+1}] = Y[z_1, \ldots, z_k, z'_{k+1}]).$$

The construction of Z_k proceeds by induction on k. To define Z_1 we notice that $\operatorname{card} Y[\,] \leqslant \lambda$, since $Y[\,] \subseteq \mathscr{P}^n(X^m)$. For a set $Y \in Y[\,]$ we denote by α_Y the least ordinal α such that $Y = Y[z_1]$ for some $z_1 \in \mathbb{V}_\alpha$. By **ZFC** replacement there is a function $f : Y[\,] \to \mathrm{Ord}$ such that $\alpha_Y = f(Y)$ for all $Y \in Y[\,]$. Hence by **ZFC** axiom of choice there is a function $\zeta : Y[\,] \to \mathbb{V}$ such that $\zeta(Y) \in \mathbb{V}_{f(Y)}$ and $Y = Y[\zeta(Y)]$ for all $Y \in Y[\,]$. Now let $Z_1 = \{\zeta(Y) : Y \in Y[\,]\}$.

To construct Z_{k+1} (provided that the sets $Z_1, ..., Z_k$ each of cardinality $\leqslant\lambda$ are already defined) we consider the set

$$W = \{\langle z_1, \ldots, z_k, Y[z_1, \ldots, z_k, z_{k+1}]\rangle : z_1 \in Z_1 \& \ldots \& z_k \in Z_k \& z_{k+1} \in \mathbb{V}\}.$$

Clearly $W \subseteq Z_1 \times \ldots \times Z_k \times \mathscr{P}^{n-k}(X^m)$, therefore W has cardinality $\leqslant\lambda$. As above there is a function $\zeta : W \to \mathbb{V}$ such that

$$Y[z_1, \ldots, z_k, \zeta(z_1, \ldots, z_k, Y)] = Y$$

whenever $z_i \in Z_i$, $1 \leqslant i \leqslant k$, and $Y \in Y[z_1, ..., z_k]$. We define

$$Z_{k+1} = \{\zeta(z_1, \ldots, z_k, Y) : \langle z_1, \ldots, z_k, Y\rangle \in W\}.$$

Finally we note that the condition (5) is expressible by an internal formula with standard parameters because φ is a formula of such a kind. So, by transfer, the sets Z_k may be chosen to be standard. This completes the proof of the claim. □

What is more, transfer again allows us to rewrite the condition (5) as follows:

(5^{st}) *for all* k, $0 \leqslant k < n$, *and all standard* $z_i \in Z_i$, $1 \leqslant i \leqslant k$,

$$\forall^{\mathrm{st}} z'_{k+1}\ \exists^{\mathrm{st}} z_{k+1} \in Z_{k+1}\ (Y[z_1, \ldots, z_k, z_{k+1}] = Y[z_1, \ldots, z_k, z'_{k+1}]).$$

Now we turn directly to the proof of Theorem 3.1. The sets Z_k are already constructed, so only the equivalence of Theorem 3.1 remains to be proved. Let us denote its left-hand side and right-hand side by $\mathscr{L}_k(z_1, ..., z_k, \mathbf{x})$ and $\mathscr{R}_k(z_1, ..., z_k, \mathbf{x})$ respectively, and consider an auxiliary formula

$$\mathrm{Q}^{\mathrm{st}}_{k+1} Y_{k+1} \in Y_k\ \mathrm{Q}^{\mathrm{st}}_{k+2} Y_{k+2} \in Y_{k+1} \ldots \mathrm{Q}^{\mathrm{st}}_n Y_n \in Y_{n-1}\ (\mathbf{x} \in Y_n).$$

We denote this formula by $\Psi_k(Y_k, \mathbf{x})$.

Assertion 2. $\mathscr{R}_k(z_1, ..., z_k, \mathbf{x}) \leftrightarrow \mathscr{L}_k(z_1, ..., z_k, \mathbf{x}) \leftrightarrow \Psi_k(Y[z_1, ..., z_k], \mathbf{x})$ *for all* $\mathbf{x} \in X^m$ *and all standard* $z_i \in Z_i$, $1 \leqslant i \leqslant k$.

Proof. We proceed by reverse induction on k.

The case $k = n$ (the base of induction). In the absence of quantifiers all is clear:

$$\mathscr{R}_n(z_1, \ldots, z_n, \mathbf{x}) \leftrightarrow \mathscr{L}_n(z_1, \ldots, z_n, \mathbf{x}) \leftrightarrow \varphi(z_1, \ldots, z_n, \mathbf{x}) \leftrightarrow$$
$$\leftrightarrow \mathbf{x} \in Y[z_1, \ldots, z_n] \leftrightarrow \Psi_n(Y[z_1, \ldots, z_n], \mathbf{x}).$$

The step from $k+1$ to k; $1 \leqslant k < n$. Suppose that $\mathrm{Q}^{\mathrm{st}}_{k+1}$ is $\exists^{\mathrm{st}}$, and put $Y_k = Y[z_1, ..., z_k]$.

We prove that $\mathscr{L}_k \to \Psi_k$. Let standard z_{k+1} be such that $\mathscr{L}_{k+1}(z_1, ..., z_k, z_{k+1}, \mathbf{x})$ holds. Then $\Psi_{k+1}(Y_{k+1}, \mathbf{x})$ holds too for $Y_{k+1} = Y[z_1, ..., z_k, z_{k+1}]$ (by the induction hypothesis). We note that $Y_{k+1} \in Y_k$ and Y_{k+1} is standard because $z_1, ..., z_k, z_{k+1}$ are standard. Therefore $\Psi_k(Y_k, \mathbf{x})$ is true.

We prove that $\Psi_k \to \mathcal{R}_k$. Let standard $Y_{k+1} \in Y_k$ be such that $\Psi_{k+1}(Y_{k+1}, \mathbf{x})$ holds. It follows from the definition of Y_k by transfer that there is a standard set z_{k+1} such that $Y_{k+1} = Y[z_1, ..., z_k, z_{k+1}]$. What is more, one may choose such a standard z_{k+1} as a member of Z_{k+1}; this is guaranteed by (5^{st}). Thus $\mathcal{R}_{k+1}(z_1, ..., z_k, z_{k+1}, \mathbf{x})$ by the induction hypothesis. Therefore $\mathcal{R}_k(z_1, ..., z_k, \mathbf{x})$ holds too because $z_{k+1} \in Z_{k+1}$.

The assertion $\mathcal{R}_k \to \mathcal{L}_k$ is evident.

The case $\mathbf{Q}^{\mathrm{st}}_{k+1}$ is $\forall^{\mathrm{st}}$ does not differ from the one we have just considered.

This ends the proof of Theorem 3.1. □

3.2. ***Corollary*** [**BIST**]. *Let X be a standard set. Suppose that $\Phi(x, y)$ is an ext-prenex formula with bounded parameters and fewer than n external quantifiers. Let θ' be* max *of orders of all the nonstandard parameters of Φ, $\theta = \max\{\theta', \operatorname{card} X\}$, and $\lambda = \exp^n\theta$. Then there is a standard set Y of cardinality $\leqslant\lambda$ such that*

$$\forall x \in X\ [\exists^{\mathrm{st}}y\ \Phi(x, y) \to \exists^{\mathrm{st}}y \in Y\ \Phi(x, y)].$$

Proof. Assume that Φ takes the form

$$\mathrm{Q}_2^{\mathrm{st}}z_2\ \ \mathrm{Q}_3^{\mathrm{st}}z_3 \ldots \mathrm{Q}_n^{\mathrm{st}}z_n\ \ \varphi(y, z_2, z_3, \ldots, z_n, x),$$

φ is internal, and every quantifier $\mathrm{Q}_i^{\mathrm{st}}$ is $\exists^{\mathrm{st}}$ or $\forall^{\mathrm{st}}$. It will be convenient to rename the variable y by z_1. We define $\mathrm{Q}_1^{\mathrm{st}}$ as $\exists^{\mathrm{st}}$. Applying Theorem 3.12 for $k = 0$ (to bound all quantifiers) and then for $k = 1$ (to re-bound all quantifiers except Q_1) we obtain the required set $Y = Z_1$. □

3.3. Proof of Theorem 6. Part (a) asserts that **Uniq**, $\mathbf{Repl}_{1,2,3,4,5}$, $\mathbf{Chc}_{1,2}$, $\mathbf{BRepl}_4$, $\mathbf{BChc}_4$, $\mathbf{Coll}_1$ and $\mathbf{Coll}_2$(st Φ) are true in **IST** for ext-prenex core formulae with standard parameters.

$\mathbf{Coll}_2$(st Φ) follows immediately from Corollary 3.2. Further $\mathbf{Coll}_1$ follows from $\mathbf{Coll}_2$(st Φ) provided only standard parameters are allowed. Hence $\mathbf{Chc}_1$ and $\mathbf{Repl}_1$ hold as well by Theorem 2.7. Of course, $\mathbf{Chc}_2$ follows from $\mathbf{Chc}_1$ and the same is true for **Repl**. Finally $\mathbf{Repl}_{3,4}$ follow from $\mathbf{Repl}_5$ and $\mathbf{BRepl}_4$ from $\mathbf{BChc}_4$. So the only things to prove are **Uniq**, $\mathbf{Repl}_5$ and $\mathbf{BChc}_4$.

To prove **Uniq**, let $\Phi(x)$ be an ext-prenex formula with standard parameters and suppose that there is a unique x such that $\Phi(x)$ holds. Applying Theorem 9 (which will be proved later in §6), we conclude that x is bounded. Hence x is a member of a standard set X. We may assume that Φ is a Σ_2^{st} formula (otherwise use 3.1 and 2.4). The result follows from Theorem 2.11.

We verify $\mathbf{Repl}_5$ for ext-prenex Φ with standard parameters:

$$\mathbf{Repl}_5\text{:}\ \ \forall^{\mathrm{st}}x\ \ \exists!y\ \ \Phi(x, y) \to \exists\tilde{y}\ \ \forall^{\mathrm{st}}x\ \ \Phi(x, \tilde{y}(x)).$$

It follows from the **Uniq** just proved that the unique y such that $\Phi(x, y)$ holds is standard whenever x is standard. Hence $\forall^{\mathrm{st}}x\ \exists!^{\mathrm{st}}y\ \Phi(x, y)$ follows from the left-hand side of $\mathbf{Repl}_5$. We denote by Ψ the (internal) formula obtained from Φ by deleting the "st" superscript from all quantifiers. Then $\Phi(x,y) \leftrightarrow \Psi(x, y)$

for all standard x, y by transfer and ext-prenexity of Φ. One can derive successively from the left-hand side of **Repl**$_5$ the next three assertions:

$$(6) \qquad \forall^{st} x\ \exists!^{st} y\ \Psi(x, y);\ \forall^{st} x\ \exists! y\ \Psi(x, y);\ \forall x\ \exists! y\ \Psi(x, y).$$

There is a set H such that $\mathbb{S} \subseteq H$ (see 2.9). Then a function $\tilde{y}$ exists such that $\tilde{y}(x)$ is defined and $\Psi(x, \tilde{y}(x))$ holds for all $x \in H$. Comparing the first and the second assertions, we see that $\tilde{y}(x)$ is standard for all standard x. Hence we may go back again to Φ and obtain the right-hand side of **Repl**$_5$.

Finally we consider

$$\textbf{BChc}_4\text{: } (\mathrm{st}\ X,\ Y)\ \forall^{st} x \in X\ \exists y \in Y\ \Phi(x, y) \to \\ \to \exists \tilde{y}\ \forall^{st} x \in X\ [\Phi(x, \tilde{y}(x))\ \&\ \tilde{y}(x) \in Y],$$

where Φ is an ext-prenex formula with standard parameters. Note that the variables x and y have standard domains X and Y. Thus one can convert Φ to Σ_2^{st} form by applying 3.1 and 2.4. Then we use 2.12.

(b) We are going to prove

$$\textbf{Chc}_4\text{: } \quad (\mathrm{st}\ X)\ \forall^{st} x \in X\ \exists y\ \Phi(x, y) \to \exists \tilde{y}\ \forall^{st} x \in X\ \Phi(x, \tilde{y}(x)),$$

where $\Phi(x, y)$ is $\exists^{st} a\ \forall^{st} b\ \varphi(x, y, a, b)$, φ is an internal formula with standard parameters. Changing the places of the variables y and a on the left-hand side and using idealization, we convert the left-hand side of **Chc**$_4$ to the form

$$\forall^{st} x \in X\ \exists^{st} a\ [\forall^{stfin} B\ \exists y\ \forall b \in B\ \varphi(x, y, a, b)].$$

Note that the expression in square brackets is ext-prenex. Thus by Corollary 3.2 there is a standard set A such that

$$\forall^{st} x \in X\ \exists^{st} a \in A\ [\forall^{stfin} B\ \exists y\ \forall b \in B\ \varphi(x, y, a, b)]$$

holds. Thus $\forall^{st} x \in X\ \exists y\ \exists^{st} a \in A\ \forall^{st} b\ \varphi(x, y, a, b)$. Here the variable a ranges over a standard set. An appeal to Theorem 2.12 (I) completes the proof.

Part (c) of Theorem 6 has already been proved in §2. □

§4. Consistency

This section contains proofs of Theorems 1A, 1B. All we need is the following "inner model" theorem:

4.1. ***Theorem*** [IST]. *Let $\varkappa$ be a standard infinite cardinal such that $\mathbb{V}_\varkappa$ is a model of* **ZFC**. *Then $\mathbb{V}_\varkappa$ is a model of* **IST** *plus* **Sep**$_3$ + **Repl**$_{2,3,4,5}$ + **Chc**$_{2,3,4,5}$ + + **BRepl**$_4$ + **BChc**$_4$ + **Uniq**.

If moreover $\varkappa$ is strongly inaccessible, then **Repl**$_1$, **Chc**$_1$, **Coll**$_1$ *and* **Coll**$_2$(st φ) *also hold in $\mathbb{V}_\varkappa$.*

Proof. We define $V = \mathbb{V}_\varkappa$. A *V-bounded formula* means any st-$\in$-formula that contains external quantifiers only of type $\exists^{st} v \in V$, $\forall^{st} v \in V$. Certainly any V-bounded formula is ext-bounded. Note that any claim about the truth within V can be expressed by a V-bounded formula.

We first prove transfer in V, that is,

$$\exists x \in V\ \Phi(x) \to \exists^{\mathrm{st}} x \in V\ \Phi(x),$$

where Φ is an internal formula with standard parameters (only parameters from V are essential here, however the result remains true for arbitrary standard parameters). Applying the **IST** transfer to the formula $\Phi(x)\ \&\ x \in V$, we obtain the required implication. Similar reasonings provide the verification of **I** and **S**.

So far as the additional hypotheses are concerned, it suffices to prove only **Uniq**, $\mathbf{Chc}_5$, $\mathbf{Chc}_1$ and $\mathbf{Coll}_2$(st Φ) within V. (The rest of the hypotheses follow from them, partially by Theorem 2.7.) The key point is that each formula relativized to V is in fact ext-bounded (with the set V), hence Σ_2^{st}. Thus one may apply the results of §2.

1. Uniq. Let $\Phi(x)$ be a V-bounded st-$\in$-formula with standard parameters. Assume that $\exists! x \in V\ \Phi(x)$. The unique x given by the formula $\Phi(x)$ is standard by Theorems 2.4 (applied to the formula $\Phi(x)\ \&\ x \in V$) and 2.11.

2. We prove the following:

$$\mathbf{Chc}_5\text{: } \forall^{\mathrm{st}} x \in V\ \exists y \in V\ \Phi(x,\ y) \to \exists \tilde{y} \in V\ \forall^{\mathrm{st}} x \in V\ \Phi(x,\ \tilde{y}(x)),$$

where Φ is a V-bounded st-$\in$-formula. Again by Theorems 2.4 and 2.12 there is a function $\tilde{y}$ such that

$$\forall^{\mathrm{st}} x \in V\ [\tilde{y}\ (x) \in V\ \&\ \Phi(x,\ \tilde{y}\,(x))]. \tag{7}$$

However it is not still clear that $\tilde{y} \in V$. To overcome this difficulty, let H be a finite set containing all standard members of V (see 2.9 for the existence of such a set H). We define

$$D = H \cap V \cap \operatorname{dom} \tilde{y};\ \ E = \{x \in D\colon\ \tilde{y}\ (x) \in V\};\ \ f = \tilde{y} \restriction E.$$

Evidently f is a function with finite domain $E \subseteq V$ and range also $\subseteq V$. Hence $f \in V$ by the finiteness. Finally the property (7) holds for f provided it is true for $\tilde{y}$.

3. Now, assuming the strong inaccessibility of $\varkappa$, we prove that

$$\mathbf{Chc}_1\text{: } \forall^{\mathrm{st}} x \in X\ \exists^{\mathrm{st}} y \in V\ \Phi(x,\ y) \to \exists^{\mathrm{st}} \tilde{y} \in V\ \forall^{\mathrm{st}} x \in X\ \Phi(x,\ \tilde{y}\,(x)),$$

where $X \in V$ is standard and Φ is a st-$\in$-formula. Using Theorem 2.3, we get a function $\tilde{y} : X \to V$ such that $\forall^{\mathrm{st}} x \in X\ \Phi(x, \tilde{y}(x))$ holds. The assumed inaccessibility of $\varkappa$ confirms that $\tilde{y} \in V$.

4. Again assume the inaccessibility of $\varkappa$ and prove $\mathbf{Coll}_2$(st Φ) for a V-bounded st-$\in$-formula $\Phi(x, y)$ with standard parameters. Let $X \in V$ be standard. By Theorem 2.4 there is a Σ_2^{st} formula $\Psi(x, y)$ which is equivalent to the formula $\Phi(x, y)$ & $y \in V$. Then, using Corollary 3.2, we obtain a standard set $Y \subseteq V$ such that card $Y \leqslant \exp^3(\theta)$, where θ = card X (clearly $\theta < \varkappa$), and

$$\forall x \in X \quad [\exists^{\mathrm{st}} y \in V\, \Phi(x, y) \to \exists^{\mathrm{st}} y \in Y \cap V\, \Phi(x, y)].$$

Note that $\exp^3(\theta) < \varkappa$ by the inaccessibility, hence $Y \in V$.

This completes the proof of Theorem 4.1. □

4.2. The proof of Theorem 1B. Theorem 1A follows immediately from 4.1 because **ZFC** and **IST** remain equiconsistent when we add the existence of a strongly inaccessible cardinal to both of them.

We consider Theorem 1B. Of course, the existence of a cardinal $\varkappa$ such that $\mathbb{V}_\varkappa$ is a **ZFC** model is outside **IST**. However, one may enlarge **IST** by a special constant $\varkappa$ and by the additional axiom "$\varkappa$ is a standard cardinal" and the list of axioms of type "A holds in $\mathbb{V}_\varkappa$" for all the **ZFC** axioms A. We denote the enlargment by $\mathbf{IST}_\varkappa$. Of course, $\mathbf{IST}_\varkappa$ is not the same as adding to **IST** the single axiom which says that $\mathbb{V}_\varkappa$ is a model of **ZFC**. In fact one can easily show that $\mathbf{IST}_\varkappa$ is a conservative (hence equiconsistent) extension of **IST**.

Moreover, $\mathbf{IST}_\varkappa$ is strong enough to prove that all the **IST** axioms as well as all the hypotheses of the list $\mathbf{Sep}_3$, $\mathbf{Repl}_{2,3,4,5}$, $\mathbf{Chc}_{2,3,4,5}$, $\mathbf{BRepl}_4$, $\mathbf{BChc}_4$, **Uniq** of Theorem 4.1 hold in $\mathbb{V}_\varkappa$. This reasoning is completely analogous to the one presented below and we leave it to the reader.

§5. Bounded set theory

This section is devoted to Theorems 11, 12, 13, which concern bounded set theory **BST**. We recall that **BST** contains all the **ZFC** axioms together with transfer **T**, standardization **S**, bounded idealization

BI: (st A_0, int Φ) $\forall^{\mathrm{st\,fin}} A \subseteq A_0\, \exists x\, \forall a \in A\ \Phi(x, a) \leftrightarrow \exists x\, \forall^{\mathrm{st}} a \in A_0\ \Phi(x, a)$

(A_0 is standard, Φ is an internal formula), and the bounded sets axiom **B** : $\forall x\, \exists^{\mathrm{st}} X\, (x \in X)$. We recall that bounded sets are those that belong to a standard set.

Let $\mathbb{B}$ denote the class of all bounded sets and bd denote the formula of boundedness, that is,

$$x \in \mathbb{B} \leftrightarrow \mathrm{bd}\, x \leftrightarrow \exists^{\mathrm{st}} X\, (x \in X).$$

Clearly, $\mathbb{S} \subseteq \mathbb{B}$, so a standard set is bounded (for if x is standard, then $X = \{x\}$ is standard too by 2.1). Hence $\mathbb{S} \subseteq \mathbb{B} \subseteq \mathbb{V}$. Both inclusions are strict in **IST**. Indeed, firstly, any nonstandard integer belongs to $\mathbb{B}$ but not to $\mathbb{S}$; secondly, a set H such that $\mathbb{S} \subseteq H$ (see 2.9) is not bounded.

Sometimes it is useful to know that a bounded set is the same as a *subset* of a standard set. Indeed,

$$x \in X \leftrightarrow x \subseteq \bigcup X \quad \text{and} \quad x \subseteq Y \leftrightarrow x \in \mathscr{P}(Y),$$

where the sets $\bigcup X$, $\mathscr{P}(Y)$ are standard provided X, Y are standard.

To compare the possibilities of **IST** and **BST** as a basis for treating nonstandard mathematics in different fields, we note that the fact that **BST** contains a smaller piece of idealization does not have any influence on most of the applications. Indeed, any research branch of mathematics has its own "universe", that is, a certain standard set large enough to contain all the sets that might be considered within the branch. All the applications of idealization within the chosen branch are just of the kind **BI** rather than full **I**; the set A_0 serves as the "universe" mentioned above. Therefore **BST** is not really weaker than **IST** as a tool for nonstandard arguments. Nevertheless **BST** is much more complete than **IST** as regards the hypotheses we study, as Theorem 12 shows.

To close this short metamathematical digression, we notice that the bounded sets axiom is sometimes involved as a definition of internality in a study of nonstandard superstructures, see Lindstrøm [21].

5.1. *Proof of Theorem* 11. Let φ be any st-$\in$-formula. By φ^{bd} we denote the formula obtained by replacing every internal (see 1.1) quantifier $\exists$ or $\forall$ in φ by $\exists^{\mathrm{bd}}$ or $\forall^{\mathrm{bd}}$ ("there is a bounded ...", "for all bounded ..."). Clearly the truth in $\mathbb{B}$ of a st-$\in$-formula φ with bounded parameters is equivalent to the truth of φ^{bd} in $\mathbb{V}$. It suffices to show the following: if A is an axiom of **BIST**, then A^{bd} is provable in **IST**. The first of the **BST** axioms we consider is transfer. We prove that

$$\exists^{\mathrm{bd}} x\ \Phi^{\mathrm{bd}}(x) \to \exists^{\mathrm{st}} x\ \Phi^{\mathrm{bd}}(x)$$

for an internal formula Φ with standard parameters. Here one cannot immediately refer to the **IST** transfer because Φ^{bd} is not an internal formula. Nevertheless the following lemma allows us to delete the superscript bd from Φ and therefore completes the proof of transfer.

5.2. ***Lemma*** **[IST].** $\Psi \leftrightarrow \Psi^{\mathrm{bd}}$ *for all internal formulae* Ψ *having bounded parameters.*

Proof. The proof is by induction on the number of logical signs in Ψ. As usual, only the step $\exists$ needs special consideration. We prove that

$$\exists z\ \psi(z) \to \exists^{\mathrm{bd}} z\ \psi(z)$$

for all internal ψ with bounded parameters. The "ordered n-tuple" **ZFC** function reduces the case of many parameters to the case of a single parameter. Thus let ψ contain the single parameter p_0. We fix a standard set P such that $p_0 \in P$. The preceding formula takes the form

$$\exists z\ \psi(z, p_0) \to \exists^{\mathrm{bd}} z\ \psi(z, p_0),$$

where $\psi(z, p)$ is an internal "parameter-free" formula with only z and p as free variables. By **ZFC** collection there is a set Z such that

$$\forall p \in P\,[\exists z\ \psi(z,p) \to \exists z \in Z\ \psi(z,p)].$$

We note that one may choose a *standard* Z with this property by transfer (in **IST**). Setting $p = p_0$ we obtain the required result. □

Hence transfer in $\mathbb{B}$ has been checked. It follows that all the **ZFC** axioms hold in $\mathbb{B}$ (being true in $\mathbb{S}$). Standardization in $\mathbb{B}$ follows immediately from standardization in $\mathbb{V}$. Clearly the bounded sets axiom **B** is valid in $\mathbb{B}$. Only **BI** remains to be proved:

$$\mathbf{BI}\text{: } \forall^{\mathrm{stfin}} A \subseteq A_0\ \exists^{\mathrm{bd}} x\ \forall a \in A\ \Phi(x, a) \leftrightarrow \exists^{\mathrm{bd}} x\ \forall^{\mathrm{st}} a \in A_0\ \Phi(x, a),$$

where A_0 is standard, and Φ is a internal formula with bounded parameters. The superscript bd can be deleted from Φ by Lemma 5.2.

Changing parameters to free variables as above, one can prove the existence of a standard set X such that

$$\forall A \subseteq A_0\ [\exists x\ \forall a \in A\ \Phi(x, a) \to \exists x \in X\ \forall a \in A\ \Phi(x, a)].$$

Moreover, one may demand that if the right-hand side of **BI** holds, then X contains an element x such that $\forall^{\mathrm{st}} a \in A_0\ \Phi(x, a)$.

We use the following example of **I** for the formula $a \in A_0 \to \Phi(x, a)\ \&\ x \in X$:

$$\forall^{\mathrm{stfin}} A \subseteq A_0\ \exists x \in X\ \forall a \in A\ \Phi(x, a) \leftrightarrow \exists x \in X\ \forall^{\mathrm{st}} a \in A_0\ \Phi(x, a).$$

Clearly its left-hand side is equivalent to the left-hand side of **BI** above, just as the right-hand side is equivalent to the right-hand side of **BI** by the choice of X. □

5.3. Proof of Theorem 13. We turn to the theorem which shows that **BST** reduces all external formulae to a Σ_2^{st} form. Note that **IST** provides the reduction only for those formulae that are either ext-prenex (see §3) or ext-bounded (see §2).

Thus let $\Phi(x_1, x_2, \ldots, x_n)$ be a "parameter-free" st-$\in$-formula with only $x_1, x_2, \ldots, x_n$ as free variables. We claim that there is a "parameter-free" Σ_2^{st} formula $\Psi(x_1, x_2, \ldots, x_n)$ such that

$$\forall x_1\ \forall x_2 \ldots \forall x_n\ [\Phi(x_1, x_2, \ldots, x_n) \leftrightarrow \Psi(x_1, x_2, \ldots, x_n)]$$

(provable in **BST**).

The proof is carried out by induction on the number of logical signs in Φ. As above (see the proof of Theorem 2.4), it suffices to go through steps $\neg$ and $\exists$. Let $\mathbf{x}$ denote $x_1, x_2, \ldots, x_n$.

The step $\neg$. We search for a Σ_2^{st} formula $\Psi(\mathbf{x})$ that is equivalent to the formula

$$\forall^{\mathrm{st}} a\ \exists^{\mathrm{st}} b\ \varphi(\mathbf{x}, a, b), \text{ where } \varphi \text{ is internal,}$$

taken as $\Phi(\mathbf{x})$. Whenever X is standard, there is (by Theorem 3.1) a pair of standard sets A, B such that

$$(8) \qquad \forall \mathbf{x} \in X^n \, [\Phi(\mathbf{x}) \leftrightarrow \forall^{\mathrm{st}} a \in A \; \exists^{\mathrm{st}} b \in B \; \varphi(\mathbf{x}, a, b)].$$

What is more, the proof of Theorem 3.1 gives an internal formula $\chi(X, A, B)$ such that the following two assertions hold:

a) $\forall^{\mathrm{st}} X \; \exists^{\mathrm{st}} A \; \exists^{\mathrm{st}} B \; \chi(X, A, B)$, and

b) $\forall^{\mathrm{st}} X \; \forall^{\mathrm{st}} A \; \forall^{\mathrm{st}} B \; [\chi(X, A, B) \to$ (8) is true].

(To be more exact, χ expresses the sentence (5^{st}) from §3.)

Applying the axiom **B** we have

$$\Phi(\mathbf{x}) \leftrightarrow \exists^{\mathrm{st}} X \; \exists^{\mathrm{st}} A \; \exists^{\mathrm{st}} B \; [\mathbf{x} \in X^n \; \& \; \chi(X, A, B) \; \& \\ \& \; \forall^{\mathrm{st}} a \in A \; \exists^{\mathrm{st}} b \in B \; \varphi(\mathbf{x}, a, b)].$$

Changing the second line to $\exists^{\mathrm{st}} \tilde{b} \in {}^A B \; \forall^{\mathrm{st}} a \in A \; \varphi(\mathbf{x}, a, \tilde{b}(a))$ (Theorem 2.3 is used) and making some evident transformations, one can obtain the required formula Ψ.

The step $\exists$. We need a Σ_2^{st} formula that is equivalent to

$$\Phi(\mathbf{x}) =_{\mathrm{def}} \exists u \; \exists^{\mathrm{st}} a \; \forall^{\mathrm{st}} b \;\; \varphi(\mathbf{x}, u, a, b),$$

φ is internal. The following equivalence is true by the bounded sets axiom **B**:

$$\Phi(\mathbf{x}) \leftrightarrow \exists^{\mathrm{st}} X \; [\mathbf{x} \in X^n \; \& \; \exists u \in X \; \exists^{\mathrm{st}} a \in X \; \forall^{\mathrm{st}} b \; \varphi(\mathbf{x}, u, a, b)].$$

Hence we conclude as above that for some internal formula $\chi(X, B)$

$$\Phi(\mathbf{x}) \leftrightarrow \exists^{\mathrm{st}} X \; \exists^{\mathrm{st}} B \; [\mathbf{x} \in X^n \; \& \; \chi(X, B) \; \& \\ \& \; \exists^{\mathrm{st}} a \in X \; \exists u \in X \; \forall^{\mathrm{st}} b \in B \; \varphi(\mathbf{x}, u, a, b)].$$

Finally we use idealization **BI** to the block of quantifiers $\exists u \; \forall^{\mathrm{st}} b$ and obtain the required formula Ψ by some simple transformations.

This completes the proof of Theorem 13. $\square$

5.4. Proof of Theorem 12. It suffices to prove (in **BST**) only the following hypotheses: **Uniq**, **Coll**, **Chc$_1$** and **Chc$_4$**.

Uniq. We consider a st-$\in$-formula $\Phi(x)$ with standard parameters with only x free and suppose that $\exists! x \; \Phi(x)$. One may assume that Φ is a Σ_2^{st} formula by Theorem 13. We now use Theorem 2.11.

Coll. Let $\Phi(x, y)$ be a st-$\in$-formula with arbitrary (bounded) parameters and only x, y free. Prove that for every X there is a standard set Y such that

$$\forall x \in X \; [\exists y \; \Phi(x, y) \to \exists y \in Y \; \Phi(x, y)].$$

One may assume that X is standard by the bounded sets axiom **B**.

We note that the formula

$$\Psi(x, z) =_{\mathrm{def}} \mathrm{st}\, z \; \& \; \exists y \in z \; \Phi(x, y)$$

is equivalent to some Σ_2^{st} formula by Theorem 13. Hence by 3.2 there is a standard Z such that

$$\forall x \in X\ [\exists z\ \Psi(x,\ z) \rightarrow \exists z \in Z\ \Psi(x,\ z)].$$

The set $Y = \bigcup Z = \{y : \exists z \in Z\ (y \in z)\}$ is as required. The standardness of Y follows from transfer.

To prove $\mathbf{Chc}_1$ for some standard X and a st-$\in$-formula $\Phi(x, y)$, apply **Coll** to the formula st y & $\Phi(x, y)$, obtaining a standard set Y such that

$$\forall x \in X\ [\exists^{st} y\ \Phi(x,\ y) \rightarrow \exists^{st} y \in Y\ \Phi(x,\ y)].$$

Finally we apply Theorem 2.3.

At last we prove the following:

$$\mathbf{Chc}_4\text{:}\ \forall^{st} x \in X\ \exists y\ \Phi(x,\ y) \rightarrow \exists \tilde{y}\ \forall^{st} x \in X\ \Phi(x,\ \tilde{y}(x)),$$

where X is standard. One may assume that Φ is Σ_2^{st} as above, therefore $\Phi(x, y)$ is $\exists^{st} a\ \forall^{st} b\ \varphi(x, y, a, b)$, where φ is internal. By collection there is a standard set Y such that the left-hand side of $\mathbf{Chc}_4$ is equivalent to $\forall^{st} x \in X\ \exists y \in Y\ \Phi(x, y)$. Theorem 3.1 gives a pair of standard sets A, B satisfying

$$\Phi(x,\ y) \leftrightarrow \exists^{st} a \in A\ \ \forall^{st} b \in B\ \ \varphi(x,\ y,\ a,\ b)$$

whenever $x \in X$, $y \in Y$. Theorem 2.12 ends the proof. □

§6. The hierarchy theorem

This short section contains the proof of Theorem 10. We shall prove two forms of the theorem. The first form deals with standard parameters, while the second allows arbitrary parameters. Unfortunately the author has not succeeded in proving the common extension of these two variants.

We recall that $\Phi(X)$ is the formula $\exists^{st} a\ \forall^{st} b\ (\langle a, b\rangle \in X)$ in the formulation of our Theorem 10.

6.1. ***Theorem.*** *Let $\Psi(X, p_1, \ldots, p_n)$ be a Π_2^{st} formula without parameters and with only $X, p_1, \ldots, p_n$ free. Then*

(a) [**IST**] $\forall^{st} p_1 \ldots \forall^{st} p_n \neg \forall X\ [\Phi(X) \leftrightarrow \Psi(X,\ p_1,\ \ldots,\ p_n)]$;
(b) [**IST** + $\mathbf{Repl}_3$] $\forall p_1 \ldots \forall p_n \neg \forall X\ [\Phi(X) \leftrightarrow \Psi(X, p_1,\ \ldots,\ p_n)]$.

It is worth presenting a consequence of (b) and Theorem 1A. Given a "parameter-freee" Π_2^{st} formula $\Psi(X, p_1, \ldots, p_n)$, one cannot prove in **IST** that

$$\exists p_1 \ldots \exists p_n\ \forall X\ [\Phi(X) \leftrightarrow \Psi(X,\ p_1, \ldots, p_n)].$$

(Indeed the negation of this is consistent with **IST** being a consequence of the consistent hypothesis $\mathbf{Repl}_3$ in **IST**.) Hence (b) claims that Φ is not provably equivalent in **IST** to any Π_2^{st} formula with arbitrary parameters, while (a) claims that Φ is provably non-equivalent to any Π_2^{st} formula with standard parameters. Of course, the second is stronger than the first. It would be better to prove (b) in **IST** without $\mathbf{Repl}_3$.

One can replace $\mathbf{Repl}_3$ in (b) by a slightly more natural hypothesis $\mathbf{Sep}_3$, since $\mathbf{Repl}_3$ is equivalent to $\mathbf{Sep}_3$ in **IST**.

Proof. We may assume that the list $p_1, \ldots, p_n$ contains in fact only one variable p; thus let $\Psi(X, p)$ be $\forall^{st} u\ \exists^{st} v\ \psi(X, u, v, p)$, where ψ is internal. We fix a set p and suppose that $\Phi(X) \to \Psi(X, p)$ holds for all X, that is,

$$\forall X\ [\exists^{st} a\ \forall^{st} b\ (\langle a, b\rangle \in X) \to \forall^{st} u\ \exists^{st} v\ \psi(X, u, v, p)].$$

This is transformable to

$$\forall^{st} a\ \forall^{st} u\ \forall X\ \exists^{st} b\ \exists^{st} v\ [\langle a, b\rangle \in X \to \psi(X, u, v, p)].$$

Further, applying idealization, we have

$$\forall^{st} a\ \forall^{st} u\ \exists^{stfin} B\ \exists^{stfin} V\ \forall X\ [\forall b \in B\ (\langle a, b\rangle \in X) \to \\ \to \exists v \in V\ \psi(X, u, v, p)].$$

Taking $a = u$ and using the fact that for every standard B there is a standard ordinal α such that $B \subseteq \mathbb{V}_\alpha$, we get a slightly weaker assertion

$$(9)\quad \forall^{st} u\ \exists^{st} \alpha \in \mathrm{Ord}\ \exists^{stfin} V\ \forall X\ [\forall b \in \mathbb{V}_\alpha\ (\langle u, b\rangle \in X) \to \\ \to \exists v \in V\ \psi(X, u, v, p)].$$

Now it is necessary to separate the cases (a) and (b).

6.2. Standard case.

We assume that p is standard. Let $\varphi(u, \alpha, V)$ denote the formula

$$\forall X\ [\forall b \in \mathbb{V}_\alpha\ (\langle u, b\rangle \in X) \to \exists v \in V\ \psi(X, u, v, p)].$$

The preceding formula takes the form

$$\forall^{st} u\ \exists^{st} \alpha\ \ \exists^{stfin} V\ \varphi(u, \alpha, V).$$

Let $\alpha(u)$ be the least ordinal α satisfying $\exists^{fin} V\ \varphi(u, \alpha, V)$, if such an α exists, or else $\alpha(u) = 0$. When u is a standard set the ordinal $\alpha(u)$ and the set $B(u) = \mathbb{V}_{\alpha(u)}$ are standard by Theorem 2.1. We prove the following claim:

$$(10)\quad \forall^{st} u\ \forall X\ [\forall b \in B(u)\ (\langle u, b\rangle \in X) \to \exists^{st} v\ \psi(X, u, v, p)].$$

Indeed, let u be standard and let X_0 be such that $\langle u, b\rangle \in X_0$ for all $b \in B(u)$. We set $\alpha = \alpha(u)$. By transfer and the definition of $\alpha(u)$ there is a *standard* finite set V satisfying $\varphi(u, \alpha, V)$, that is,

$$\forall X\ [\forall b \in B(u)\ (\langle u, b\rangle \in X) \to \exists v \in V\ \psi(X, u, v, p)].$$

Taking $X = X_0$, we obtain $\exists v \in V\ \psi(X_0, u, v, p)$ and then $\exists^{st} v\ \psi(X_0, u, v, p)$ by 2.8 (all elements of a standard finite set are standard). This ends the proof of the claim (10). $\square$

Let H be a set containing all standard sets (see 2.9). We define X by

$$X = \{\langle u,\ b\rangle:\quad u \in H \,\&\, b \in B\,(u)\}.$$

Clearly $\langle u,\ B(u)\rangle \notin X$ for all u. Hence $\neg\,\Phi(X)$, because $B(u)$ is standard whenever u is standard.

However, if u is standard, then $\forall b \in B(u)\ (\langle u,\ b\rangle \in X)$ by the definition of X, hence $\exists^{st} v\ \psi(X,\ u,\ v,\ p)$ by the claim (10). Thus $\Psi(X,\ p)$ is true. Hence a set X satisfying $\Psi(X,\ p)\ \&\ \neg\,\Phi(X)$ has been constructed. This ends the standard case.

6.3. Nonstandard case.
The reasoning of 6.2 fails at the point where we assert that $\alpha(u)$ is standard provided u is standard (this is in general wrong for a nonstandard p). We overcome this obstacle with the help of the additional assumption **Repl**$_3$.

Firstly we consider again the statement (9). Applying idealization, we obtain

$$\forall^{st} u\ \exists^{st}\alpha\ \forall X\ [\forall b \in \mathbb{V}_\alpha\ (\langle u,\ b\rangle \in X) \rightarrow \exists^{st} v\ \psi(X,\ u,\ v,\ p)].$$

Let $\varphi(u,\ \alpha)$ denote the formula on the right of $\exists^{st}\alpha$; thus the last assertion takes the form $\forall^{st} u\ \exists^{st}\alpha\ \varphi(u,\ \alpha)$.

We now consider the new formula $\varphi'(u,\ \alpha)$ which says that α is the least standard ordinal satisfying $\varphi(u,\ \alpha)$ (one may choose the least standard α correctly by Theorem 2.5). Hence

$$\varphi'(u,\ \alpha)\ =_{\mathrm{def}}\ \varphi(u,\alpha)\ \&\ \forall^{st}\gamma < \alpha\ \neg\ \varphi(u,\ \gamma\,).$$

Clearly $\forall^{st} u\ \exists!^{st}\alpha\ \varphi'(u,\ \alpha)$ and $\varphi'(u,\ \alpha) \rightarrow \varphi(u,\ \alpha)$ for all standard $u,\ \alpha$. Hence by **Repl**$_3$ there is a function F such that for all standard u the value $F(u)$ is defined, is an ordinal, and $\varphi(u,\ F(u))$ holds.

The set $B(u)\ =\ \mathbb{V}_{F(u)}$ is also standard provided u is standard. We come to the claim 10 of 6.2 and complete the proof in the same way. □

Problem 14. Prove (b) in **IST** without any additional assumption.

Let us compare the theorem just proved with some classical hierarchy theorems. Certainly the given proof is very far from the usual "universal set" reasoning. Rather it slightly resembles a topological proof of the existence of the set $\mathbb{F}_\sigma$, but not $\mathbb{G}_\delta$. The author was not able to carry out the "universal set" construction. However, there is an evident candidate for the Σ^{st}_n formula solving the hierarchy problem:

Problem 15. Prove that the Σ^{st}_n-formula

$$\exists^{st} a_1\ \forall^{st} a_2\ \exists^{st} a_3\ \forall^{st} a_4\ \ldots\ \exists\ (\forall)^{st} a_n\ [\langle a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n\rangle \in X]$$

is not equivalent in **IST** to any Π^{st}_n-formula.

§7. Truth definability

It is the purpose of this section to prove Theorem 5 concerning the truth definition for internal formulae with standard parameters. As a consequence we shall prove Theorem 2, which highlights the special status of $\mathbf{Repl}_1$, $\mathbf{Chc}_1$, $\mathbf{Coll}_1$, $\mathbf{Coll}_1(\mathrm{st}\ \Phi)$ among other hypotheses.

7.1. Coding the language.

To prove Theorem 5 we use the well known technical tool of coding the formulae of $\in$-language by finite sequences of a special kind and then constructing the satisfaction function.

Firstly we assume for simplicity that $\in$-formulae may contain only the following logical signs: $\neg$, &, $\exists$, $\in$, $=$, and of course brackets (,), the variables υ and υ_i, $i \in \mathbb{N}$, and finally parameters, that is, arbitrary sets replacing free variables. Note that the signs $\vee$, $\forall$, $\rightarrow$, $\leftrightarrow$ which we did not mention are easily expressible by $\neg$, &, $\exists$.

We denote by $\ulcorner\Phi\urcorner$ the sequence obtained by replacing in Φ

$$\text{each sign } \neg,\ \&,\ \exists,\ \in,\ =,\ (,)\text{ by integers } 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6;$$
$$\text{each variable } \upsilon_k \text{ by } 8+k \text{ and } \upsilon \text{ by } 7;$$
$$\text{each parameter } p\ (p \in \mathbb{V}) \text{ by } \langle 0, p\rangle \text{ (the ordered pair).}$$

Thus $\ulcorner\Phi\urcorner$ is a finite sequence of special type. $\ulcorner\Phi\urcorner$ is sometimes called the *translation of* Φ. We put

$$\mathrm{Form} = \{\ulcorner\Phi\urcorner : \Phi \text{ is a (well-formed) } \in\text{-formula with arbitrary parameters}\};$$
$$\mathrm{Form}_X = \{\ulcorner\Phi\urcorner \in \mathrm{Form}: \text{ all parameters of } \Phi \text{ are members of } X\}.$$

We say that a formula ψ is *subordinate* to φ if ψ is a subformula of φ in which some (maybe none or all) free variables have been replaced by arbitrary parameters. For example, φ itself is subordinate to φ; $\varphi(p)$ for all p and $\varphi(\upsilon)$ (υ is free) are subordinate to $\exists\upsilon\ \varphi(\upsilon)$. We define

$$\mathrm{Form}[\varphi] = \{\ulcorner\psi\urcorner : \psi \text{ is subordinate to } \varphi\};$$
$$\mathrm{Form}_X[\Phi] = \mathrm{Form}_X \cap \mathrm{Form}[\varphi].$$

For example, $\ulcorner\varphi(p)\urcorner \in \mathrm{Form}_X[\exists\upsilon\ \varphi(\upsilon)]$ whenever $p \in X$.

Further we distinguish the translations of closed formulae:

$$\mathrm{CForm} = \{\ulcorner\varphi\urcorner \in \mathrm{Form}: \ \varphi \text{ is a closed formula}\},$$

and CForm_X, $\mathrm{CForm}[\varphi]$, $\mathrm{CForm}_X[\varphi]$ in the same manner.

Note that a translation $\ulcorner\varphi\urcorner$ is standard (as a finite sequence) if and only if φ has only standard parameters and the number of logical signs of φ is standard too.

Now the key definition. We denote by Sat(T) the conjunction of the following five st-$\in$-formulae:

1) $T \subseteq \mathrm{CForm}$;
2) $\forall^{\mathrm{st}} p\ \forall^{\mathrm{st}} q\ [(\ulcorner p = q\urcorner \in T \leftrightarrow p = q) \,\&\, (\ulcorner p \in q\urcorner \in T \leftrightarrow p \in q)]$;
3) $\forall^{\mathrm{st}} \ulcorner\varphi\urcorner\ \forall^{\mathrm{st}} \ulcorner\psi\urcorner\ [\ulcorner\varphi \,\&\, \psi\urcorner \in T \leftrightarrow (\ulcorner\varphi\urcorner \in T \,\&\, \ulcorner\psi\urcorner \in T)]$;
4) $\forall^{\mathrm{st}} \ulcorner\varphi\urcorner \in T\ \forall^{\mathrm{st}} \ulcorner\psi\urcorner \in \mathrm{CForm}\,[\varphi]\ (\ulcorner\neg\psi\urcorner \in T \leftrightarrow \ulcorner\psi\urcorner \notin T)$;
5) $\forall^{\mathrm{st}} \ulcorner\varphi\,(v)\urcorner\ [\ulcorner\exists v\ \varphi\,(v)\urcorner \in T \leftrightarrow \exists^{\mathrm{st}} p\ (\ulcorner\varphi\,(p)\urcorner \in T)]$.

Any set T satisfying Sat(T) is adapted to the definition of truth within the universe $\mathbb{S}$ of all standard sets.

7.2. ***Lemma*** [**IST**]. *Either of the two conditions*

$$\exists T\ [\mathrm{Sat}\,(T) \,\&\, \ulcorner\varphi\urcorner \in T];\ \forall T\ [\mathrm{Sat}\,(T) \to \ulcorner\neg\varphi\urcorner \notin T]$$

is necessary and sufficient for any closed $\in$-formula φ with standard parameters to be true in $\mathbb{S}$ (or, what is the same, in $\mathbb{V}$).

To be more precise, we claim that

$$\forall^{\mathrm{st}} x_1 \ldots \forall^{\mathrm{st}} x_n[\varphi\,(x_1, \ldots, x_n) \leftrightarrow \exists T\ [\mathrm{Sat}\,(T) \,\&\, \ulcorner\varphi(x_1, \ldots, x_n)\urcorner \in T] \leftrightarrow \\ \leftrightarrow \forall T\ [\mathrm{Sat}\,(T) \to \ulcorner\neg\varphi(x_1, \ldots, x_n)\urcorner \notin T]$$

is provable in **IST** whenever $\varphi(v_1, \ldots, v_n)$ is a "parameter-free" $\in$-formula with only $v_1, \ldots, v_n$ as free variables. Thus either of the formulae

$$\exists T\ [\mathrm{Sat}\,(T) \,\&\, \ulcorner\varphi\urcorner \in T];\ \forall T\ [\mathrm{Sat}\,(T) \to \ulcorner\neg\varphi\urcorner \notin T]$$

can be taken as $\tau(\ulcorner\varphi\urcorner)$ for Theorem 5. Hence the only thing to prove is the lemma.

Proof. The proof is based on two auxiliary claims from which the lemma clearly follows.

***Assertion* 1.** *For every closed $\in$-formula φ with standard parameters there is a set T* satisfying Sat(T) *and containing at least one* (*in fact exactly one*) *of the translations* $\ulcorner\varphi\urcorner$, $\ulcorner\neg\varphi\urcorner$.

Proof. We replace all the parameters occurring in φ by free variables. Let $\varphi(v_1, \ldots, v_n)$ be the formula we obtain, and let

$$\varphi(v_{i_1}, \ldots, v_{i_{n(i)}}),\ 1 \leqslant i \leqslant m,\ i_n \in \mathbb{N}$$

be the list of all its subformulae (including φ itself). We take a set H containing all standard sets and define

$$T_i = \{\ulcorner\varphi_i(x_1, \ldots, x_{n(i)})\urcorner : x_1, \ldots, x_{n(i)} \in H \,\&\, \varphi_i(x_1, \ldots, x_{n(i)})\} \cup \\ \cup \{\ulcorner\neg\varphi_i(x_1, \ldots, x_{n(i)})\urcorner : x_1, \ldots, x_{n(i)} \in H \,\&\, \neg\, \varphi_i(x_1, \ldots, x_{n(i)})\}.$$

The set $T = \bigcup_{1 \leqslant i \leqslant m} T_i$ is as required. $\square$

***Assertion* 2.** *Let φ be a closed $\in$-formula with standard parameters. If $\ulcorner\varphi\urcorner \in T$ and* Sat(T) *holds, then φ is true in $\mathbb{S}$ (as well as in $\mathbb{V}$).*

Proof. We proceed by induction on the number of signs in $\ulcorner\varphi\urcorner$. The base of induction (that is, the case φ is either $x \in y$ or $x = y$) is justified by part 2) of Sat, while induction steps are based on parts 3), 4), 5). The only non-trivial case is the step $\neg$. So let us assume that $\ulcorner\neg\varphi\urcorner \in T$ and try to prove that φ is false.

Firstly we note that $\ulcorner\varphi\urcorner \notin T$ by 4).

Case 1: φ is either $x \in y$ or $x = y$; x, y are standard. Then $x \notin y$ (respectively $x \neq y$) by 2) and the induction hypothesis (we recall that $\ulcorner\varphi\urcorner \notin T$). Hence φ is false.

Case 2: φ is ψ & χ. At least one of $\ulcorner\psi\urcorner$, $\ulcorner\chi\urcorner$ is not a member of T by 3). Let $\ulcorner\psi\urcorner \notin T$, say. Then $\ulcorner\neg\psi\urcorner \in T$ by 4). So $\neg\psi$ is true by the induction hypothesis. Hence ψ is false.

Case 3: φ is $\neg\psi$. Then 4) implies that $\ulcorner\psi\urcorner \in T$ because $\ulcorner\varphi\urcorner = \ulcorner\neg\psi\urcorner \notin T$. So ψ is true, therefore φ is false.

Case 4: φ is $\exists v\, \psi(v)$. It suffices to prove that $\psi(x)$ is false whenever x is standard. We note that $\ulcorner\psi(x)\urcorner \notin T$ by 5) (because $\ulcorner\varphi\urcorner \notin T$). Hence $\ulcorner\neg\psi(x)\urcorner \in T$ by 4). Thus $\neg\psi(x)$ by the induction hypothesis. $\square$

This completes the proof of the lemma and Theorem 5. $\square$

7.3. Proof of Theorem 2. It sufficies to prove that Cons **ZFC** is implied by $\mathbf{Coll}_1$(st Φ) (that is, $\mathbf{Coll}_1$ with only standard parameters allowed in the core formula) in **IST**.

Given a "parameter-free" $\in$-formula $\varphi(v_1, \ldots, v_m)$, it is a theorem of **ZFC** that there is an ordinal $\varkappa$ such that $\mathbb{V}_\varkappa$ is an elementary submodel of the universe $\mathbb{V}$ with respect to the formula φ, that is,

$$\forall p_1 \in \mathbb{V}_\varkappa \ldots \forall p_m \in \mathbb{V}_\varkappa[\varphi(p_1, \ldots, p_m) \leftrightarrow \varphi^\varkappa(p_1, \ldots, p_m)],$$

where $\varphi^\varkappa$ is the relativization of φ to $\mathbb{V}_\varkappa$. One more fact to note here is that we may choose a *standard* $\varkappa$ of such a kind by using transfer.

We fix a reasonable enumeration $\langle\varphi_k : k \in \mathbb{N}\rangle$ of all closed $\in$-formulae and define a st-$\in$-formula $\Phi(k, \varkappa)$ which says that k is a standard integer and $\varkappa$ is the least (standard) ordinal such that $\mathbb{V}_\varkappa$ is an elementary submodel of $\mathbb{V}$ for all φ_i, $1 \leqslant i \leqslant k$. The precise definition of $\Phi(k, \varkappa)$ is as follows:

$k \in \mathbb{N}$ & $\varkappa \in$ Ord & st k & st $\varkappa$ & $\exists T$ [Sat (T) & T *contains the translation* $\ulcorner\varkappa$ *is the least ordinal such that* $\mathbb{V}_\varkappa$ *is an elementary submodel of* $\mathbb{V}$ *for all* φ_i, $1 \leqslant i \leqslant k\urcorner$.

Lemma 7.2 shows that the precise definition of Φ corresponds to the informal one given above. Hence the following holds:

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists!^{\mathrm{st}} \varkappa \in \mathrm{Ord}\ \Phi(k, \varkappa).$$

Applying $\mathbf{Coll}_1$(st Φ), we get a standard $f : \mathbb{N} \to$ Ord such that $\Phi(k, f(k))$ holds for all standard $k \in \mathbb{N}$. We put $\lambda = \sup_{k\in\mathbb{N}} f(k)$ and show that $\mathbb{V}_\lambda$ is a model of **ZFC**.

One may prove this in $\mathbb{S}$ by transfer. Thus, coming back to $\mathbb{V}$, it is sufficient to prove that each standard (in the sense that $\ulcorner A \urcorner$ is standard) axiom A of **ZFC** holds in $\mathbb{V}_\lambda$.

Let a standard k be such that A and all subformulae of A are contained in the list φ_i, $1 \leqslant i \leqslant k$. Then for all standard $n \geqslant k$ the set $\mathbb{V}_{f(n)}$ is an elementary submodel of $\mathbb{V}$ with respect to A and each subformula of A. It follows that $\mathbb{V}_\lambda$ is also an elementary submodel of $\mathbb{V}$ as regards A. Hence A is true in $\mathbb{V}_\lambda$. This completes the proof of Theorem 2. □

§8. Full collection

Unlike the hypotheses of separation, replacement and choice, collection is valid in **IST** for all core formulae; this assertion is exactly the same as Theorem 4. This section contains the proof together with the proof of two corollaries. The first is our Theorem 9 and the second is the corollary mentioned in section 1.13 of the Introduction.

8.1. Beginning of the proof.
Thus we try to prove **Coll** for a st-$\in$-formula

$$\Phi(x, y) =_{\text{def}} \mathbf{Q}_2 x_2 \mathbf{Q}_3 x_3 \ldots \mathbf{Q}_n x_n \ \varphi(x, y, x_2, x_3, \ldots, x_n),$$

where φ is a quantifier-free formula and each $\mathbf{Q}_i$ is a quantifier of four possible kinds: $\exists^{\text{st}}$, $\forall^{\text{st}}$, $\exists$ or $\forall$. Arbitrary parameters are allowed in Φ. Let us fix a set X. It suffices to find a set Y such that

$$\forall x \in X \ [\exists y \ \Phi(x, y) \rightarrow \exists y \in Y \ \Phi(x, y)].$$

It will be more convenient to rename the variables x, y by x_0, x_1 respectively. Also let $\mathbf{Q}_1$ be $\exists$ (this corresponds to the formula $\exists y \ \Phi(x, y)$).

The following way of reasoning partially resembles the proof of Theorem 3.1, though mixing of two kinds of quantifiers (internal $\exists$, $\forall$ and external $\exists^{\text{st}}$, $\forall^{\text{st}}$) gives some additional difficulties.

Let us fix a cardinal θ. The role of θ will become clear later.

The key definition. For all $k \leqslant n$ we define the set C_k, and then for each sequence $x_0, \ldots, x_k$ define $F(x_0, \ldots, x_k) \in C_k$. The definition is arranged so that F will be internal at all levels k. The construction depends on the chosen θ, though the notation does not reflect the dependence in a clear form.

The definition goes by reverse induction on $k = n, n-1, \ldots, 1, 0$.

The base of induction: $k = n$. We define $C_n = \{0, 1\}$ and

$$F(x_0, \ldots, x_n) = \begin{cases} 1 \text{ if } \varphi(x_0, \ldots, x_n) \text{ holds,} \\ 0 \text{ if } \varphi(x_0, \ldots, x_n) \text{ fails.} \end{cases}$$

The inductive step. We suppose that $k < n$ and the set C_{k+1} together with the values $F(x_0, \ldots, x_k, x_{k+1}) \in C_{k+1}$ for all $x_0, \ldots, x_k, x_{k+1}$ are already defined (and the map F is internal).

The external case: $\mathbf{Q}_{k+1}$ is either $\exists^{st}$ or $\forall^{st}$. We put

$$C_k = \mathscr{P}(\mathbb{V}_\theta \times C_{k+1}) = \mathscr{P}(\{\langle x, c\rangle : x \in \mathbb{V}_\theta \,\&\, c \in C_{k+1}\}),$$

and

$$F(x_0, \ldots, x_k) = \{\langle x_{k+1}, c\rangle : x_{k+1} \in \mathbb{V}_\theta \,\&\, c = F(x_0, \ldots, x_k, x_{k+1})\}$$

for all $x_0, \ldots, x_k$. Hence

$$F(x_0, \ldots, x_k, x_{k+1}) = c \leftrightarrow \langle x_{k+1}, c\rangle \in F(x_0, \ldots, x_k)$$

whenever $x_{k+1} \in \mathbb{V}_\theta$. One may rewrite the last equivalence in the form

$$F(x_0, \ldots, x_k, x_{k+1}) = F(x_0, \ldots, x_k)(x_{k+1}),$$

again provided that $x_{k+1} \in \mathbb{V}_\theta$.

The internal case: $\mathbf{Q}_{k+1}$ is either $\exists$ or $\forall$. For example, for $k = 0$, $\mathbf{Q}_1$ is $\exists$ by the definition of $\mathbf{Q}_1$. We put $C_k = \mathscr{P}(C_{k+1})$ and

$$F(x_0, \ldots, x_k) = \{F(x_0, \ldots, x_k, x_{k+1}) : x_{k+1} \in \mathbb{V}\}$$

for all $x_0, \ldots, x_k$. If F is an internal map at level $k+1$ and each $F(x_0, \ldots, x_k, x_{k+1})$ is a member of C_{k+1}, then each value $F(x_0, \ldots, x_k)$ is also an internal set (a member of C_k) by the ZFC separation, although the domain of the variable x_{k+1} is not restricted by any set. Also for the same reason F remains internal at level k.

8.2. *Lemma.* *Let $x_0, \ldots, x_k$ and $x'_0, \ldots, x'_k$ be such that*

$$F(x_0, \ldots, x_k) = F(x'_0, \ldots, x'_k).$$

If $\mathbf{Q}_{k+1}$ is an external quantifier, then

$$F(x_0, \ldots, x_k, x_{k+1}) = F(x'_0, \ldots, x'_k, x_{k+1})$$

for all $x_{k+1} \in \mathbb{V}_\theta$. If $\mathbf{Q}_{k+1}$ is internal, then for all x_{k+1} there exists x'_{k+1} such that

$$F(x_0, \ldots, x_k, x_{k+1}) = F(x'_0, \ldots, x'_k, x'_{k+1}).$$

Proof. The internal case is clear. Further, if $\mathbf{Q}_{k+1}$ is external, then by definition

$$F(x_0, \ldots, x_k, x_{k+1}) = F(x_0, \ldots, x_k)(x_{k+1}) = F(x'_0, \ldots, x'_k)(x_{k+1}) = \\ = F(x'_0, \ldots, x'_k, x_{k+1}). \qquad \square$$

Before one more lemma is formulated, let us look at the notion of bounded and unbounded ordinals. We recall that a set x is *bounded* if and only if x is a member of a standard set X. When restricted to the class of ordinals, the notion of boundedness can be reformulated in certain more convenient forms, for example:

$$\theta \in \mathrm{Ord} \text{ is unbounded } \leftrightarrow \forall^{\mathrm{st}}\alpha \in \mathrm{Ord}\,(\alpha < \theta) \leftrightarrow \forall^{\mathrm{st}}x\ (x \in \mathbb{V}_\theta)$$

(the simple proof is left to the reader).

8.3 ***Lemma.*** *Suppose that the ordinal θ which we have fixed above is unbounded. Let $k \leqslant n$ and let x_i, x_i', $i \leqslant k$, be such that*

$$F(x_0, \ldots, x_k) = F(x_0', \ldots, x_k').$$

Then

$$\mathbf{Q}_{k+1}x_{k+1} \ldots \mathbf{Q}_n x_n\ \varphi(x_0, \ldots, x_k, x_{k+1}, \ldots, x_n) \leftrightarrow$$
$$\leftrightarrow \mathbf{Q}_{k+1}x_{k+1}' \ldots \mathbf{Q}_n x_n'\ \varphi(x_0', \ldots, x_k', x_{k+1}', \ldots, x_n').$$

Proof. We argue by induction on $k = n, n-1, \ldots, 1, 0$. The case $k = n$ is evident by the definition of F (the string of quantifiers is empty). Now the induction step. We prove the lemma for some $k < n$ provided it is true for $k+1$. One may consider only the case when $\mathbf{Q}_{k+1}$ is either $\exists$ or $\exists^{\mathrm{st}}$ (the case of universal quantifiers does not differ essentially).

Thus let x_{k+1} be such that the following holds:

$$\mathbf{Q}_{k+2}x_{k+2} \ldots \mathbf{Q}_n x_n\ \varphi(x_0, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_n).$$

Also we assume that if $\mathbf{Q}_{k+1}$ is $\exists^{\mathrm{st}}$, then x_{k+1} is standard (hence x_{k+1} belongs to $\mathbb{V}_\theta$ by the fact that θ is unbounded). The preceding lemma gives a set x_{k+1}' such that

$$F(x_0, \ldots, x_k, x_{k+1}) = F(x_0', \ldots, x_k', x_{k+1}'),$$

and in addition if $\mathbf{Q}_{k+1}$ is $\exists^{\mathrm{st}}$, then $x_{k+1}' = x_{k+1}$ and x_{k+1}' is standard. Thus by the induction hypothesis the following is true:

$$\mathbf{Q}_{k+2}x_{k+2}' \ldots \mathbf{Q}_n x_n'\ \varphi(x_0', \ldots, x_k', x_{k+1}', x_{k+2}', \ldots, x_n').$$

This completes the proof of the right-hand side of the required equivalence. □

We now turn back to the proof of Theorem 4. We recall that x_1 is y and x_0 is x. Therefore Lemma 8.3 for $k = 1$ takes the form:

8.4. ***Corollary.*** *Let θ be unbounded. If $F(x, y) = F(x', y')$, then the following holds*: $\Phi(x, y) \leftrightarrow \Phi(x', y')$. □

Assume that θ is in fact unbounded (the existence of unbounded ordinals follows from 2.9). By **ZFC** collection we get a set Y such that

$$(11)\quad \forall c \in C_1\ \forall x \in X\ [\exists y\ (c = F(x, y)) \to \exists y \in Y\ (c = F(x, y))]$$

holds (the equality $c = F(x, y)$ is internal). We recall that $F(x, y) \in C_1$ for all x, y. Hence one may rewrite (11) as follows:

$$\forall x \in X\ \forall y\ \exists y' \in Y\ [F(x, y) = F(x, y')].$$

Then

$$\forall x \in X \ \forall y \ \exists y' \in Y \ [\Phi(x, y) \leftrightarrow \Phi(x, y')]$$

by 8.4. This ends the proof of the theorem. □

8.5. Comments.
In fact we have proved something stronger than the assertion of Theorem 8.1. Namely, given a st-$\in$-formula $\Phi(x, y)$, there is an internal formula $\Phi^*(\theta, X, \lambda)$ such that

a) $\forall \theta \in \mathrm{Ord} \ \forall X \ \exists! \lambda \in \mathrm{Ord} \ \Phi^*(\theta, X, \lambda)$;

b) if θ is unbounded, $\lambda \in \mathrm{Ord}$ and $\Phi^*(\theta, X, \lambda)$ holds, then

$$\forall x \in X \ [\exists y \ \Phi(x, y) \rightarrow \exists y \in \mathbb{V}_\lambda \ \Phi(x, y)].$$

In addition Φ^* has the same list of parameters as Φ has.

The formula we have in mind is as follows:

"$\theta \in \mathrm{Ord}$ & $\lambda \in \mathrm{Ord}$ & λ *is the least ordinal such that the assertion* (11) *holds for* $Y = \mathbb{V}_\lambda$, *that is,*

$$\forall c \in C_1 \ \forall x \in X \ [\exists y \ (c = F(x, y)) \rightarrow \exists y \in \mathbb{V}_\lambda (c = F(x, y))],$$

where C_1 *and* F *are constructed from the given* θ."

This more exact form of collection serves as a key tool in the proof of uniqueness for the class of bounded sets.

8.6. ***Theorem*** **[IST]** (= Theorem 9). *Let* $\Phi(x)$ *be a* st-$\in$-*formula with bounded parameters. Suppose that there is a unique* x *satisfying* $\Phi(x)$. *Then this unique* x *is bounded.*

Proof. One may assume that Φ contains a single parameter $p_0 \in P$, where P is a standard set. Thus x is the unique set satisfying $\Phi(x, p_0)$.

We denote by λ_0 the least ordinal λ such that $x \in \mathbb{V}_\lambda$. All we need to prove is that λ_0 is bounded, that is, $\lambda_0 < \gamma$ for a standard ordinal γ. We are going to use the formula Φ^* given by 8.5. Thus Φ^* is such that

a) $\forall \theta \in \mathrm{Ord} \ \exists! \lambda \in \mathrm{Ord} \ \Phi^*(\theta, P, \lambda)$. We denote by $\lambda(\theta)$ the unique λ satisfying $\Phi^*(\theta, P, \lambda)$. Then $\lambda(\theta)$ is standard whenever θ is standard (by transfer; Φ^* has no parameters by the choice of $\Phi(x, p)$;

b) if θ is unbounded and $\Phi^*(\theta, P, \lambda)$, then $x \in \mathbb{V}_\lambda$ (for $p_0 \in P$).

Then $\lambda(\theta) \geqslant \lambda_0$ by the choice of λ_0, provided that θ is unbounded. Hence $\forall \theta \in \mathrm{Ord} \ [\forall^{\mathrm{st}} \gamma \ (\gamma < \theta) \rightarrow \lambda(\theta) \geqslant \lambda_0]$. This is equivalent to $\forall \theta \ \exists^{\mathrm{st}} \gamma \ [\gamma \geqslant \theta$ or $\lambda(\theta) \geqslant \lambda_0]$. Now we apply idealization. There is a standard finite $\Gamma \subseteq \mathrm{Ord}$ such that

$$\forall \theta \in \quad \exists \gamma \in \Gamma \ [\gamma \geqslant \theta \ \text{ or } \ \lambda(\theta) \geqslant \lambda_0].$$

Finally we denote by γ_0 the largest ordinal in Γ; γ_0 is standard by Theorem 2.1 and $\forall \theta \ [\gamma_0 \geqslant \theta$ or $\lambda(\theta) \geqslant \lambda_0]$ holds. Therefore $\lambda(\gamma_0 + 1) \geqslant \lambda_0$. However, $\gamma_0 + 1$ is standard, hence $\lambda(\gamma_0 + 1)$ is standard as well. □

8.7. The undefinability of truth.

Now we turn to the proof of the corollary mentioned in 1.13. Assume the contrary, that $\tau(x)$ is a st-$\in$-formula such that for each internal formula $\Phi(x_1, ..., x_n)$ the following is provable in **IST**:

$$\forall x_1 \ldots \forall x_n \; [\Phi(x_1, \ldots, x_n) \leftrightarrow \tau\,(\ulcorner\Phi(x_1, \ldots, x_n)\urcorner)].$$

The formula τ expresses the truth of all internal formulae. To derive a contradiction, we denote by $T(x, y)$ the formula

x is the translation $\ulcorner\varphi(\upsilon)\urcorner$ *of some internal* $\varphi(\upsilon)$ *with only* υ *free (parameters are allowed)* & $\tau(\ulcorner\varphi(y)\urcorner)$.

Let $T^*(\theta, X, \lambda)$ be the formula that corresponds to T by the comments 8.5. We consider the formula $\Phi(\theta, y)$, which says (informally) that y is not contained in $\mathbb{V}_\lambda$ whenever λ satisfies $T^*(\theta+\omega, \mathbb{V}_{\theta+\omega}, \lambda)$. (We recall that ω is the least infinte ordinal.) Then $\forall\theta \in \mathrm{Ord}\; \exists y\; \Phi(\theta, y)$ is true by the choice of T^*. We shall show that the formula Φ leads to a contradiction.

To see this we fix an unbounded ordinal θ, put $X = \mathbb{V}_{\theta+\omega}$, and let λ be the unique ordinal satisfying $T^*(\theta+\omega, \mathbb{V}_{\theta+\omega}, \lambda)$. We define Y by $Y = \mathbb{V}_\lambda$. Then

$$\forall x \in X \; [\exists y \;\; T(x, y) \rightarrow \exists y \in Y \;\; T(x, y)].$$

Now we take $x = \ulcorner\Phi(\theta, \upsilon)\urcorner$. Clearly $x \in X = \mathbb{V}_{\theta+\omega}$, hence

$$\exists y \;\; T(x, y) \rightarrow \exists y \in Y \;\; T(x, y).$$

Claim 1. The left-hand side is true.

Indeed, let y be such that $\Phi(\theta, y)$ holds. Then $\tau(\ulcorner\Phi(\theta, y)\urcorner)$ holds as well by the choice of the formula τ. □

Claim 2. The right-hand side is false.

Indeed, let $y \in Y$ satisfy $T(x, y)$. Then $\tau(\ulcorner\Phi(\theta, y)\urcorner)$ holds by the definition of T. Hence $\Phi(\theta, y)$ is true by the choice of τ. But $\Phi(\theta, y)$ says that $y \notin Y = \mathbb{V}_\lambda$ whenever $T^*(\theta+\omega, \mathbb{V}_{\theta+\omega}, \lambda)$ holds. □

The contradiction we have reached completes the proof of Corollary 1.13. Note that in fact the following is proved: for each st-$\in$-formula $\tau(x)$ there is an internal formula $\Phi(\theta, y)$ such that

$$\exists\theta \;\exists y \;\neg\, [\Phi(\theta, y) \leftrightarrow \tau\,(\ulcorner\Phi(\theta, y)\urcorner)]$$

is a theorem of **IST**. □

§9. Independence

This section contains the proof of Theorems 3, 7, 8. We shall construct a model of **IST** in whch the hypothesis $\mathbf{Repl}_2$ fails at $X = \mathbb{N}$ and at some "parameter-free" formula Φ. All other hypotheses mentioned in Theorem 3 fail as well as $\mathbf{Repl}_2$ in such a model. To build up the required model we use a special ground model of **ZFC** and a special way of arranging its nonstandard extensions, neither of which is the same as in Nelson [28].

9.1. The ground model.
We assume (in **ZFC**) the existence of a cardinal θ such that $\mathbb{V}_\theta$ is a model of **ZFC**. Of course, this assumption is outside **ZFC**. But it is taken for the sake of convenience only. One might get rid of it in the same way as in §4, that is, by considering an appropriate extension of **ZFC**.

Thus, let θ be an infinite cardinal satisfying the claim that $\mathbb{V}_\theta$ is a model of **ZFC**. Moreover we suppose that θ is the *least* cardinal of such a kind.

Finally we assume that the well-known set theoretic *axiom of constructibility* $\mathbb{V} = \mathbb{L}$ holds. The essential consequence of $\mathbb{V} = \mathbb{L}$ here is that a certain relation $<_L$ well orders the whole universe $\mathbb{V}$ in such a way that the following two properties hold:

1) given a cardinal θ, $<_L$ well orders $\mathbb{V}_\theta$ with order type θ;

2) given a cardinal θ, $<_L$ restricted to $\mathbb{V}_\theta$ is $\in$-definable in $\mathbb{V}_\theta$.

We fix a "natural" enumeration $\varphi_k(v_1, ..., v_{m(k)})$, $k \in \mathbb{N}$, of all "parameter-free" $\in$-formulae with a clear indication of the list of its free variables. It is not hard to prove in **ZFC** that for each integer n there is a cardinal $\varkappa < \theta$ such that $\mathbb{V}_\varkappa$ is an elementary submodel of $\mathbb{V}_\theta$ with respect to all sentences of type

$$\varphi_k(p_1, \ldots, p_{m(k)}),\ k \leqslant n,\ p_i \in \mathbb{V}_\varkappa .$$

Let $\varkappa_n$ denote the least cardinal $\varkappa$ of such a kind. Clearly, $\varkappa_n \leqslant \varkappa_{n+1}$ for all n, $\varkappa = \sup\{\varkappa_n : n \in \mathbb{N}\}$ is a cardinal, and $\mathbb{V}_\varkappa$ is an elementary submodel of $\mathbb{V}_\theta$ with respect to all $\in$-formulae with parameters from $\mathbb{V}_\varkappa$, that is, $\mathbb{V}_\varkappa$ is a model of **ZFC**. Thus in fact $\varkappa = \theta$, hence $\theta = \sup_{n\in\mathbb{N}} \varkappa_n$.

The set $M = \mathbb{V}_\varkappa$ will be taken as a ground model of **ZFC** for the construction of the **IST** model we need. The way of extending M is connected with the use of *definable* functions as elements of an ultrapower. Let us recall some notions concerning definability.

Firstly let the letter V denote the set $\mathbb{V}_\theta$ (as well as M). We use two different marks for a single set because of the two different roles that $\mathbb{V}_\theta$ plays in our reasoning, that is, the ground model and the "universe of definability".

Note that $\varkappa_n \in V$ for all n. Indeed it suffices to prove that $\varkappa_n < \theta$. We suppose to the contrary that $\varkappa_n = \theta$ for all $n \geqslant n_0$, $n_0 \in \mathbb{N}$. Then considering that V is a model of **ZFC** and taking $n = n_0+1$, we find a cardinal $\varkappa \in V$ (hence $\varkappa < \theta$) such that $\mathbb{V}_\varkappa$ is an elementary submodel of V with respect to all φ_k, $k \leqslant n$. Hence $\varkappa_n \leqslant \varkappa < \theta$, a contradiction.

We recall that $\mathrm{Def}(V)$ usually denotes the set of all sets $X \subseteq V$ definable in V. More exactly, $X \in \mathrm{Def}(V)$ if and only if

$$X = \{z \in V\colon \varphi^V(z)\} = \{z \in V\colon \varphi(z) \text{ is true in } V\}$$

for some $\in$-formula φ with parameters from V having a single free variable z. The superscript V in φ^V means the *relativization* of φ to V, that is, each quantifier $\exists z$ or $\forall z$ in φ takes the form $\exists z \in V$ or $\forall z \in V$.

Lemma. *The sequence* $\langle \varkappa_n : n \in \mathbb{N}\rangle$ *does not belong to* $\mathrm{Def}(V)$.

Proof. Assume to the contrary that there is an $\in$-formula $\varphi(n, \varkappa)$ (parameters from V are allowed) satisfying

$$\forall n \in \mathbb{N}\ \forall \varkappa \in \mathbb{V}\ [\varkappa = \varkappa_n \leftrightarrow \varphi^V(n, \varkappa)].$$

Let n be such that 1) each parameter occurring in φ is a member of $\mathbb{V}_{\varkappa_n}$, and 2) $\mathbb{V}_{\varkappa_n}$ is an elementary submodel of V with respect to the formulae $\exists \varkappa\ \varphi(\mathrm{v}, \varkappa)$ (v free) and $\varphi(\mathrm{v}, \varkappa)$ (v, $\varkappa$ free). Then $\exists \varkappa\ \varphi(n, \varkappa)$ holds in V (to see this take $\varkappa = \varkappa_n$), therefore it holds in $\mathbb{V}_{\varkappa_n}$ too. Hence there is $\varkappa \in \mathbb{V}_{\varkappa_n}$ satisfying $\varphi(n, \varkappa)$ in $\mathbb{V}_{\varkappa_n}$ as well as in V. This is possible only in the case $\varkappa = \varkappa_n$. Thus $\varkappa_n \in \mathbb{V}_{\varkappa_n}$, a contradiction. □

In fact the sequence of the cardinals $\varkappa_n$ will serve as a basis for destroying **Repl$_2$** in the **IST** model we shall construct. The main idea is to build up a nonstandard extension of M using only those functions from the index set into M that are in $\mathrm{Def}(V)$. One may hope that the sequence $\langle \varkappa_n : n \in \mathbb{N}\rangle$ will not penetrate into an extension of such a kind. On the other hand, Theorem 5 ensures that the map $n \mapsto \varkappa_n$ will be definable in the extension by some (external) formula.

Now we turn to details.

9.2. Index set and the ultrafilter.

We define

$$I = \mathscr{P}^{\mathrm{fin}}(M) = \{i \subseteq M\colon i \text{ is finite}\}.$$

I will be the *index* set. Clearly $I \in \mathrm{Def}(V)$. The following theorem gives the ultrafilter we need.

Theorem. *There is an ultrafilter U over I possessing the following two properties*:

(A) $\{i \in I : a \in i\} \in U$ *whenever* $a \in M$;

(B) $\{p \in M : \{i : \langle i, p\rangle \in P\} \in U\}$ *is in* $\mathrm{Def}(V)$ *whenever* $P \subseteq I \times M$, $P \in \mathrm{Def}(V)$.

Proof. Firstly we let U_0 be the collection of all sets of type $\{i \in I : a \in i\}$, where $a \in M$. It is evident that U_0 has the *finite intersection property* (f.i.p.), which says that the intersection of any finite subcollection of sets from U_0 is not empty.

Secondly we fix an enumeration $\chi_k(i, p)$, $k \geqslant 1$, of all $\in$-formulae with only two free variables having no parameters. We recall that V is well-ordered by the order relation $<_L$ so that the order type of V is θ. Let p_α ($\alpha < \theta$) be the αth element of V with respect to $<_L$. The sequence $\langle p_\alpha : \alpha < \theta\rangle$ belongs to $\mathrm{Def}(V)$ because $<_L$ restricted to V belongs to $\mathrm{Def}(V)$. We define

$$A_k(\alpha) = \{i \in I\colon\ \chi_k(i, p_\alpha) \text{ is true in } V\} \text{ and } C_k(\alpha) = I \setminus A_k(\alpha).$$

Claim. There is a sequence $\langle \rho(k, \alpha) : k \geqslant 1 \ \&\ \alpha < \theta\rangle$ such that
1) each $\rho(k, \alpha)$ is either 0 or 1;
2) given $k \geqslant 1$, the subsequence $\langle \rho(k, \alpha) : \alpha < \theta\rangle$ is in $\mathrm{Def}(V)$;
3) given $m \geqslant 1$ and $\gamma \leqslant \theta$, the set

$$U_{m\gamma} = \{A_k(\alpha) : k \leqslant m \ \&\ [k = m \to \alpha < \gamma] \ \&\ \rho(k, \alpha) = 1\} \cup$$
$$\cup\ \{C_k(\alpha) : k \leqslant m \ \&\ [\, k = m \to \alpha < \gamma] \ \&\ \rho(k, \alpha) = 0\}$$

satisfies the f.i.p.

Proof of the claim. The key idea is that whenever U' is a f.i.p. collection and $X \subseteq I$, at least one of the sets X, $I \setminus X$ can be added to U' without destroying the f.i.p.; moreover one can organize the way of choosing between $A_k(\alpha)$ and $C_k(\alpha)$ at each state $\langle k, \alpha\rangle$ within Def(V). The routine construction of $\rho(k, \alpha)$ by induction on k and on α when k is fixed is left to the reader. □

Finally we define $U_\infty = \bigcup_{m \in \mathbb{N}} U_{m\theta}$. Then U_∞ has the f.i.p., hence one can enlarge U_∞ to an ultrafilter U over I. The set U is as required. □

9.3. The quantifier "there exist U-many".
One can use this logical tool to simplify considerably the technical framework of applying the properties (A) and (B) of the ultrafilter U given by the preceding theorem. We define the new quantifier $\mathbf{Q} = \mathbf{Q}_U$ by

$$\mathbf{Q}i\ \varphi(i) \text{ if and only if } \{i \in I : \varphi(i) \text{ is true in } M\} \in U.$$

One can easily check the following properties of $\mathbf{Q}$ by using the properties (A) and (B) of U and the usual properties of any ultrafilter:

(Q1) if $a \in M$, then $\mathbf{Q}i\ (a \in i)$;

(Q2) if $P \subseteq I \times M$, $P \in \mathrm{Def}(V)$, then $\{p \in M : \mathbf{Q}i\ (\langle i, p\rangle \in P)\} \in \mathrm{Def}(V)$;

(Q3) if $\forall i\ [\varphi(i) \to \psi(i)]$, then $\mathbf{Q}i\ \varphi(i) \to \mathbf{Q}i\ \psi(i)$;

(Q4) $\mathbf{Q}i\ \varphi(i) \ \&\ \mathbf{Q}i\ \psi(i) \leftrightarrow \mathbf{Q}i\ [\varphi(i) \ \&\ \psi(i)]$;

(Q5) $\mathbf{Q}i\ \neg\ \varphi(i) \leftrightarrow \neg\ \mathbf{Q}i\ \varphi(i)$;

(Q6) $\varphi \leftrightarrow \mathbf{Q}i\ \varphi$ whenever i is not free in φ;

(Q7) $\forall i\ \varphi(i) \to \mathbf{Q}i\ \varphi(i) \to \exists i\ \varphi(i)$.

9.4. The extension.
Let $r \geqslant 1$. We define

$$I^r = I \times I \times \ldots \times I \ (r \text{ factors } I);$$
$$M^r = \{f \in \mathrm{Def}(V)\colon f \text{ is a function}, f : I^r \to M\};$$

in addition we put $I^0 = \{0\}$ and $M^0 = \{\langle 0, z\rangle\colon z \in M\}$. We also define $^*M = \bigcup_{r \geqslant 0} M^r$. If $f \in {}^*M$, then let $r(f)$ denote the unique r satisfying $f \in M^r$.

Further, if $f \in {}^*M$, $q \geqslant r = r(f)$, $\mathbf{i} = \langle i_1, \ldots, i_r, \ldots, i_q\rangle \in I^q$, then we define $f[\mathbf{i}] = f(i_1, \ldots, i_r)$. Note that $f[\mathbf{i}] = f(\mathbf{i})$ whenever $r = q$. In addition we put $f[\mathbf{i}] = z$ for $f = \langle 0, z\rangle \in M^0$.

Let $f, g \in {}^*M$ and $r = \max\{r(f), r(g)\}$. We define

$$f \in^* g \ \text{ if and only if } \ \mathrm{Q}i_r\mathrm{Q}i_{r-1} \ldots \mathrm{Q}i_1 \ \ (f[\mathbf{i}] \in g[\mathbf{i}]);$$
$$f =^* g \ \text{ if and only if } \ \mathrm{Q}i_r\mathrm{Q}i_{r-1} \ldots \mathrm{Q}i_1 \ \ (f[\mathbf{i}] = g[\mathbf{i}]);$$

of course, $\mathbf{i}$ denotes the sequence $i_1, \ldots, i_r$.

Let $^*s = \langle 0, s\rangle$ for all $s \in M$; clearly $^*s \in M^0$.

Finally we give the definition of standardness in $^*M$ by:

$$^*\mathrm{st}\, f \text{ if and only if there is } s \in M \text{ such that } f \ {}^*\!= {}^*s.$$

Thus up to the $^*{=}$ the level M^0 is just the standard part of $^*M$.

The truth of all st-$\in$-formulae in $^*M$ is defined in the sense of replacing the logical symbols $=$, $\in$, st by the relations $^*{=}$, $^*{\in}$, *st respectively.

Now the last definitions. Let Φ be a formula with parameters from $^*M$. We define $r(\Phi) = \max\{r(f)\colon f \text{ is occurring in } \Phi\}$. If in addition $r \geqslant r(\Phi)$ and $\mathbf{i} \in I^r$, then let $\Phi[\mathbf{i}]$ denote the result of replacing each f occurring in Φ by $f[\mathbf{i}]$. Clearly $\Phi[\mathbf{i}]$ is a formula with parameters from M.

9.5. ***Theorem*** (the Łoš theorem). *Let Φ be an $\in$-sentence with parameters from $^*M$ and suppose that $r \geqslant r(\Phi)$. Then*

$$\Phi \ \textit{ is true in } \ M^* \leftrightarrow \mathrm{Q}i_r \ldots \mathrm{Q}i_1 \ (\Phi[i_1, \ldots, i_r] \ \textit{ is true in } \ M).$$

Proof. The proof goes by induction on the logical complexity of Φ. The case of elementary formulae $f = g$, $f \in g$ immediately follows from the definition. Now the induction step.

As usual, it suffices to consider only the steps $\neg$, &, $\exists$. The first two of them do not require any discussion (apply the properties (Q4, Q5, Q6) of the quantifier Q).

The step $\exists$. We prove the theorem for a formula $\exists x\ \Phi(x)$ assuming the result holds for $\Phi(f)$ whenever $f \in {}^*M$. Let $r = r(\Phi)$.

The direction $\to$. Suppose that $\exists x\ \Phi(x)$ holds in $^*M$. Then $\Phi(f)$ holds in $^*M$ for some $f \in {}^*M$. We define p by $p = \max\{r, r(f)\}$. To convert the following reasoning into more convenient form, let **i** and **j** denote sequences of type

$$i_1, \ldots, i_r \ (\in I^r) \text{ and } i_1, \ldots, i_r, \ldots, i_p \ (\in I^p)$$

respectively. Let **Qi** and **Qj** denote the sequences of quantifiers of the forms

$$\mathrm{Q}i_r \ldots \mathrm{Q}i_1 \quad \text{and} \quad \mathrm{Q}i_p \ldots \mathrm{Q}i_r \ldots \mathrm{Q}i_1.$$

Then $\mathbf{Qj}\ \Phi(f)[\mathbf{j}]$ holds by the induction hypothesis. Note that $\Phi(f)[\mathbf{j}] \to \exists x\ \Phi(x)[\mathbf{j}]$ for all **j**. Hence $\mathbf{Qj}\exists x\ \Phi(x)[\mathbf{j}]$ by (Q3). But $\exists x\ \Phi(x)[\mathbf{j}]$ coincides (graphically) with $\exists x\ \Phi(x)[\mathbf{i}]$ because $r(\exists x\ \Phi(x)) = r \leqslant p$. Hence, deleting the superfluous quantifiers by (Q6), we obtain $\mathbf{Qi}\exists x\ \Phi(x)[\mathbf{i}]$.

The direction $\leftarrow$. We suppose that $\mathbf{Qi}\exists x\ \Phi(x)[\mathbf{i}]$ holds. The following set P belongs to $\mathrm{Def}(V)$ by (Q2):

$$P = \{\langle \mathbf{i}, z\rangle \in I^r \times M : \Phi(\langle 0, z\rangle)[\mathbf{i}] \text{ is true in } M\}.$$

(Note that $\langle 0, z\rangle = {}^*z \in {}^*M$ and ${}^*z[\mathbf{i}] = z$ for all **i**.) For each $\mathbf{i} \in I^r$ let $f(\mathbf{i})$ be the $<_L$-least $z \in M$ such that $\langle \mathbf{i}, z\rangle \in P$. (If such a set z exists: otherwise define $f(\mathbf{i}) = 0$.) Then $f \in \mathrm{Def}(V)$ by the definability of $<_L$, hence $f \in {}^*M$. Further, we note that

$$\forall \mathrm{i} \in I^r\ (\exists x\ \Phi(x)\ [\mathrm{i}] \to \Phi(f)\ [\mathrm{i}]),$$

therefore $\mathbf{Qi}\exists x\ \Phi(x)[\mathbf{i}] \to \mathbf{Qi}\ \Phi(f)[\mathbf{i}]$. We recall that the left-hand side of the last implication has been assumed to be true. Hence the right-hand side is also true. Then $\Phi(f)$ holds in ${}^*M$ by the induction hypothesis. □

Corollary. *Let* φ *be an* $\in$*-sentence with parameters from* M. *Suppose that* ${}^*\varphi$ *is obtained from* φ *by replacing each* $p \in M$ *by* ${}^*p$. *Then* φ *holds in* M *if and only if* ${}^*\varphi$ *holds in* ${}^*M$.

Proof. Clearly ${}^*\varphi[\mathbf{i}]$ coincides with φ. □

9.6. *Theorem.* $\langle {}^*M, {}^*{=}, {}^*{\in}, {}^*\mathrm{st}\rangle$ *is a model of* IST.

Proof. The preceding corollary says that transfer holds in ${}^*M$. Hence all the ZFC axioms also hold in ${}^*M$, being true in M. Standardization is ensured by $y \subseteq x \in M \to y \in M$. Thus all that remains to be proved is idealization.

Let $\varphi(x, a)$ be an $\in$-formula with parameters from ${}^*M$. We take $r = r(\varphi)$ and prove the following:

$$\forall^{\mathrm{stfin}} A\ \exists x\ \forall a \in A\quad \varphi(x, a) \to \exists x\ \forall^{\mathrm{st}} a\ \varphi(x, a)$$

in ${}^*M$. (The implication $\leftarrow$ does not need special consideration because it follows from standardization, see 2.8.) One may rewrite the left-hand side of I by the Loš theorem in the form

$$\forall^{\mathrm{fin}} A \subseteq M\ \mathrm{Q}i_r \ldots \mathrm{Q}i_1\ \exists x\ \forall a \in A\ \ (\varphi(x, a)\,[i_1, \ldots, i_r]).$$

We recall that I consists of all finite subsets of M, and thus replace the variable A by i, implying that $i \in I$. We further define $\widetilde{A} : I^{r+1} \to M$ by $\widetilde{A}(i_1, ..., i_r, i) = i$. Then $\widetilde{A} \in {}^*M$. Now the left-hand side of **I** takes the form

$$\forall i \; \mathbf{Q} i_r \, . \, . \, \mathbf{Q} i_1 \; (\exists x \; \forall a \in \widetilde{A} \; \varphi(x, \; a)) \, [i_1, \; . \; . \; ., \; i_r, \; i].$$

Changing $\forall i$ to $\mathbf{Q}i$ we obtain $\exists x \; \forall a \in \widetilde{A} \; \varphi(x, a)$ in ${}^*M$, again by the Łoš theorem. Hence to derive the right-hand side of **I** it suffices to prove that ${}^*a \in \widetilde{A}$ in ${}^*M$ for all $a \in M$. This is equivalent to

$$\mathbf{Q} i \, \mathbf{Q} i_r \, . \, . \, . \, \mathbf{Q} i_1 \; (a \in \widetilde{A} \, [i_1, \; . \; . \; ., \; i_r, \; i]),$$

by the Łoš theorem, and further to $\mathbf{Q}i \; \mathbf{Q}i_r \; ... \; \mathbf{Q}i_1 \; (a \in i)$ by the definition of $\widetilde{A}$. We apply (**Q1**) and complete the proof. □

9.7. The violation of the hypotheses in $^*M$.
Coming back to the definition of $\varkappa_n$ in 9.1, we see that for each integer n there is a certain $\in$-formula $\Phi_n(\varkappa)$ by which $\varkappa_n$ has been defined, that is,

$$\forall \varkappa \; [\varkappa = \varkappa_n \leftrightarrow \Phi_n(\varkappa)] \text{ is true in } V.$$

Let $\tau(...)$ be the truth formula of Theorem 5. We denote by $\Phi(n, \varkappa)$ the formula $\tau(\ulcorner\Phi_n(\varkappa)\urcorner)$ & st $\varkappa$ & $n \in \mathbb{N}$ & st n.

Lemma. *The following case of* **Repl**$_2$ *fails in the model* $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N} \; \exists! \varkappa \;\; \Phi(n, \; \varkappa) \to \exists f \; \forall^{\mathrm{st}} n \in \mathbb{N} \; \Phi(n, \; f(n)).$$

Proof. We verify the truth of the left-hand side in ${}^*M$. We fix an integer n and prove $\exists!^{\mathrm{st}}\varkappa \; \Phi({}^*n, \varkappa)$ in ${}^*M$. To obtain the existence of $\varkappa$, we take $\varkappa = \varkappa_n$. Then $\Phi_n(\varkappa)$ holds in M, hence $\Phi_n({}^*\varkappa)$ holds in ${}^*M$ by transfer. Therefore $\Phi({}^*n, {}^*\varkappa)$ holds in ${}^*M$ by the definition of Φ. To justify the uniqueness we suppose that $\Phi({}^*n, \varkappa')$ is true in ${}^*M$. Then $\varkappa'$ is standard in ${}^*M$, hence one may assume that $\varkappa' = {}^*\varkappa$ for some $\varkappa \in M$. Turning the preceding argument into the reverse direction, we reach $\varkappa = \varkappa_n$.

We verify the falsity of the right-hand side. Suppose on the contrary that $f \in {}^*M$ satisfies $\forall \varkappa \; [\Phi({}^*n, \varkappa) \leftrightarrow \varkappa = f({}^*n)]$ in ${}^*M$ for all integers n. Let $r = r(f)$. The Łoš theorem shows that

$$\varkappa = \varkappa_n \leftrightarrow \mathbf{Q} i_r \, . \, . \, . \, \mathbf{Q} i_1 \;\; ({}^*\varkappa = f\,({}^*n)) \;\; [i_1, \; . \; . \; ., \; i_r].$$

However $f \in \mathrm{Def}(V)$, the map $s \mapsto {}^*s$ also belongs to $\mathrm{Def}(V)$, and the action of **Q** does not lead out of $\mathrm{Def}(V)$. Hence the map $n \mapsto \varkappa_n$ is in $\mathrm{Def}(V)$ too. This contradicts Lemma 9.1. The proof is completed. □

Now we are assured that **Sep**$_3$, **Repl**$_i$, **Chc**$_i$, $i = 1, 2, 3, 4, 5$, **Coll**$_1$, **Coll**$_2$(st Φ) fail in the model ${}^*M$. We prove that **BRepl**$_4$ and **BChc**$_4$ also fail.

Let H be a finite set such that $\mathbb{S} \subseteq H$ (see 2.9). Let ν denote the number of elements of H (thus ν is a nonstandard integer), and let $K = \{1, 2, ..., \nu\}$.

Finally let h be a $1-1$ map K onto H. We claim that **BRepl**$_4$ is false in $^*M$ for $X = Y = \mathbb{N}$ and that the formula

$$\Psi(n,\ k) =_{\mathrm{def}} k \Subset K\ \&\ \ \mathrm{st}\, h\ (k)\ \&\ \Phi(n,\ h\ (k))$$

holds (Φ is as above). In other words, the following fails in $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists! k \in \mathbb{N}\ \Psi(n,\ k) \to \exists \tilde{k}\ \forall^{\mathrm{st}} n \in \mathbb{N}\ [\tilde{k}(n) \in \mathbb{N}\ \&\ \Psi(n,\ \tilde{k}(n))].$$

Indeed we suppose that the right-hand side is true in $^*M$ for some $\tilde{k}$. We define $f(n) = h(\tilde{k}(n))$ for all n. Then $\Phi(n, f(n))$ is true in $^*M$ for all standard n—a contradiction with the lemma.

This ends the proof of Theorem 3. □

It is evident that nonstandard parameters play an essential role in our arguments with regard to **BRepl**$_4$ (hence to **BChc**$_4$). In fact I do not know whether **BRepl**$_4$ and **BChc**$_4$ are false in $^*M$ or in any other model of **IST** for a core formula without nonstandard parameters.

9.8. The complexity of violating formulae.
Now we are able to prove Theorems 7 and 8. One can easily verify that the formula Sat from §7 can be transformed to Π_2^{st} form (to be more precise, Sat is equivalent in **IST** to some Π_2^{st} formula). Hence the truth formula τ is transformable to $\exists\Pi_2^{\mathrm{st}}$ form as well as to $\forall\Sigma_2^{\mathrm{st}}$ form (see the definiton of τ before the beginning of the proof of Lemma 7.2). Hence Φ and Ψ from 9.7 are in fact (equivalent to some) formulae of type $\exists\Pi_2^{\mathrm{st}}$ as well as of type $\forall\Sigma_2^{\mathrm{st}}$. This ends the proof of part (a) of Theorem 7 and the claim of Theorem 8 which is related to **BRepl**$_4$.

To prove part (b) of Theorem 7, we consider another formula:

$$\varphi(n,\ T)\ =_{\mathrm{def}} n \in \mathbb{N}\ \&\ \mathrm{st}\ n\ \&\ \mathrm{Sat}(T)\ \&\ \exists^{\mathrm{st}} \varkappa\ (\lceil \Phi_n(\varkappa) \rceil \in T)$$

of type Π_2^{st}. Thus we claim that the next sentence fails in $^*M$:

$$\forall^{\mathrm{st}} n \in \mathbb{N}\ \exists T\ \varphi(n,\ T) \to \exists \widetilde{T}\ \forall^{\mathrm{st}} n \in \mathbb{N}\ \varphi(n,\ \widetilde{T}(n)).$$

To verify the truth of the left-hand side, we fix some n. A set $T \in {}^*M$ such that $\varphi({}^*n,\ T)$ is true in $^*M$ can be obtained by applying (in $^*M$) the claim 2 from the proof of Lemma 7.2 to the formula $\Phi_n({}^*\varkappa_n)$.

To verify the falsity of the right-hand side, we suppose on the contrary that $\widetilde{T} \in {}^*M$ satisfies $\Phi({}^*n,\ \widetilde{T}({}^*n))$ for all n. Then

$$\varkappa = \varkappa_n \leftrightarrow \mathrm{Q}i_r\ .\ .\ .\ \mathrm{Q}i_1\ \ (\ulcorner\Phi_n({}^*\varkappa)\urcorner \Subset \widetilde{T}\,({}^*n))\ [i_1,\ .\ .\ .,\ i_r]$$

is true in M for all n, $\varkappa$. Hence the map $n \mapsto \varkappa_n$ belongs to $\mathrm{Def}(V)$. This again contradicts Lemma 9.1.

Thus the proof of Theorem 7(b) is also complete. □

Finally, to prove Theorem 8 (**BChc**$_4$), we set $X = \mathbb{N}$, $Y = \mathscr{P}^{\mathrm{fin}}(\mathbb{N})$ and use the following core formula:

$$\psi\ (n,\ \ t)\ =_{\mathrm{def}} t \sqsubseteq K\ \&\ \ \varphi(n,\ \ h\text{''}t).$$

(For K and h look at 9.7; $h\text{''}t = \{h(k) : k \in t\}$.) □

§10. Final comments. Externalization as a general way to new problems

The external forms of separation, replacement, choice, and collection, and the uniqueness property, which we consider above, do not cover the list of all interesting external analogues of classical **ZFC** theorems. In fact there are many set-theoretic sentences of interest in the investigations related to our work. As a topic of demonstration we choose the external cardinality.

We recall that sets X, Y are *equipotent* (or have the same cardinality)—symbolically $X \approx Y$—if there is a $1-1$ map $f: X$ onto Y. Two meanings of the notion of a map are possible:

a) as a set of ordered pairs such that ...

b) as a relation defined by some formula and such that ...

In the usual set theory **ZFC** they are the same, but not the same in **IST** because external formulae do not always define sets. One may expect unusual effects when external maps are allowed in the definition of equipotency. Indeed the sets

$$X = \{1, 2, \ldots, n\}, \quad Y = \{1, 2, \ldots, n, n+1\}$$

have different cardinalities n and $n+1$ but they are equipotent in the external sense in the case when n is infinitely large. To see this, let us define

$$f(k) = \begin{cases} k & \text{for standard } k \le n \\ k+1 & \text{for nonstandard } k \le n. \end{cases}$$

The (external) map f is $1-1$ X onto Y.

Thus finite cardinals n and $n+1$ are externally equipotent. The same is true for pairs n^2 and $(n+1)^2 = n^2+2n+1$ and in general for n^r and $(n+k)^r$, n infinitely large, k and r standard ($n, k, r \in \mathbb{N}$). The author tried to prove that $n \approx 2n$ externally (it should be sufficient for each pair of infinitely large integers to be externally equipotent) but did not succeed.(1)

No such "cardinality-mixing" external constructions are known for infinite cardinals. One can make the following hypothesis:

NEC: for each pair of infinite sets X, Y if card $X \neq$ card Y (in the usual sense), then there is no external $1-1$ map of X onto Y. (The correct formulation is left for the reader.)

Problem 16. Prove that NEC is consistent with IST.

The approach discussed above is directed from internal to external maps. Now let us consider another approach, that is, from outer to external sets. Let M be a model of **IST**. *Outer* with respect to M means any set and any relation in the "real world" $\mathbb{V}$, not necessarily a member of M or definable in M, while *external* means definable inside M with a st-$\in$-formula. Thus the

(1)This question has been solved by Henson and Ross, see B. Živaljević, J. Symbolic Logic **55** (1990), 604–614. (Added to the translation.)

outer cardinality of a set $X \in M$ is the real cardinality of the set of all M-members of X. It is known how to build up nonstandard models in which all internal infinite sets have the same outer cardinality, see Ross [33], as well as models of another kind, where (hyper)finite sets have different outer cardinalities, see Miller [26]. (Though it is not quite clear whether one may combine the constructions of Ross and Miller with Nelson's adequate ultralimits.)

The external variant of this outer property is contained in the following hypothesis:

EC: for each pair of infinite sets X, Y there is an external $1-1$ map $f: X$ onto Y.

Here the exact formulation is necessary:

$$\forall^{\inf} X\ \forall^{\inf} Y\ \exists p\,(\{\langle x,\ y\rangle\colon\ \Phi(x,\ y,\ p)\} \text{ is a } 1-1 \text{ map of } X \text{ onto } Y)$$

for some st-$\in$-formula Φ with only x, y, p free. (Of course, the straightforward expression of type $\forall X\ \forall Y\ \exists \Phi$ is incorrect.)

Problem 17. Prove that **EC** is consistent with **IST** for some Φ.

Many similar problems, deeper and more interesting, may be obtained in the same way.

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