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Topologies generated by effectively Suslin sets, and their applications in descriptive set theory

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Contents

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Preface

In 1994 the mathematical community marked the 100th anniversary of the birth of Mikhail Suslin, a Russian mathematician of great talent and tragic destiny. Suslin was born in Saratov province on 15 (3) November 1894, graduated from Moscow University in 1917, taught in Ivanovo-Voznesensk Polytechnic Institute for some time, and died of typhoid in his native place in October 1919.[1]

When the author received an honorary offer, at the conference devoted to the memory of Suslin, held in Saratov in the summer of 1994, to write an article for *Uspekhi Matematicheskikh Nauk* about one of the modern sections in the foundations of mathematics where Suslin's ideas have found a new realization, he did not hesitate long before he chose the Suslin sets to be his subject.[2] Neither did the choice of a specific topic cause much hesitation: ***topologies generated by effectively Suslin sets***, the set-theoretic technique that arose in the late 1970s as a synthesis of some ideas of the topology of complete metric spaces (and close to them) and 'effective' descriptive set theory, the technique that has now become, after a number of remarkable successes, perhaps the most popular method in descriptive set theory. It is typical, for example, that one of the two main lecture courses at the international conference of topologists in Amsterdam (August 1994) was devoted mostly to applications of this method in topology and descriptive set theory.[3]

Only the choice of the style for the article gave rise to some hesitation: should it be a wide survey of results or a more confined presentation, but showing, to some extent, the structure of the proofs. The author has chosen the second alternative, assuming that such exposition of the method[4] in an accessible journal of the highest rank will better help Russian mathematicians to become acquainted with applications of the topology of effectively Suslin sets.

It is a pleasure for the author to mention the help of V. M. Tikhomirov and V. A. Uspenskii in writing and publishing this paper.

Introduction

Generally, the simpler the sets we consider, the easier it is to prove their properties. Suppose that one or another property that we are interested in has been proved for all 'simple', say, open sets of the real line $\mathbb{R}$, the Baire space $\mathcal{N} = \omega^\omega$, or some other Polish space,[5] but we wish to extend the result to more complicated sets, for instance, Borel or Suslin sets. As a rule, this requires, of course, more sophisticated techniques. However, there is another approach.

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[1]The article by V. Igoshin in this issue gives material on Suslin's biography. The article [7] offers an extensive analysis of Suslin's mathematical activity. In [6] Tikhomirov covers important historical and mathematical moments connected with activities of a group of young Moscow mathematicians under the guidance of N. N. Luzin, to which Suslin belonged. All this makes it unnecessary to go any further into these themes in this article.

[2]During Suslin's life, only one mathematical work was published under his name: the note [25] in *Comptes Rendus*, one of the most authoritative journals of that time. This is where the Suslin sets were introduced in mathematics; Suslin himself called them ***the sets*** (A). They are known also under the name the A-sets, or the sets of the class $\mathbf{\Sigma}^1_1$.

[3]The material of the course delivered by A. S. Kechris is presented in his article [16].

[4]Apparently the first in Russian.

[5]That is, a complete separable metric space, often with the additional condition of the absence of isolated points: a *perfect* Polish space.

Namely, we change the topology of the space so that the more complicated sets of interest to us become, for example, open in the new topology. It is necessary, of course, to find out what is the form of the property that we wish to prove in the new topology, and also to make sure that this new topology is good enough to allow the argument about 'simple' sets that was allowed in the old Polish topology.

This approach turned out to be very successfully realized in descriptive set theory via a topology whose idea is due to Gandy, but whose first remarkable application[6] was obtained by Harrington. This is the topology with base formed by Σ^1_1-sets, or, more generally, sets of class $\Sigma^1_1(p)$, where $p \in \mathcal{N}$ is fixed (the properties of such a topology in practice do not depend on the choice of p).

Explanation of terminology. The class Σ^1_1 consists of the sets that can be obtained by an A-operation on *effective* (in our case, computable) families of closed sets of the space in question, which makes it natural to call the sets of Σ^1_1 *effectively Suslin*. In a more general sense, this term extends to the set of any class $\Sigma^1_1(p)$, as soon as the point p of the Baire space is fixed.

To demonstrate applications of the topologies generated by Suslin sets we have chosen three theorems (all of them are among the most important of quite recent results in descriptive set theory): these are Silver's theorem on the number of equivalence classes of the co-Suslin (or, which is the same, $\mathbf{\Pi}^1_1$) equivalence relations (§3), the theorem of Harrington, Kechris, and Louveau on the Glimm–Effros dichotomy for the Borel equivalence relations (§4), and Louveau's theorem on representation of plane Borel sets with vertical sections of a certain additive Borel class as a countable union of Borel sets with sections of lower class (§5).

Before that, §2 contains an introduction to the class of topologies under consideration, and §1 presents the main definitions and some principal theorems of effective descriptive set theory[7] used in the article.

This explains, in general, how the paper is organized.

The choice we have made in connection with the style of exposition has resulted in some disregard for *applications* of the method, especially those related to measure theory and Borel transformation groups, in favour of material on 'pure' descriptive set theory, even though these allegedly applied problems mainly inspired the development of the method itself. This circle of topics is covered by Kechris in [16].

§1. On effective descriptive set theory

This section is an introduction to terminology and a brief survey of some important theorems of descriptive set theory, which, in principle, is sufficient for a topologist, say, who has worked with the descriptive theory and has some idea about recursion theory, to follow the presentation.

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[6] *A new proof of Silver's theorem on co-Suslin equivalence relations*; see §3 below. It should be noted that neither Gandy nor Harrington has published materials on their pioneering research of this subject; their priority has been recognized and become known from the work of successors.

[7] Certainly, this field of mathematics, like probably any other, includes essential 'folklore' knowledge, which is sometimes hard to formalize, and here, in particular, would require more space than is available in this article. Therefore, preliminary acquaintance with effective descriptive set theory, in the scope, say, of the article [5] by Martin, would help to understand the technical details of our presentation.

1.1. Spaces and sets. We consider sets of spaces $\mathcal{X} = \omega^k \times \mathcal{N}^m$, where k, m are natural numbers, $\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers, and $\mathcal{N} = \omega^\omega$ is the *Baire space*. Points of these spaces (in particular, of $\mathcal{N}$) are denoted by the letters p, q, x, y, z, or sometimes a, b, c, and natural numbers by the letters i, j, k, l, m, n.

It is sometimes convenient to treat sets $P \subseteq \mathcal{X} = \omega^k \times \mathcal{N}^m$ as $k+m$-*ary relations*; thus, the formula $P(n_1, \dots, n_k, x_1, \dots, x_m)$ expresses the same as the formula $\{n_1, \dots, n_k, x_1, \dots, x_m\} \in P$. We shall not use this style when a confusion with a function may arise or when $k + m = 1$.

If we fix a discrete metric on ω (for example, $\delta(k, l) = 1$ for $k \neq l$), and the Baire metric $\rho(x, y) = n^{-1}$ on $\mathcal{N}$, where $n = \max\{n' : x \upharpoonright n' = y \upharpoonright n'\}$ for $x \neq y \in \mathcal{N}$, then the spaces $\mathcal{X} = \omega^k \times \mathcal{N}^m$ under consideration become separable complete metric spaces; this metric and the topology generated by it will be called *Polish* metric and topology in the sequel.

1.2. The classical hierarchies. The *Borel hierarchy* of classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, $\mathbf{\Delta}^0_\xi$, $1 \leqslant \xi < \omega_1$, is defined by transfinite induction on ξ:

- $\mathbf{\Sigma}^0_1$ = all the open sets of a given space;
- $\mathbf{\Pi}^0_\xi$ = the complements of the sets in $\mathbf{\Sigma}^0_\xi$; $\mathbf{\Delta}^0_\xi = \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$;
- $\mathbf{\Sigma}^0_\xi$ = countable unions of the sets in $\mathbf{\Pi}^0_\eta$, $1 \leqslant \eta < \xi$ for $\xi > 1$.

In particular, $\mathbf{\Pi}^0_1$ = the closed sets, $\mathbf{\Sigma}^0_2 = \mathbb{F}_\sigma$, $\mathbf{\Pi}^0_2 = \mathbb{G}_\delta$, and so on.

We consider also the initial level of the *projective* hierarchy:

- $\mathbf{\Sigma}^1_1$ is the collection of all the *projections* of $\mathbf{\Pi}^0_1$-sets. In other words, a set $X \subseteq \mathcal{X}$ belongs to $\mathbf{\Sigma}^1_1$ if there is a (closed) $\mathbf{\Pi}^0_1$-set $P \subseteq \mathcal{X} \times \mathcal{N}$ such that $X = \{x \in \mathcal{X} : \exists y \in \mathcal{N}\, P(x, y)\}$.
- $\mathbf{\Pi}^1_1$ = the complements of the sets in $\mathbf{\Sigma}^1_1$; $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$.

By Suslin's theorem, $\mathbf{\Delta}^1_1$ = all the Borel sets.

1.3. Enumeration of bases. All the spaces under consideration have a countable base; thus, given an enumeration of all the base sets, every open set is determined by a set of natural numbers, the indices of the base sets. The idea of the effective theory is to consider only *computable* unions of base sets.

The question of how all this depends on the choice of a particular enumeration of the base sets is of no interest here; we simply fix one specific enumeration for each space.

Natural numbers. We put $B_{n+1}[\omega] = \{n\}$, and separately, $B_0[\omega] = \varnothing$ (the empty set).

The Baire space. Here we have to put in some effort.

Let $\ulcorner k, l \urcorner = 2^k(2l+1) - 1$ for $k, l \in \omega$ (an arithmetic 'pair'). The enumeration $\omega^{<\omega} = \{s_n : n \in \omega\}$ of the set $\omega^{<\omega}$ of all finite sequences of natural numbers is defined as follows: $s_0 = s_1 = \Lambda$ (the empty sequence), and if $n = \ulcorner k, l \urcorner \geqslant 2$, then $s_n = s_k{}^\wedge l$. We put $B_{n+1}[\mathcal{N}] = \{x \in \mathcal{N} : s_n \subset x\}$, and separately, $B_0[\mathcal{N}] = \varnothing$.

The general case: the space $\mathcal{X} = \omega^l \times \mathcal{N}^m$. Let $n = \pi_1^{k_1} \cdots \pi_{l+m}^{k_{n+m}}$, where π_i is the ith prime number. We put

$$B_n[\mathcal{X}] = B_{k_1}[\omega] \times \cdots \times B_{k_l}[\omega] \times B_{k_{l+1}}[\mathcal{N}] \times \cdots \times B_{k_{l+m}}[\mathcal{N}]$$

for $n \geqslant 1$, and separately, $B_0[\mathcal{X}] = \varnothing$ (this coincides with $B_1[\mathcal{X}]$).

1.4. The effective hierarchy.

- Σ^0_1 (in the space $\mathcal{X}$) is the class of all sets of the form $X = \bigcup_n B_{f(n)}[\mathcal{X}]$, where $f \in \mathcal{N}$ is a computable function.
- Let $p \in \mathcal{N}$. Then $\Sigma^0_1(p)$ is the class of all sets of the form $X = \bigcup_n B_{f(n)}[\mathcal{X}]$, where $f \in \mathcal{N}$ is a function computable with respect to p.

(It can be assumed that a *computable* function $f\colon \omega \to \omega$ is a function such that there is a computer program for its evaluation. If we speak about a more general concept of a program, which includes the use of values $p(n)$ of a given function p (which itself may fail to be computable), this is the computability with respect to p. In general, some acquaintance with such notions is assumed.)

- Π^0_1 = the complements of the sets in Σ^0_1; $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$.
- Σ^1_1 = the projections of the Π^0_1-sets.
- Π^1_1 = the complements of sets in Σ^1_1; $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- The classes $\Pi^0_1(p)$, $\Delta^0_1(p)$, $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$ are defined similarly.

The initial classes Σ^0_1 and $\Sigma^0_1(p)$ for any p, and hence all the remaining classes Γ^1_i, $\Gamma^1_i(p)$,[8] contain only countably many sets, unlike their 'classical' prototypes $\mathbf{\Gamma}^i_{\dots}$.

We deliberately omit here the definition of the classes Γ^0_ξ for $2 \leqslant \xi < \omega_1$; this question is not simple[9] and will be considered later, in §5.

This classification is by definition extended to sets of natural numbers as well as subsets of the spaces ω^k; in particular, since every point $x \in \mathcal{N}$ is a function from $\mathcal{N}$ to $\mathcal{N}$, that is, a subset of ω^2 (as a graph), this gives a meaning to expressions of the type $x \in \Sigma^1_1(p)$, where x and p belong to $\mathcal{N}$.

1.5. Relationships between the classes. For any p and $i = 0, 1$, we have $\Sigma^i_1(p) \not\subseteq \Pi^i_1(p)$; on the other hand, $\Sigma^0_1(p) \cup \Pi^0_1(p) \subseteq \Delta^1_1(p)$.

Moreover, the hierarchy is *monotone* in p in the sense that if $q \in \Delta^0_1(p)$, then $\Gamma^i_1(q) \subseteq \Gamma^i_1(p)$. The classes Γ^i_1 are identical to $\Gamma^i_1(p)$ for $p \in \Delta^0_1$; in particular, $\Gamma^i_1 = \Gamma^i_1(\mathbf{0})$, where the point $\mathbf{0} \in \mathcal{N}$ is determined by the condition $\mathbf{0}(k) = 0$ for all k.

Connection with the classical hierarchy: $\mathbf{\Gamma}^i_1 = \bigcup_{p \in \mathcal{N}} \Gamma^i_1(p)$.

Substitution of parameters: if $P(x, y, z, \dots)$ is a relation of the class $\Gamma^1_1(p)$ and $x_0 \in \mathcal{N}$ belongs to $\Delta^1_1(p)$, then the relation $Q(y, z, \dots)$ defined as $P(x_0, y, z, \dots)$ also belongs to $\Gamma^1_1(p)$. (As usual, $\Gamma = \Sigma$, Π or Δ.)

There are simple rules (see, for example, the book by Shoenfield [8], Chapter 7.8) that allow evaluation of the class of a set obtained as a result of certain operations applied to sets whose classes are known. These rules can be conveniently represented by treating the sets as relations (see §1.1) so that the language of logic can be used.

Negation. The negation of a $\Sigma^i_1(p)$-relation is a relation of the class $\Pi^i_1(p)$, and vice versa. In terms of sets, this expresses the fact that the sets of $\Sigma^i_1(p)$ and $\Pi^i_1(p)$ are mutually complementary.

[8] It is customary to denote by Γ and $\mathbf{\Gamma}$ any of the classifiers Σ, Π, Δ or $\mathbf{\Sigma}$, $\mathbf{\Pi}$, $\mathbf{\Delta}$, respectively.

[9] Apart from relatively small, for example, finite values of ξ, in which case the definition is quite elementary.

Conjunction and disjunction. All the classes defined in §1.4 are closed with respect to these operations (in the finite form).

Quantifiers of the type ω. The classes $\Sigma^0_1(p)$ and all $\Gamma^1_1(p)$, $\Gamma = \Sigma$, Π, Δ, are closed with respect to the quantifier $\exists n \in \omega$. In other words, if $P(n,x,y,\dots)$ is a $\Sigma^0_1(p)$-relation, say, then the relation $Q(x,y,\dots) \longleftrightarrow \exists n \in \omega\, P(n,x,y,\dots)$ belongs to the same class $\Sigma^0_1(p)$.

The classes $\Pi^0_1(p)$ and all $\Gamma^1_1(p)$ are closed with respect to the quantifier $\forall z \in \omega$.

Quantifiers of the type $\mathcal{N}$. The classes $\Sigma^0_1(p)$ and $\Sigma^1_1(p)$ are closed with respect to the quantifier $\exists\, z \in \mathcal{N}$. The classes $\Pi^0_1(p)$ and $\Pi^1_1(p)$ are closed with respect to the quantifier $\forall z \in \mathcal{N}$.

These rules will be systematically used in our presentation.

1.6. Uniformization, reduction, and separation.

Uniformization. Let $P \subseteq \mathcal{X} \times \mathcal{Y}$. We say that a set $Q \subseteq P$ *uniformizes* P if, firstly, the projection of Q to $\mathcal{X}$ coincides with the projection of P to $\mathcal{X}$, that is, formally, for any $x \in \mathcal{X}$, if $\exists\, y\, P(x,y)$, then $\exists\, y\, Q(x,y)$, and, secondly, Q is a *uniform set*, that is, a set whose intersection with any 'vertical line' $x = x_0$ contains at most one point. A uniform set can be understood as the graph of a function that maps a subset of $\mathcal{X}$ into $\mathcal{Y}$.

Principle 1 (Novikov–Kondo–Addison; see [8], Chapter 7.11). *Let* $P \subseteq \mathcal{X} \times \mathcal{Y}$ *be a* $\Pi^1_1(p)$*-set,* $p \in \mathcal{N}$. *Then there is a* $\Pi^1_1(p)$*-set* $Q \subseteq P$ *uniformizing* P.

There are known important cases in which uniformization can be achieved via a set of the class Δ^1_1. We give one example, of special interest.

Corollary 2. *If, under the conditions of Principle* 1, $\mathcal{Y} = \omega$, *whereas* $X = \{x : \exists\, y\, P(x,y)\}$ *(that is, the projection of* P *to* $\mathcal{X}$*) is a* $\Sigma^1_1(p)$*-set, then the uniformizing set* Q *can be chosen in the class* $\Delta^1_1(p)$.

Proof. We take an arbitrary $\Pi^1_1(p)$-set $Q \subseteq P$ uniformizing P. Then Q automatically belongs to $\Sigma^1_1(p)$ since

$$Q(x,k) \longleftrightarrow x \in X \,\&\, \forall\, k' \neq k\ \neg Q(x,k').$$

Reduction and separation. We say that a pair of sets A', B' *reduces* the pair A, B if $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \varnothing$, but $A' \cup B' = A \cup B$. We say that a set C *separates* A from B if $A \subseteq C$ and $B \cap C = \varnothing$.

Corollary 3 [Reduction and separation]. *Let* $p \in \mathcal{N}$. *Any pair of* $\Pi^1_1(p)$*-sets* A, B *(in the same space) can be reduced by a pair of* $\Pi^1_1(p)$*-sets. If* A *and* B *are disjoint* $\Sigma^1_1(p)$*-sets, then there is a separating* $\Delta^1_1(p)$*-set.*

Proof. The set $P = (A \times \{0\}) \cup (B \times \{1\})$ belongs to $\Pi^1_1(p)$. We uniformize P by a set $Q \subseteq P$ in $\Pi^1_1(p)$; then $A' = \{x : Q(x,0)\}$ and $B' = \{x : Q(x,1)\}$ are $\Pi^1_1(p)$-sets, which proves the reduction. To obtain the separation, it suffices to perform the reduction of the complements.

1.7. Enumeration of the classes $\Delta^1_1(p)$. We recall that in each class $\Delta^1_1(p)$ there are only countably many sets. One of the most important technical tools of the effective theory is the existence of special enumerations of these classes.

Principle 4 [Part 1: enumeration of Δ^1_1-sets]. *There are a Π^1_1-set $\mathbf{W} \subseteq \mathcal{N} \times \omega$ and an indexed family $\langle \mathbf{D}_n(p) : \langle p, n\rangle \in \mathbf{W}\rangle$ of sets $\mathbf{D}_n(p) \subseteq \mathcal{N}$ such that*

(i) *for any $p \in \mathcal{N}$, the set $\{\mathbf{D}_n(p) : \langle p, n\rangle \in \mathbf{W}\}$ is identical to the collection of all $\Delta^1_1(p)$-sets $X \subseteq \mathcal{N}$;*

(ii) *the following sets belong to Π^1_1:*

$$\{\langle p, n, x\rangle : \mathbf{W}(p,n) \,\&\, x \in \mathbf{D}_n(p)\} \quad \textit{and} \quad \{\langle p, n, x\rangle : \mathbf{W}(p,n) \,\&\, x \notin \mathbf{D}_n(p)\}.$$

[Part 2: enumeration of Δ^1_1-points]. *There are a Π^1_1-set $\mathbf{w} \subseteq \mathcal{N} \times \omega$ and an indexed family $\langle \mathbf{d}_n(p) : \langle p, n\rangle \in \mathbf{w}\rangle$ of points $\mathbf{d}_n(p) \in \mathcal{N}$ such that*

(iii) *for any $p \in \mathcal{N}$, the set $\{\mathbf{d}_n(p) : \langle p, n\rangle \in \mathbf{w}\}$ is identical to the collection of all $\Delta^1_1(p)$-points $x \in \mathcal{N}$;*

(iv) *the following sets belong to Π^1_1:*

$$\{\langle p, n, \mathbf{d}_n(p)\rangle : \langle p, n\rangle \in \mathbf{w}\} \quad \textit{and} \quad \{\langle p, n, x\rangle : \langle p, n\rangle \in \mathbf{w} \,\&\, x \neq \mathbf{d}_n(p)\}.$$

Thus, for any $p \in \mathcal{N}$, we have the enumeration of the family of all $\Delta^1_1(p)$-sets $X \subseteq \mathcal{N}$ in the form $\{\mathbf{D}_n(p) : n \in \mathbf{W}(p)\}$, where $\mathbf{W}(p) = \{n : \mathbf{W}(p,n)\}$ is a set of the class $\Pi^1_1(p)$, and the similar enumeration of the collection of all $\Delta^1_1(p)$-points $x \in \mathcal{N}$. Both enumerations have convenient characteristics of definability that are close to $\Delta^1_1(p)$-definability. (It is known that, in reality, a $\Delta^1_1(p)$-enumeration is impossible in this situation.)

Proof. [10] *Part* 1. Using the technique of constructing universal sets (see [5], Theorem 4.9) we can produce a pair of Π^1_1-sets $U, V \subseteq \mathcal{N} \times \omega \times \mathcal{N}$ that is doubly universal in the sense that, for any $p \in \mathcal{N}$, if sets $X, Y \subseteq \mathcal{N}$ belong to $\Pi^1_1(p)$, then there is an $n \in \omega$ such that $X = U_{pn} = \{x : U(p,n,x)\}$ and $Y = V_{pn} = \{x : V(p,n,x)\}$.

Corollary 3 gives a pair of Π^1_1-sets $U' \subseteq U$, $V' \subseteq V$, that reduces the pair U, V; in particular, $U' \cap V' = \varnothing$. Let $\mathbf{W} = \{\langle p, n\rangle : U'_{pn} \cup V'_{pn} = \mathcal{N}\}$ and $\mathbf{D}_n(p) = U'_{pn} = \{x : U'(p,n,x)\}$ for $\langle p, n\rangle \in \mathbf{W}$. To prove that the second set mentioned in (ii) belongs to Π^1_1, one must use the fact that, for $\langle p, n\rangle \in \mathbf{W}$, the sets U'_{pn} and V'_{pn} are the complements of each other.

Part 2. Here the calculations are somewhat different. We start with a Π^1_1-set $U \subseteq \mathcal{N} \times \omega \times \omega^2$ that is universal in the sense that, for any $p \in \mathcal{N}$, if a set $X \subseteq \omega^2$ belongs to $\Pi^1_1(p)$, then there is a natural number n such that $X = U_{pn} = \{\langle k, l\rangle : U(p,n,k,l)\}$. We uniformize U as a subset of $(\mathcal{N} \times \omega \times \omega) \times \omega$ by a set $V \subseteq U$ of the class Π^1_1. The main feature is that if U_{pn} is *already* a function defined on ω, that is, a point of $\mathcal{N}$, then $V_{pn} = U_{pn}$. Now we achieve the goal by putting

$$\mathbf{w} = \{\langle p, n\rangle : U_{pn} \in \mathcal{N}\} = \{\langle p, n\rangle : \forall\, k\ \exists!\, l\ U(p,n,k,l)\}$$

[10] We give an outline of the proof here because, firstly, this is a key fact, and, secondly, it is difficult to offer a reasonable reference to a source in Russian.

and $\mathbf{d}_n(p) = V_{pn} = U_{pn}$ for $\langle p,n\rangle \in \mathbf{w}$. To prove that the sets in (iv) belong to Π^1_1, one must keep in mind that, for $\langle p,n\rangle \in \mathbf{w}$,

$$x = \mathbf{d}_n(p) \longleftrightarrow \forall\, k,l \in \omega\ [U(p,n,k,l) \to x(k) = l]$$
$$\longleftrightarrow \forall\, k,l \in \omega\ [x(k) = l \to U(p,n,k,l)].$$

Corollary 5. *Let $R(x,p,\dots)$ be a Π^1_1-relation. Then the relation $\exists\, x \in \Delta^1_1(p)$ $R(x,p,\dots)$ also belongs to Π^1_1.*

Proof. Applying Principle 4, we obtain

$$\exists\, x \in \Delta^1_1(p)\ R(x,p,\dots) \longleftrightarrow \exists\, n\ [\mathbf{w}(p,n)\ \&\ \forall\, x\ (x = \mathbf{d}_n(p) \to R(x,n,\dots))].$$

Consider the right-hand side. The quantifier $\exists\, n$ is $\exists\, n \in \omega$, that is, it does not count. It remains to observe that $\mathbf{w}(p,n)$ is a Π^1_1-relation, and the equality $x = \mathbf{d}_n(p)$ can be expressed by the Σ^1_1-formula $\neg(x \neq \mathbf{d}_n(p))$.

Corollary 6. *The set $\Delta = \big\langle \langle p,x\rangle \in \mathcal{N}^2 : x \in \Delta^1_1(p)\big\rangle$ belongs to Π^1_1.*

Proof. $\Delta(p,x) \longleftrightarrow \exists\, x' \in \Delta^1_1(p)\ (x = x')$.

§2. Topologies generated by effectively Suslin sets

Now we have the terminology needed to formally introduce this family of topologies.[11]

2.1. Topology.

Definition 7. $\mathcal{T}$ is the topology whose base is formed by all Σ^1_1-sets of a given space. Similarly, for $p \in \mathcal{N}$, $\mathcal{T}(p)$ is the topology generated by the $\Sigma^1_1(p)$-sets of a given space.

These topologies are rather unusual. $\mathcal{T}(p)$ is stronger than the Polish topology, since all the Baire intervals belong to $\Sigma^1_1(p)$. However, $\mathcal{T}(p)$ makes some singletons open; more precisely, $\{x\}$ is open in $\mathcal{T}(p)$ if $x \in \Delta^1_1(p)$. On the other hand, $\mathcal{T}(p)$ has a countable base.

Furthermore, these topologies are not Polish. In fact, a simple property of topologies metrizable by a complete metric is that every closed set is $\mathbb{G}_\delta$. Consider an arbitrary Π^1_1-set that is not a $\mathbf{\Sigma}^1_1$-set. (The existence of such sets is easily proved in the framework of the effective theory.) This set is closed in $\mathcal{T}(p)$ by definition, but it cannot be $\mathbb{G}_\delta$, since any $\mathbb{G}_\delta$ in the sense of $\mathcal{T}(p)$ is $\mathbf{\Sigma}^1_1$ in the sense of Polish topology.

At the same time, some derivatives of the completeness of Polish spaces are inherited by the topologies $\mathcal{T}(p)$. In particular, we shall see later (Corollary 13) that all these topologies are Baire topologies.

[11] A significant part of the material of this section is taken from the articles by Martin and Kechris [20], Louveau [18], Kechris [16], Harrington, Kechris, and Louveau [12], and the book by Mansfield and Weitkamp [19] where more complete information can be found.

For a proof of this property it is necessary to have a method of constructing decreasing sequences of sets with non-empty intersection in the given topology, a method similar, say, to the one provided by a complete metric. In the classical presentations of the subject (see footnote 11) this is done via Choquet games. We use a different technique, which is closer to the methods of complete metric spaces, but we consider also the method based on Choquet games.

As is customary, in what follows we consider only the topology $\mathcal{T}$; however, all the results hold for any topology $\mathcal{T}(p)$, with obvious changes as appropriate.

2.2. Ensuring the non-emptiness of intersections. The following definition introduces a structure that will quite successfully replace complete metrizability for the topology $\mathcal{T}$.

Definition 8. We call a collection of families of sets $\mathcal{X}_m$ $(m \in \omega)$ a *Polish network* if

(1) each $\mathcal{X}_m$ is a family of non-empty open subsets of a given space;
(2) each $\mathcal{X}_m$ is *dense* in the topology: for every non-empty open set X there is $Y \in \mathcal{X}_m$, $Y \subseteq X$;
(3) *compactness*: if $X_m \in \mathcal{X}_m$ and $X_{\leqslant m} = \bigcap_{k \leqslant m} X_k \neq \varnothing$ for all m, then the intersection $\bigcap_{m \in \omega} X_m$ contains exactly one point.

For example, in the case of $\mathcal{N}$ with the usual topology we define a Polish network by $\mathcal{X}_m = \{\mathcal{N}_s : s \in \omega^m\}$, where $\mathcal{N}_s = \{x \in \mathcal{N} : s \subseteq x\}$ for $s \in \omega^{<\omega}$.

Lemma 9. *The topology $\mathcal{T}$ has a Polish network that satisfies the following additional condition of 'genericness':*

(4) for any Σ^1_1-set $A \subseteq \mathcal{N}$ there is an $m \in \omega$ such that each $X \in \mathcal{X}_m$ satisfies the requirement: $X \subseteq A$ or else $X \cap A = \varnothing$.

Proof. Consider the space $\mathcal{N}^{1+\omega} = \mathcal{N} \times \mathcal{N}^\omega$. It is convenient to represent points $\vec{x} \in \mathcal{N}^{1+\omega}$ in the form $\vec{x} = \langle x, x_0, x_1, x_2, \dots \rangle$. For $P \subseteq \mathcal{N}^{1+\omega}$ we put

$$\operatorname{pr} P = \{x : \exists \vec{x} = \langle x, x_0, \dots \rangle \in P\} \text{ and } \operatorname{pr}_n P = \{x_n : \exists \vec{x} = \langle x, x_0, \dots, x_n, \dots \rangle \in P\}$$

for all n. Also, we put $\vec{x} \restriction_{\leqslant n} = \langle x, x_0, x_1, \dots, x_n \rangle$ for any point $\vec{x} = \langle x, x_0, x_1, \dots, x_n, \dots \rangle \in \mathcal{N}^{1+\omega}$ and $\operatorname{pr}_{\leqslant n} P = \{\vec{x} \restriction_{\leqslant n} : \vec{x} \in P\}$ for $P \subseteq \mathcal{N}^{1+\omega}$.

We define $\mathcal{B}_n$ as the family of all Π^0_1-sets $B \subseteq \mathcal{N}^{1+\omega}$ of the form

$$B = \operatorname{pr}^{-1}_{\leqslant n}(B) = \{\vec{x} \in \mathcal{N}^{1+\omega} : \vec{x} \restriction_{\leqslant n} \in B'\}, \quad \text{where } B' \subseteq \mathcal{N}^{1+n} \text{ is a } \Pi^0_1\text{-set},$$

for every n. It is clear that $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is countable; let $\mathcal{B} = \{\mathrm{B}^m : m \in \omega\}$, where any $B \in \mathcal{B}$ has infinitely many numbers m such that $B = \mathrm{B}^m$.

Let $m \in \omega$. We introduce the collection $\mathcal{X}_m$ of all *non-empty* Σ^1_1-sets $X = \operatorname{pr} B \subseteq \mathcal{N}$, where the (non-empty) $B \in \mathcal{B}$ satisfies the following conditions:

(i) $\operatorname{diam} \operatorname{pr}_k B \leqslant m^{-1}$ for all $k \leqslant m$ and $\operatorname{diam} \operatorname{pr} B \leqslant m^{-1}$;[12]
(ii) either $X \cap \operatorname{pr} \mathrm{B}^m = \varnothing$, or $B \subseteq \mathrm{B}^m$.

We prove that the families $\mathcal{X}_m$ form a Polish network. We need the following result.

[12]The diameter $\operatorname{diam} X$ of a set $X \subseteq \mathcal{N}$ is understood in the sense of a fixed complete metric on $\mathcal{N}$; see §1.

Proposition 10. *Suppose that $Q \in \mathcal{B}$ and $X \subseteq \operatorname{pr} Q$ is a non-empty Σ^1_1-set. Then there is a set $B \in \mathcal{B}$, $B \subseteq Q$, such that* $\operatorname{pr} B = X$.

Proof. There is a Π^0_1-set $F \subseteq \mathcal{N}^2$ satisfying $X = \operatorname{pr} F = \{x : \exists y\, F(x,y)\}$. Let $Q \in \mathcal{B}_n$. Then $B = \{\vec{x} \in Q : \langle x, x_{n+1}\rangle \in F\}$ belongs to $\mathcal{B}_{n+1}$ (since $Q \in \mathcal{B}_n$) and $\operatorname{pr} B = X$.

We now return to the proof of the lemma.

Verify the density of $\mathcal{X}_m$: condition (2) of Definition 8. Suppose that a Σ^1_1-set $X \subseteq \mathcal{N}$ is non-empty; it is required to find a $Y \in \mathcal{X}_m$ such that $Y \subseteq X$.

Case 1: $X \cap \operatorname{pr} \mathrm{B}^m = \varnothing$. Then $X = \operatorname{pr} Q$ for a suitable $Q \in \mathcal{B}_0$, since $X \in \Sigma^1_1$. Taking the intersection of Q with a 'Baire cube' of sufficiently small diameter in $\mathcal{N}^{1+\omega}$, we obtain a non-empty Π^0_1-set $B \in \mathcal{B}$, $B \subseteq Q$, satisfying (i). It remains to put $Y = \operatorname{pr} B$.

Case 2: $X' = X \cap \operatorname{pr} \mathrm{B}^m \neq \varnothing$. We apply Proposition 10 to the sets X' and $Q = \mathrm{B}^m$. We have a subset $Q' \in \mathcal{B}$, $Q' \subseteq Q$, such that $X' = \operatorname{pr} Q'$. It remains to take the intersection of Q' with a suitable 'cube' to obtain a non-empty set $B \subseteq Q'$ satisfying (i).

Verify the compactness: condition (3). Thus, suppose that $X_m \in \mathcal{X}_m$ and $\bigcap_{k \leqslant m} X_k$ is non-empty for each m; we prove that the intersection $\bigcap_m X_m$ contains exactly one point. It suffices to show that, whatever $m = m_0$, there is an infinite sequence $m_0 < m_1 < m_2 < \cdots$ such that $\bigcap_n X_{m_n}$ is non-empty; in fact, because of condition (i) every non-empty intersection of this kind has exactly one point, and so any two points obtained in such a way coincide.

Thus, let $m_0 \in \omega$. By induction, we find numbers m_n and sets $P_n \in \mathcal{B}_{m_n}$ satisfying the conditions $P_{n+1} \subseteq P_n$, $X_{m_n} = \operatorname{pr} \Pi_n$ and $\operatorname{diam} \operatorname{pr}_k P_n \leqslant m_n^{-1}$ for any $k \leqslant m_n$. Then the intersection $\bigcap_n P_n$ is non-empty (each P_n is closed in the Polish topology of the space $\mathcal{N}^{1+\omega}$, and the diameters converge to zero), that is, the intersection $\bigcap_n X_{m_n}$ is non-empty as well.

Since $X_{m_0} \in \mathcal{X}_{m_0}$, there is a set $P \in \mathcal{B}$ for which conditions (i) and (ii) hold for $m = m_0$ and the equality $X_{m_0} = \operatorname{pr} P$ is fulfilled. Let $P_0 = P$.

We assume that m_n and $P_n \in \mathcal{B}_{m_n}$ have been already defined. Then $P_n = \mathrm{B}^{(m)}$ for a suitable $m > m_n$. We recall that $X_m \in \mathcal{X}_m$; therefore, there is a set $B \in \mathcal{B}$ satisfying (i) and (ii) for this m, and $X_m = \operatorname{pr} B$. We remark that the case $X_m \cap \operatorname{pr} \mathrm{B}^{(m)}$ in (ii) is impossible here. In fact, $\operatorname{pr} \mathrm{B}^{(m)} = \operatorname{pr} P_n = X_{m_n}$; however, $X_m \cap X_{m_n}$ is non-empty by the choice of the sets X_m. Consequently, $B \subseteq \mathrm{B}^{(m)} = P_n$. It remains to take $m_{n+1} = m$ and $P_{n+1} = B$.

Finally, the 'genericness' requirement (4) is ensured by (ii).

Remark 11. It is easy to see that the Polish network $\{\mathcal{X}_m : m \in \omega\}$ given by the lemma satisfies the following monotonicity condition: if Y is a Σ^1_1-set and $\varnothing \neq Y \subseteq X \in \mathcal{X}_m$, then $Y \in \mathcal{X}_m$.

2.3. Some corollaries. We consider applications of Lemma 9, in particular, to some derivative topologies.

Definition 12. Let $n \geqslant 1$. We denote by $\mathcal{T}_n$ the $\mathcal{T}$-topology of the space $\mathcal{N}^n$. (We note that $\{\mathcal{N}^n; \mathcal{T}_n\}$ is homeomorphic to $\langle \mathcal{N}; \mathcal{T} \rangle$.)

$\mathcal{T}^n$ is the Tikhonov product of n copies of $\mathcal{T} = \mathcal{T}_1$.

$\mathcal{T}_{n+m}$ is the product $\mathcal{T}_n \times \mathcal{T}_m$, a topology on $\mathcal{N}^{n+m}$.

Corollary 13. *All the topologies $\mathcal{T}_n$ and $\mathcal{T}^n$ are Baire.*[13]

Proof. First, a product of two topologies that have a Polish network has a Polish network (consisting of Cartesian products of the sets that form the networks of the factors). Then, to prove that a topology with a Polish network is Baire, we use an elementary proof of the Baire property for complete metric spaces.

The topology $\mathcal{T}_n$ contains $\mathcal{T}^n$ and, in fact, is strictly stronger than $\mathcal{T}^n$; for example, the diagonal $\Delta(\mathcal{N}) = \{\langle x, x \rangle : x \in \mathcal{N}\}$ is open in $\mathcal{T}_2$, but not in $\mathcal{T}^2$. However, the product of topologies is sufficiently dense in the product topology.

Lemma 14. *Suppose that a set $V \subseteq \mathcal{N}$ is $\mathcal{T}$-open, and $D \subseteq V$ is of the second category in V in the sense of $\mathcal{T}$. Then $D \times D$ is dense in $V \times V$ in the sense of $\mathcal{T}_2$.*

Proof. For simplicity we assume that $V = \mathcal{N}$. Let $\bigcap_n D_n \subseteq D$, where all the sets $D_n \subseteq \mathcal{N}$ are $\mathcal{T}$-open and dense in $\mathcal{N}$. Then every set $D'_n = D_n \times \mathcal{N}$ is open and dense in the sense of $\mathcal{T}_2$. (For the projection $\operatorname{pr} A$ of any Σ^1_1-set A is a Σ^1_1-set in $\mathcal{N}$.) Similarly, every set $D''_n = \mathcal{N} \times D_n$ is open and dense. At the same time, we have $\bigcap_n (D'_n \cap D''_n) \subseteq D \times D$. It remains to apply Corollary 13.

The next lemma will be used later, in examining Borel equivalence relations.

Let R be an equivalence relation on $\mathcal{N}$. We put

$$\mathsf{R}^{(n)} = \{\langle x_1, \dots, x_n \rangle : \forall i \, (x_i \, \mathsf{R} \, x_{i+1})\}.$$

For any n and m we denote by $\langle \mathsf{R}^{(n+m)}; \mathcal{T}_{n+m} \rangle$ the set $\mathsf{R}^{(n+m)} \subseteq \mathcal{N}^{n+m}$ with topology inherited from the space $\langle \mathcal{N}^{n+m}; \mathcal{T}_{n+m} \rangle$.

Lemma 15. *Let R be a Σ^1_1-equivalence relation on $\mathcal{N}$, $n' \leqslant n$ and $m' \leqslant m$. Then the projection $\pi \colon \mathcal{N}^n \times \mathcal{N}^m$ to $\mathcal{N}^{n'} \times \mathcal{N}^{m'}$ is an open continuous map of $\langle \mathsf{R}^{(n+m)}; \mathcal{T}_{n+m} \rangle$ onto $\langle \mathsf{R}^{(n'+m')}; \mathcal{T}_{n'+m'} \rangle$.*

Proof. For simplicity, let $m = n = 2$, $m' = n' = 1$; then $\pi(x_1, x_2; y_1, y_2) = \langle x_1, y_1 \rangle$. We omit an easy verification of continuity and concentrate our efforts on the proof of the openness, where, in particular, a special structure of the set $\mathsf{R}^{(n+m)}$ will be important. Consider a pair of $\mathcal{T}_2$-base sets, that is, Σ^1_1-sets $U, V \subseteq \mathcal{N}^2$. We prove that the projection $O = \pi\big((U \times V) \cap \mathsf{R}^{(4)}\big)$ is $\mathcal{T}^2$-open in $\mathsf{R}^{(2)} = \mathsf{R}$. Using the properties of closedness of the effective classes (see §1.5), we conclude that the sets

$$U' = \{x_1 : \exists x_2 \, [U(x_1, x_2) \,\&\, x_1 \, \mathsf{R} \, x_2]\} \quad \text{and} \quad V' = \{y_1 : \exists y_2 \, [V(y_1, y_2) \,\&\, y_1 \, \mathsf{R} \, y_2]\}$$

belong to Σ^1_1, that is, are $\mathcal{T}$-open. But $O = (U' \times V') \cap \mathsf{R}$.

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[13]The *Baire topologies* are characterized by the following property: all sets of the second category are dense. A *set of the second category* is a set containing an intersection of countably many dense open sets. (A direct proof that $\mathcal{T}$ is a Baire topology is given in [20].)

2.4. Choquet spaces. Now we present the more traditional technique of investigating the $\mathcal{T}$-topologies based on Choquet games.

A *Choquet game* $C_{\mathcal{X}}$ in a topological space $\mathcal{X}$ proceeds as follows. There are two players, β and α, and β starts the game. The *moves* in the game are non-empty open subsets of $\mathcal{X}$, and each move must be a subset of the previous move of the opponent. We obtain a sequence of moves

$$U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots \qquad (U_n \text{ and } V_n \text{ are open}), \qquad (1)$$

where U_n are the moves of the player β, and V_n those of α. Each player, before his next move, knows everything that has happened in the game till this move.

Finally, the winner is α if $\bigcap_n U_n \ (= \bigcap_n V_n) \ \neq \varnothing$.

We call $\mathcal{X}$ a *Choquet space*, or a *space with the Choquet property*, if α has a winning strategy.[14]

A modified, *strong* Choquet game $C'_{\mathcal{X}}$ gives an additional chance to the player β. Namely, with every move U_n, β makes an extra move $x_n \in U_n$, which forces α to respond by V_n such that $x_n \in V_n$ (and, as before, $V_n \subseteq U_n$).

The result is determined as in the game $C_{\mathcal{X}}$: α wins if $\bigcap_n V_n \neq \varnothing$.

Finally, $\mathcal{X}$ is called a space with *the strong Choquet property* if α has a winning strategy in the game $C'_{\mathcal{X}}$.

Lemma 16. *Complete metric spaces and the spaces with a Polish network have the strong Choquet property.*

Proof. In the case of a complete metric space, α can play in such a way that the *closure* of V_n is a subset in U_n of diameter less than n^{-1}. If there is a Polish network $\{\mathcal{X}_m : m \in \omega\}$, then all that the player α has to do is to make each move V_n so that $V_n \in \mathcal{X}_n$.

The strong Choquet property, in turn, implies a certain 'regularity', which allows us to deduce some usual consequences of the completeness.

Proposition 17 (Harrington, Kechris, and Louveau [12]).

1. *The strong Choquet property implies the Choquet property.*
2. *A Choquet space is a Baire space.*
3. *If X and Y are Choquet spaces, then $X \times Y$ is also a Choquet space. The same is true for spaces with the strong Choquet property.*
4. *If $\mathcal{X}$ has the strong Choquet property, then any non-empty $\mathbb{G}_\delta$-set $X \subseteq \mathcal{X}$ has the strong Choquet property in the inherited topology.*

Proof. Assertion 1 is obvious: the $C_{\mathcal{X}}$-strategy for the player α is his $C'_{\mathcal{X}}$-strategy with moves x_n disregarded.

2. Suppose that every $D_n \subseteq \mathcal{X}$ is open and dense, and U is open and non-empty; we prove that $I = U \cap \bigcap_n D_n$ is also non-empty. Let β play the game $C_{\mathcal{X}}$ making

[14] That is, a rule that prescribes how α must play, depending on the play of β, so that α will win, no matter how β plays. Technically, this notion is realized as a function τ defined on finite sequences of open sets $\langle U_0, \ldots, U_n \rangle$ with values also among open sets, so that $V_n = \tau(U_0, \ldots, U_n) \subseteq U_n$. See [20] on infinite games with perfect information, one of which is the game $C_{\mathcal{X}}$.

moves $U_n \subseteq D_n$ (we use the fact that the D_n are open and dense) and the initial move $U_0 \subseteq D_0 \cap U$. Then the winning strategy for α proves that I is non-empty.

3. Exercise: α 'splits' the game $C_{\mathcal{X}\times\mathcal{Y}}$ into the games $C_{\mathcal{X}}$ and $C_{\mathcal{Y}}$ by restricting each move U_n of the opponent to the product $U_n^x \times U_n^y \subseteq U_n$, where U_n^x and U_n^y are open in $\mathcal{X}$ and $\mathcal{Y}$, respectively.

4. Let $X = \bigcap_n G_n$, where each G_n is open in $\mathcal{X}$. To win the game C'_X, the player α acts as follows. Let $\langle U_0, x_0\rangle, \dots, \langle U_n, x_n\rangle$ be the initial moves of β in the game C'_X. Then $x_i \in U_i = X \cap \mathcal{U}_i$, where the sets $\mathcal{U}_i \subseteq \mathcal{X}$ are open for all $i \leqslant n$. Suppose that the $C'_{\mathcal{X}}$-strategy for α recommends the move $\mathcal{V}_n$ in response to the moves $\langle \mathcal{U}_0, x_0\rangle, \langle \mathcal{U}_1, x_1\rangle, \dots, \langle \mathcal{U}_n, x_n\rangle$ of β. Then there is an open $\mathcal{V}' \subseteq \mathcal{X}$ such that $x_n \in \mathcal{V}' \subseteq \mathcal{V}_n \cap G_n$. Now α makes the move $V_n = \mathcal{V}' \cap X$ in the game C'_X.

We conclude this section by presenting a direct proof of the Choquet property for the $\mathcal{T}$-topologies. The reader may find interesting parallels between this argument and the proof of Lemma 9.

Lemma 18. *Let $p \in \mathcal{N}$ and $n \geqslant 1$. Then the topology $\mathcal{T}$ on $\mathcal{N}^n$ has the strong Choquet property.*

Proof. Each $\mathcal{N}^n$ with the $\mathcal{T}$-topology is homeomorphic to $\langle \mathcal{N}; \mathcal{T}\rangle$; therefore, we can assume that $n = 1$. We recall that the base of $\mathcal{T}$ is formed by Σ_1^1-sets, and every such set $X \subseteq \mathcal{N}$ is the projection of some Π_1^0-set $F \subseteq \mathcal{N}^2$. This outlines the main idea of the proof: to make use of the completeness of the spaces $\mathcal{N}^m$ in the Polish metric (see §1.1).

In the notation of the proof of Lemma 9, the desired strategy for α in the game $C' = C'_{\langle \mathcal{N};\mathcal{T}\rangle}$ can be described in the following way.

Let $\langle U_0, x_0\rangle, \langle U_1, x_1\rangle, \langle U_2, x_2\rangle, \dots$ be the sequence of moves made by β. The player α, making his moves $V_0, V_1, V_2, \dots$, which in this case are Σ_1^1-sets rather than just $\mathcal{T}$-open sets, and in addition $x_n \in V_n$, constructs a sequence of Π_1^0-sets $F_n \subseteq \mathcal{N} \times \mathcal{N}^n$ so that

$$x_n \in V_n = \operatorname{pr} F_n \subseteq U_n, \quad \operatorname{pr}_{<n} F_n \subseteq F_{n-1}, \quad \operatorname{diam} \operatorname{pr}_m F_n \leqslant n^{-1} \text{ for } m \leqslant n. \tag{2}$$

By virtue of the completeness, the condition on diameters implies the existence of a unique sequence of points $\vec{x} = \langle x, x_0, x_1, x_2, x_3, \dots\rangle \in \mathcal{N}^{1+\omega}$, that satisfies $\vec{x} \restriction_{\leqslant n} \in F_n$ for all n. Then $x \in \bigcap_n V_n$, which is the required result.

It remains to check that α, by playing properly, can ensure that (2) is valid for all n. Thus, suppose that β has made his next move $\langle U_n, x_n\rangle$; here $x_n \in U_n \subseteq V_{n-1} = \operatorname{pr} F_{n-1}$. Since U_n is $\mathcal{T}$-open, there is a non-empty Σ_1^1-set $U'_n \subseteq U_n$ and a Π_1^0-set $P \subseteq \mathcal{N} \times \mathcal{N}$ such that $U'_n = \operatorname{pr} P$. Then the set $F = \{\langle x, x, y\rangle : \langle x, x\rangle \in F_{n-1} \,\&\, P(x, y)\}$ belongs to Π_1^0 and satisfies $x_n \in \operatorname{pr} F = U'_n$ and $\operatorname{pr}_{<n} F \subseteq F_{n-1}$. (We remark that $F \subseteq \mathcal{N} \times \mathcal{N}^n$.) We partition F into countably many Π_1^0-sets with projections of diameter less than n^{-1} by taking the intersections with 'small' Polish base neighbourhoods in $\mathcal{N} \times \mathcal{N}^n$. One of these sets, $F' \subseteq F$, say, satisfies $x_n \in \operatorname{pr} F'$. We put $F_n = F'$ and $V_n = \operatorname{pr} F'$ (the response of α).

§3. First application: co-Suslin equivalence relations

Before we start the technical part of this section, some discussion is necessary to present properly the meaning of the results.

We recall that Cantor's *continuum hypothesis* (CH) is the assertion that there is no cardinality strictly between the countable cardinality $\aleph_0$ and the cardinality of the continuum $\mathfrak{c} = 2^{\aleph_0}$ (which is strictly greater than $\aleph_0$). After the work of K. Gödel in the 1930s and P. Cohen in the early 1960s it has been definitively established that modern mathematics (at least as long as it is based on the Zermelo–Fraenkel set theory ZFC) does not allow the answer 'yes' or 'no' to the question: is the CH true?

However, the approach aimed at verifying CH for certain set classes turned out to be rather fruitful. In particular, Aleksandrov [9] and Hausdorff [13] proved that CH is true in the class of Borel sets (in the sense of Polish topology) of the real line $\mathbb{R}$ or, which is the same in the present case, the space $\mathcal{N}$. In other words, a Borel set $X \subseteq \mathcal{N}$ cannot have an intermediate cardinality. There are no counterexamples in the larger class of $\mathbf{\Sigma}^1_1$-sets either: this is Suslin's theorem [3], which asserts that, moreover, every uncountable $\mathbf{\Sigma}^1_1$-set has a perfect[15] subset. However, there may be some counterexamples in the class $\mathbf{\Pi}^1_1$. (See the author's work [4], where a more detailed analysis of this circle of questions is given.)

There is yet another approach to the study of CH for certain classes that consists in finding counterexamples in the form of *quotient sets* rather than *point sets*, that is, the following question is investigated: how many equivalence classes may an equivalence relation have? The situation here is somewhat similar to that for sets, but with a reverse: the number of equivalence classes for a $\mathbf{\Pi}^1_1$-relation cannot be an intermediate cardinal, whereas that for a $\mathbf{\Sigma}^1_1$-relation may (see the end of this section).

The second part of this assertion (on $\mathbf{\Sigma}^1_1$-relations) follows from what was said above about $\mathbf{\Pi}^1_1$-sets, whereas the first part is very complicated. In fact, no proof of it has been known so far that was carried out in the framework of classical methods of descriptive set theory.

Theorem 19 (Silver [23]). *Let* E *be a* $\mathbf{\Pi}^1_1$*-equivalence relation on* $\mathcal{N}$. *Then either* E *has finitely or countably many classes, or there is a perfect set of pairwise* E*-inequivalent points.*

We remark that this result automatically carries over to all the complete separable metric spaces, since every such space admits a Borel (in particular, preserving $\mathbf{\Pi}^1_1$) isomorphism onto $\mathcal{N}$.

Proof. [16] We assume that E has uncountably many equivalence classes and prove that in that case there is a perfect set of pairwise inequivalent points. The proof of this assertion consists of the idea and two technical parts.

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[15]That is, a non-empty closed set that has no isolated points in the sense of *Polish* topology. Such sets have the cardinality of the continuum.

[16]The proof given here is due to Harrington and is taken from the article [20] by Martin and Kechris, with a change in that we use the technique of Polish networks instead of Choquet games. The more complicated proof of Silver himself [23] relies on forcing technique; a similar proof is presented by Miller [21].

The idea. Suppose that we have managed to find a non-empty set $H \subseteq \mathcal{N}$, open in the *Polish* topology of $\mathcal{N}$ and such that E is of the first category on $H^2 = H \times H$; in other words, $\mathsf{E} \cap H^2 \subseteq \bigcup_n P_n$, where all the $P_n \subseteq H^2$ are nowhere dense. In this case we construct easily a system $\langle X_s : s \in 2^{<\omega} \rangle$ of non-empty open-closed sets $X_s \subseteq H$ such that

(a) $X_{s^\wedge i} \subseteq X_s$, $X_{s^\wedge 0} \cap X_{s^\wedge 1} = \varnothing$,
(b) $(X_s \times X_t) \cap \bigcup_{m \leqslant n} P_m = \varnothing$ for $s, t \in 2^n$, $s \neq t$,
(c) $\operatorname{diam} X_s \leqslant m^{-1}$ for $s \in 2^m$,

where 2^n is the collection of all sequences of 0's and 1's of length n, $2^{<\omega} = \bigcup_{n \in \omega} 2^n$, $s^\wedge i$ has the obvious meaning, and $\operatorname{diam} X$ denotes the diameter in the Polish metric. Then the set $X = \bigcap_n \bigcup_{s \in 2^n} X_s$ is the desired perfect set of pairwise inequivalent points.

Unfortunately, we cannot claim that a set H of the above-described form actually exists, in the *Polish* topology. This is where the topologies $\mathcal{T}(p)$ come into play.

Technical part 1. First, since $\mathsf{E} \in \mathbf{\Pi}^1_1$, there is a $p \in \mathcal{N}$ such that E is a $\Pi^1_1(p)$-relation. We assume that E belongs to Π^1_1; if this is not so, then simply the parameter p uniformly enters the computations, that is, say, $\mathcal{T}$ changes to $\mathcal{T}(p)$, and so on (see the remark at the end of §2.1).

We indicate a non-empty $\mathcal{T}$-open set H such that E is of the first category on $H^2 = H \times H$ in the sense of $\mathcal{T}^2$.

The idea is essentially very simple: we remove the E-equivalence classes that are open in the topology $\mathcal{T}$. In the Polish topology, this trick will not work: there is no guarantee that after the removal of open classes the rest will be an open set. However, in the Gandy–Harrington topology, as we shall see, everything will be all right due to its descriptive properties.

The *Harrington set* H is defined as follows:

$$H = \{x \in \mathcal{N} : \text{there is no } \Delta^1_1\text{-set } B \text{ such that } x \in B \subseteq [x]\},$$

where $[x] = [x]_{\mathsf{E}} = \{y : x \mathsf{E} y\}$ is the E-equivalence class of x.

We remark that H is non-empty; otherwise, every E-class would be a union of Δ^1_1-sets, and since there are only countably many of those, E would have at most countably many classes, which contradicts the assumption made at the beginning of the proof.

Lemma 20. *The set H is open in $\mathcal{T}$; moreover, H belongs to Σ^1_1.*

Proof. Indeed,

$$x \in H \longleftrightarrow \forall B \in \Delta^1_1 \; \big(x \in B \to \exists y \in B \; x \not\mathsf{E}\, y\big). \tag{3}$$

Let $p = \mathbf{0} = \omega \times \{0\}$ (identically zero, that is, say, $\Delta^1_1 = \Delta^1_1(\mathbf{0})$). Using the sets $\mathbf{W}$ and $\mathbf{D}_n$ provided by Principle 4 (part 1) in §1, we bring the right-hand side of (3) to the form

$$\forall n \; \big[\mathbf{W}(\mathbf{0}, n) \,\&\, x \in \mathbf{D}_n(\mathbf{0}) \to \exists y \; (y \in \mathbf{D}_n(\mathbf{0}) \,\&\, x \not\mathsf{E}\, y)\big].$$

This expression can easily be transformed to Σ^1_1 by using the properties of $\mathbf{W}$ and $\mathbf{D}_n$.

Lemma 21. E *is of the first category on* H^2 *in the sense of* $\mathcal{T}^2$.

Proof. We recall that E is a $\mathbf{\Pi}^1_1$, that is, a co-Suslin set in the Polish topology, and hence in $\mathcal{T}$. This implies that E has the Baire property in the sense of $\mathcal{T}$, since, as is known, the Baire property is preserved under A-operations. Thus, by virtue of the Ulam–Kuratowski theorem, it suffices to prove that $H_x = H \cap [x] = \{y \in H : x \mathsf{E} y\}$ is of the first category in the sense of $\mathcal{T}$ for any $x \in H$.

As before, H_x has the Baire property in the sense of $\mathcal{T}$. Therefore, to verify that H_x is of the first category in the sense of $\mathcal{T}$, it suffices to establish that H_x cannot be of the *second* category on any non-empty Σ^1_1-set $U \subseteq H$.

Assume the contrary: let H_x be of the second category in the sense of $\mathcal{T}$ on a non-empty Σ^1_1-set $U \subseteq H$. The set $D' = (H_x \cap U) \times (H_x \cap U)$ is dense in $U^2 = U \times U$ in the sense of $\mathcal{T}_2$, by Lemma 14. Thus, D' has a non-empty intersection with any non-empty Σ^1_1-set $P \subseteq U^2$. In particular, if the set $P = \{\langle y, z\rangle \in U^2 : y \not\mathsf{E}\, z\}$ is non-empty, then $P \cap D' \neq \varnothing$.

Suppose that $\langle y, z\rangle \in P \cap D'$. Then both y and z belong to H_x, that is, $y \mathsf{E} z$, which contradicts the assumption $\langle y, z\rangle \in P$. Therefore, P is in fact an empty set, which implies that $U \subseteq [x]$.

But in this case, the equivalence class $[x]$ itself belongs to Π^1_1, since $y \mathsf{E} x$ is equivalent to $\forall z\, [z \in U \to z \mathsf{E} y]$. Thus, the Σ^1_1-set U is included in the Π^1_1-set $[x]$. By the separation theorem (Corollary 3), there is a Δ^1_1-set B such that $U \subseteq B \subseteq [x]$. We take an arbitrary $x' \in U$. Then $x' \mathsf{E} x$, that is, we have $x' \in B \subseteq [x']$, whence $x' \notin H$. But $x' \in U \subseteq H$, a contradiction.

Technical part 2. We modify the argument at the beginning of the proof of the theorem to construct a perfect set X of pairwise E-inequivalent points, the modification being that the Polish network of the topology $\mathcal{T}$ will replace the completeness of the Polish topology of $\mathcal{N}$.

By what we have already proved, let $\mathsf{E} \cap H^2 \subseteq \bigcup_n P_n$, where each $P_n \subseteq \mathcal{N}^2$ is nowhere dense in $\mathcal{T}^2$. Let $\{\mathcal{X}_n : n \in \omega\}$ be the Polish network provided by Lemma 9 for the topology $\mathcal{T}$.

It is not difficult to construct a family of Σ^1_1-sets $X_s \subseteq H$ $(s \in 2^{<\omega})$, that satisfies conditions (a) and (b) (see above) and the following condition replacing (c):

(c′) if $s \in 2^m$, then $X_s \subseteq X'_s$ for some $X'_s \in \mathcal{X}_m$.

For any $a \in 2^\omega$, the sequence of sets $X_{a \restriction m}$, $m \in \omega$, has exactly one point x_a in its intersection, since $X_{a \restriction m} \in \mathcal{X}_m$. By construction, if $a \neq a'$, then $\langle x_a, x_{a'}\rangle \notin P_m$ for all m, so that $x_a \not\mathsf{E}\, x_{a'}$ and, in particular, $x_a \neq x_{a'}$. Therefore, the set $X = \{x_a : a \in 2^\omega\}$ is a perfect (and even homeomorphic to Cantor's discontinuum) set of pairwise E-inequivalent points.

Remark 22. We describe briefly the construction of the set of pairwise inequivalent points given in the paper [12], where a Choquet game was used to ensure the non-emptiness of intersections on each path $a \in 2^\omega$.

We construct two indexed families $\{U_s : s \in 2^{<\omega}\}$ and $\{V_s : s \in 2^{<\omega}\}$ of non-empty Σ^1_1-subsets of $\mathcal{N}$ so that, first, the family of sets U_s satisfy conditions (a), (b), and (c), and, second, for any $a \in 2^\omega$ the sequence of sets

$$U_{a\restriction 0} = U_\Lambda,\ V_{a\restriction 0} = V_\Lambda,\ U_{a\restriction 1},\ V_{a\restriction 1},\ U_{a\restriction 2},\ V_{a\restriction 2},\ \ldots,\ U_{a\restriction n},\ V_{a\restriction n},\ \ldots$$

corresponds to the winning strategy of the player α in the Choquet game $C_{\langle \mathcal{N};\mathcal{T}\rangle}$ (which exists for α by virtue of Lemma 18).[17] Then the intersection $X_a = \bigcap_n V_{a\restriction n}$ is non-empty for any $a \in 2^\omega$, and, in fact, because of condition (c) on diameters, X_a contains just one point x_a, and so on.

Equivalence relations of the class $\mathbf{\Sigma}^1_1$. This line of investigation now generates considerable interest, offering challenging problems, in particular, those connected with the possibility of obtaining analogues of the Silver theorem.

Unfortunately, Theorem 19 itself does not extend to the $\mathbf{\Sigma}^1_1$-relations. Consider a non-Borel Π^1_1-set $C \subseteq \mathcal{N}$, which is decomposed into Borel constituents: $C = \bigcup_{\alpha<\omega_1} C_\alpha$. We define the relation E by

$$x \mathrel{\mathsf{E}} y \text{ if and only if } \exists\, \alpha\ (x \in C_\alpha \,\&\, y \in C_\alpha) \vee (x \notin C \,\&\, y \notin C).$$

This is a $\mathbf{\Sigma}^1_1$-relation that does not admit perfect sets of pairwise inequivalent points by virtue of the restriction principle of the classical descriptive theory, but has $\aleph_1$ (non-empty) equivalence classes.

Burgess [10] proved that this example is sufficiently representative: in general, a relation of the class $\mathbf{\Sigma}^1_1$ such that there is no perfect set of pairwise inequivalent points—such relations are called *thin*—has at most $\aleph_1$ equivalence classes. The question whether an ***absolute*** list of classes is possible remains open so far.

The following seems to be desirable (and natural). Consider a transitive class $M \subseteq \mathrm{V}$ in the set-theoretical universe V which itself is a model of ZFC, for example, the class L of all constructible sets. We assume also that M correctly computes $\aleph_1$, that is, $\aleph_1 = \aleph_1^M$. (Otherwise, the statement of the question does not make sense.) Finally, let E be a thin $\mathbf{\Sigma}^1_1$-equivalence relation. Is it true that in this case any (non-empty) class of E has a representative in M? The positive answer was obtained by Hjorth [14] in two cases:

1) V is a generic extension of M;

2) a derivative of the hypothesis of the existence of a measurable cardinal.

However, the general case remains open.

Investigation of thin $\mathbf{\Sigma}^1_1$-relations is facilitated considerably if we introduce the additional requirement that all the equivalence classes are Borel sets of bounded rank, that is, they all belong to a single Borel class $\mathbf{\Sigma}^0_\alpha$ ($\alpha < \omega_1$); such relations are called *Luzin*, since N. N. Luzin initiated their study.[18] Although even here the statement that every Luzin $\mathbf{\Sigma}^1_1$-relation has only countably many classes is not true (the corresponding counterexamples were given by Sami in [22]), nevertheless, in this case the classes admit quite absolute numbering by countable ordinals. Moreover, under the assumption that $\mathcal{N} \cap \mathrm{L}[x]$ is countable for any $x \in \mathcal{N}$, even any Luzin $\mathbf{\Delta}^1_2$-relation has only countably many classes; see [26] and [3] (especially for the equivalences generated by partition into constituents).

[17] Formally, this means that $V_{a\restriction n} = \tau(U_{a\restriction 0}, U_{a\restriction 1}, \ldots, U_{a\restriction n})$ for any n, where τ is the winning strategy for α.

[18] See [4] and [26] on the history of the question.

§4. Classification of Borel equivalence relations

We continue the presentation of results on equivalence relations, which, in fact, admit much more thorough analysis than just the alternative: 'countably many classes or a perfect set of pairwise inequivalent points' given in §3. This analysis is connected with questions of comparison and classification of equivalence relations.

4.1. Smooth relations. This kind of equivalence relations has assumed great importance, in particular, in connection with some problems of the theory of Borel transformation groups and measure theory.

Definition 23. An equivalence relation E on the space $\mathcal{X}$ is called *smooth*[19] if there is a Borel function $f\colon \mathcal{X} \to \mathcal{N}$ such that $x \mathsf{E} y \longleftrightarrow f(x) = f(y)$.

In other words, smooth relations have Borel invariants. We can replace $\mathcal{N}$ by any other perfect Polish space $\mathcal{Y}$, since all such spaces are Borel isomorphic to $\mathcal{N}$.

The following criterion is often helpful: E is a smooth relation if and only if there is a family of Borel sets A_n such that $x \mathsf{E} y \longleftrightarrow \forall n\, (x \in A_n \longleftrightarrow y \in A_n)$ (*separating family*).

For example, some relations between matrices, arising as a result of reduction to a canonical form, belong to this type.

An example of a *non*-smooth relation is the Vitali relation E_{V} on the real line $\mathbb{R}$. To verify that E_{V} is not smooth, assume the contrary, and let the Borel function $f\colon \mathbb{R} \to \mathcal{N}$ demonstrate the smoothness of E_{V}. Then all the sections $P/x = \{y : P(x,y)\} = f^{-1}(x)$ of the plane Borel set $P = \{\langle x,y\rangle : f(y) = x\}$, that is, the Vitali equivalence classes, are countable. By the classical theorem of P. S. Novikov, in this case P admits a representation $P = \bigcup_n P_n$, where all P_n are *uniform* Borel sets. Each set $X_n = \{y : \exists\, x\, P_n(x,y)\}$ belongs to $\mathbf{\Sigma}^1_1$ (in fact, it is even a Borel set); therefore, it is Lebesgue measurable, and $\mathbb{R} = \bigcup_n X_n$, which implies that at least one of the X_n has non-zero measure. On the other hand, X_n has at most one common point with every E_{V}-equivalence class, which rapidly brings us to a contradiction, by means of Vitali's argument.

In a similar way, non-smoothness can be proved for the analogue of E_{V} in $\mathcal{N}$, the relation

$$\gamma \mathsf{E}_0 \delta \longleftrightarrow \exists\, m\, \forall\, k \geqslant m\; [\gamma(k) = \delta(k)] \tag{4}$$

on Cantor's discontinuum $\mathcal{D} = 2^\omega$, which is of principal importance in the classification of equivalence relations.

4.2. Glimm–Effros dichotomy. More precisely, we shall see that E_0 is the 'smallest' among non-smooth Borel relations: any of them contains E_0 in a certain sense.

Definition 24. Let E and E' be equivalence relations on $\mathcal{X}$ and $\mathcal{X}'$, respectively. We write $\mathsf{E} \leqslant \mathsf{E}'$ if there is a Borel function $f\colon \mathcal{X} \to \mathcal{X}'$ such that $x \mathsf{E} y \longleftrightarrow f(x)\, \mathsf{E}'\, f(y)$ for all $x, y \in \mathcal{X}$. (Such a function f, if it exists, is called a *reduction* of E to E'.)

We write $\mathsf{E} \sqsubseteq \mathsf{E}'$ if there is a one-to-one (but not necessarily 'onto') Borel function with the above property. (Such a function, if it exists, is called an *embedding* of E into E'.)

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[19]This is the term adopted in English-language literature. A verbatim translation of the Russian term would be *smoothed.*

It is clear that in this situation, if E' is smooth, then E is also smooth. Therefore, any relation E satisfying $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ is non-smooth. The following theorem shows that, moreover, the condition $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ characterizes the property of non-smoothness.

Theorem 25 (Harrington, Kechris, and Louveau [12]). [20] *Let E be a Borel equivalence relation on $\mathcal{N}$. Then one of the following two alternatives holds:*

(I) E *is a smooth relation;*

(II) $\mathsf{E}_0 \sqsubseteq \mathsf{E}$*, and even by means of a continuous f.*

Hence we can easily obtain Silver's theorem (Theorem 19) in the case of a *Borel*, and not just co-Suslin, relation E.[21] For it is easy to construct a perfect set of pairwise E_0-inequivalent points; therefore, in the case (II), E also has a perfect set of pairwise inequivalent points. On the other hand, the dichotomy of Theorem 19 for *smooth* Borel relations is readily verified.

We note that both (I) and (II) have several equivalent forms and derivatives related to Borel transformation groups and measure theory; see [12], and also [16].

As before, the theorem extends to all Polish spaces.

Proof. [22] Thus, let E be a Borel equivalence relation on $\mathcal{N}$. Then E belongs to $\Delta^1_1(p)$ for some $p \in \mathcal{N}$. As usual in such cases, we carry out the proof under the assumption that E is a Δ^1_1-relation; in the general case, p enters the argument uniformly, so that $\mathcal{T}$, say, changes to $\mathcal{T}(p)$, and so on.

The main idea of the proof is the use of the relationships between E and $\overline{\mathsf{E}}$, the closure of E in the topology $\mathcal{T}^2$ (which, we recall, is the product of two copies of $\mathcal{T}$). There are two possibilities:

Case 1: $\mathsf{E} = \overline{\mathsf{E}}$, that is, E is closed in $\mathcal{T}^2$.

Case 2: $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$.

We consider these possibilities separately. It will turn out that in the first case the relation E is smooth, and in the second, E_0 can be continuously embedded into E.

4.3. The case of closed relations. Thus, we assume that $\mathsf{E} = \overline{\mathsf{E}}$.

We define $[A]_{\mathsf{E}} = \{x : \exists y \in A\,(x\,\mathsf{E}\,y)\}$ for $A \subseteq \mathcal{N}$ (the E-*saturation* of A). A set A is called E-*invariant* if $A = [A]_{\mathsf{E}}$.

To estimate the descriptive complexity of $\overline{\mathsf{E}}$, we need the following lemma.

Lemma 26. *If Σ^1_1-sets A, B are such that $[A]_{\mathsf{E}} \cap [B]_{\mathsf{E}} = \varnothing$, then there is an E-invariant Δ^1_1-set C that separates $[A]_{\mathsf{E}}$ from $[B]_{\mathsf{E}}$.*

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[20] The dichotomy of Borel equivalence relations given by this theorem is known as the *Glimm–Effros classification*, in the name of the mathematicians who first obtained the result for relations of the class $\mathbb{F}_\sigma$. See the article [12] in connection with the history of the subject and various applications in algebra and probability theory.

[21] Theorem 19 itself can be stated as follows: if a $\mathbf{\Pi}^1_1$-relation E has uncountably many equivalence classes, then $\Delta(\mathcal{D}) \sqsubseteq \mathsf{E}$ by means of a *continuous* f, where $\Delta(\mathcal{D})$ is the equality relation on Cantor's discontinuum $\mathcal{D} = 2^\omega$.

[22] We follow the proof in [12]. Some recursive-theoretic facts have been successfully eliminated. The construction in [12] based on a Choquet game has been replaced by Polish networks.

Proof. Using the fact that $[A]_{\mathsf{E}} \in \Sigma^1_1$ whenever $A \in \Sigma^1_1$, and the separation theorem (Corollary 3), we construct an increasing sequence of sets $A = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ such that C_n is a Δ^1_1-set separating $[C_{n-1}]_{\mathsf{E}}$ from $[B]_{\mathsf{E}}$ for any n. The union $C = \bigcup_n C_n$ is E-invariant and separates $[A]_{\mathsf{E}}$ from $[B]_{\mathsf{E}}$.

It is not immediately obvious why C is Δ^1_1 (it is only clear that C is Borel); however, a more sophisticated analysis (see [12], Lemma 5.1) shows that the separation theorem has enough uniformity to allow the construction to be performed in a version that guarantees $C \in \Delta^1_1$.

Lemma 27. $\overline{\mathsf{E}}$ *is an equivalence relation of the class* Σ^1_1.

Proof. According to Lemma 26 and by the definition of $\mathcal{T}$ we have

$$x \,\overline{\mathsf{E}}\, y \longleftrightarrow \forall C \in \Delta^1_1 \; \big[C \text{ is } \mathsf{E}\text{-invariant} \to (x \in C \to y \in C)\big]. \tag{5}$$

Consequently, the (countable) family of all invariant Δ^1_1-sets C is a separating family. But the right-hand side of (5) can be brought to the Σ^1_1-form in the same manner as in the proof of Lemma 20, using Principle 4.

Thus, if $\mathsf{E} = \overline{\mathsf{E}}$, that is, E is closed in $\mathcal{T}^2$, then E is smooth.

4.4. The case of non-closed relations. We continue the proof of Theorem 25 and now consider the case $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, and show that this assumption implies $\mathsf{E}_0 \sqsubseteq \mathsf{E}$.

Since $\mathsf{E} \subseteq \overline{\mathsf{E}}$, each E-class $[x]_{\mathsf{E}} = \{y : x \,\mathsf{E}\, y\}$ is included in the $\overline{\mathsf{E}}$-class $[x]_{\overline{\mathsf{E}}} = \{y : x \,\overline{\mathsf{E}}\, y\}$, and by assumption there are $\overline{\mathsf{E}}$-classes that contain *more than one* E-class. Consider the union

$$H = \big\{x \in \mathcal{N} : [x]_{\mathsf{E}} \neq [x]_{\overline{\mathsf{E}}}\big\} = \big\{x : \exists y \; (x \,\overline{\mathsf{E}}\, y \,\&\, x \not\mathsf{E}\, y)\big\}$$

of all such classes; $H \in \Sigma^1_1$, since $\mathsf{E} \in \Delta^1_1$ and $\overline{\mathsf{E}} \in \Sigma^1_1$. The set H plays the key role here, like the other set H did in the proof of Silver's Theorem 19.

Lemma 28. *In the sense of* $\mathcal{T}^2$, E *is dense and of the first category in the* (*open in* $\overline{\mathsf{E}}$) *set* $H^2 \cap \overline{\mathsf{E}}$.

Proof. The density is obvious; let us check the category statement. Since E is a Borel set, assuming the contrary gives a pair of Σ^1_1-sets $A, B \subseteq H$ such that $(A \times B) \cap \overline{\mathsf{E}} \neq \varnothing$, and E is of the *second* category (that is, the complement of a set of the first category) on $(A \times B) \cap \overline{\mathsf{E}}$. Suppose that $A \subseteq [B]_{\overline{\mathsf{E}}}$ and $B \subseteq [A]_{\overline{\mathsf{E}}}$ (if not, we replace A by $A \cap [B]_{\overline{\mathsf{E}}}$, and similarly for B).

We claim that $A^2 \cap \overline{\mathsf{E}} \subseteq \mathsf{E}$; in other words, E and $\overline{\mathsf{E}}$ coincide on A. To prove this, consider the set $\overline{\mathsf{E}}^3 = \{\langle x, y, z\rangle : x \,\overline{\mathsf{E}}\, y \,\overline{\mathsf{E}}\, z\}$ with topology inherited from $\mathcal{T}_{2+1} = \mathcal{T}_2 \times \mathcal{T}$.

Fact 29. $\langle \overline{\mathsf{E}}^3; \mathcal{T}_{2+1}\rangle$ *has the Baire property.*

Proof. It suffices to indicate a Polish network for $\overline{\mathsf{E}}^3$ in the topology inherited from $\mathcal{T}_{2+1}$. To this end, we use the 'generic' Polish networks $\{\mathcal{X}_n : n \in \omega\}$ and $\{\mathcal{Z}_n : n \in \omega\}$ for the topologies $\mathcal{T}_2$ and $\mathcal{T}$ given by Lemma 9. The desired network $\{\mathcal{P}_n : n \in \omega\}$ for $\overline{\mathsf{E}}^3$ is determined as follows: $\mathcal{P}_n$ is the family of all *non-empty* sets of the form $P = (X \times Z) \cap \overline{\mathsf{E}}^3$, where $X \in \mathcal{X}_n$ and $Z \in \mathcal{Z}_n$. We verify the requirements of Definition 7.

Density. Suppose that the Σ^1_1-sets $X \subseteq \mathcal{N}^2$ and $Z \subseteq \mathcal{N}$ are such that the set $P = (X \times Z) \cap \overline{\mathsf{E}}^3$ is non-empty. We take an arbitrary $X' \in \mathcal{X}_n$ satisfying $X' \subseteq X \cap ([Z]_{\overline{\mathsf{E}}} \times [Z]_{\overline{\mathsf{E}}})$, and then an arbitrary $Z' \in \mathcal{Z}_n$ such that $Z' \subseteq X \cap [X']_{\overline{\mathsf{E}}}$; then $P' = (X' \times Z') \cap \overline{\mathsf{E}}^3$ belongs to $\mathcal{P}_n$ and $P' \subseteq P$.

Compactness. Let $P_m = (X_m \times Z_m) \cap \overline{\mathsf{E}}^3 \in \mathcal{P}_m$ for every m, and assume that all finite intersections are non-empty. There is a unique triple $\langle x, y, z\rangle \in \mathcal{N}^3$ such that $\langle x, y\rangle \in \mathcal{X}_n$ and $z \in \mathcal{Z}_n$ for all n. We show that $\langle x, y, z\rangle \in \overline{\mathsf{E}}^3$. On the contrary, let $x \not\!\overline{\mathsf{E}}\, z$. By Lemma 26, there is an E-invariant Δ^1_1-set C that contains x and does not contain z. Since the networks $\{\mathcal{X}_n\}$ and $\{\mathcal{Z}_n\}$ are 'generic', there are numbers m and n such that $Z_m \cap C = \varnothing$ and $X_n \subseteq C \times \mathcal{N}$. But in this case $P_n \cap P_m$ is empty, a contradiction.

We continue the proof of the inclusion $A^2 \cap \overline{\mathsf{E}} \subseteq \mathsf{E}$. Consider now the set

$$P = \{\langle x, y, z\rangle \in \overline{\mathsf{E}}^3 : x \in A \,\&\, y \in A \,\&\, z \in B\},$$

which is $\mathcal{T}_{2+1}$-open in $\overline{\mathsf{E}}^3$ and non-empty by assumption. In view of Lemma 15 the maps from $\overline{\mathsf{E}}^3$ to $\overline{\mathsf{E}}^2$ defined by $\langle x, y, z\rangle \mapsto \langle x, z\rangle$ and $\langle x, y, z\rangle \mapsto \langle y, z\rangle$ are open, and hence, by the choice of A and B, the sets $R = \{\langle x, y, z\rangle \in P : x \,\mathsf{E}\, z\}$ and $S = \{\langle x, y, z\rangle \in P : y \,\mathsf{E}\, z\}$ are of the second $\mathcal{T}_{2+1}$-category in P. If we now assume the contrary, that is, $A^2 \cap \overline{\mathsf{E}} \not\subseteq \mathsf{E}$, then the set $Q = \{\langle x, y, z\rangle \in P : x \not\!\mathsf{E}\, y\}$ will be non-empty. But Q is $\mathcal{T}_{2+1}$-open, since P is open, and $\mathsf{E} \in \Delta^1_1$. Therefore, Q has non-empty intersection with $R \cap S$ (we use the Baire property of the topology), and we arrive at a contradiction.

Thus, in fact, E coincides with $\overline{\mathsf{E}}$ on A. A simple argument shows that $[A]_{\mathsf{E}} = [A]_{\overline{\mathsf{E}}}$. (For otherwise the Σ^1_1-set $A' = [A]_{\overline{\mathsf{E}}} \setminus [A]_{\mathsf{E}}$ is non-empty, that is, $(A' \times A) \cap \overline{\mathsf{E}}$ is also non-empty. Then $(A' \times A) \cap \mathsf{E} \neq \varnothing$, since $\overline{\mathsf{E}}$ is the closure of E, a contradiction.) But this is impossible, since $A \subseteq H$.

4.5. Embedding of E_0 into E. We proceed with the proof of the theorem (the case $\mathsf{E} \subsetneq \overline{\mathsf{E}}$).

According to Lemma 28, there is a decreasing sequence of sets $W_n \subseteq H^2$, $\mathcal{T}^2$-open in $\mathcal{N}^2$, such that every $W_n \cap \overline{\mathsf{E}}$ is $\mathcal{T}^2$-dense in $H^2 \cap \overline{\mathsf{E}}$ and the intersection $\mathsf{E} \cap \bigcap_n W_n$ is empty. We can assume that the diagonal $\Delta(H) = \{\langle x, x\rangle : x \in H\}$ is disjoint from W_0 (otherwise we replace all the W_n by the differences $W_n \setminus \Delta$; the set Δ is closed, even in the Polish topology).

The embedding of E_0 into E that we present relies on the same idea as the classical construction in the article [12] by Harrington, Kechris, and Louveau. However, instead of Choquet games (which are the main tool in ensuring non-emptiness of the intersections along the branches of the splitting) we consider Polish networks, which somewhat simplifies the computation. We begin with some definitions.

Let $P \subseteq \mathcal{N}^2$; we put $\mathrm{pr}_1 P = \{x : \exists\, y\, P(x,y)\}$ and $\mathrm{pr}_2 P = \{y : \exists\, x\, P(x,y)\}$.
Let $X, Y \subseteq \mathcal{N}$ and $\mathsf{R} \subseteq \mathcal{N}^2$. We write $X \,\mathsf{R}\, Y$ if

$$\forall x \in X\, \exists y \in Y\ (x \,\mathsf{R}\, y) \quad \text{and} \quad \forall y \in Y\, \exists x \in X\ (x \,\mathsf{R}\, y).$$

By Lemma 9, there are Polish networks $\{\mathcal{X}_m : m \in \omega\}$ and $\{\mathcal{P}_m : m \in \omega\}$ for the topologies $\mathcal{T}$ on $\mathcal{N}$ and $\mathcal{T}_2$ on $\mathcal{N}^2$, respectively. We put

$$\mathcal{X}_n^* = \{X : X \text{ is a non-empty } \Sigma_1^1\text{-set and } \exists\, X' \in \mathcal{X}_n\ (X \subseteq X')\}$$

and similarly define $\mathcal{P}_n^*$ from $\mathcal{P}_n$.

We construct a family of Σ_1^1-sets X_u $(u \in 2^{<\omega})$ such that

(a) $X_u \in \mathcal{X}_{n-1}^*$ and $X_{u^\wedge i} \subseteq X_u \subseteq H$ for all n and all $u \in 2^n$ and $i = 0, 1$.

(We recall that 2^n denotes the collection of all sequences of 0's and 1's of length n, and $2^{<\omega} = \bigcup_{n\in\omega} 2^n$.) It follows that for any $a \in 2^\omega$ the intersection $\bigcap_{n\in\omega} X_{a\restriction n}$ contains a single point, which we denote by $\varphi(a)$ in the sequel, and the map φ is continuous in the sense of Polish topology.

To ensure that φ is one-to-one, and for some other purposes, we need to impose another requirement, on *pairs* $u, v \in 2^{<\omega}$.

(b) $X_u \times X_v \subseteq W_{n-1}$ for all n and each pair $u, v \in 2^n$ such that $u(n-1) \neq v(n-1)$ (that is, the last terms of u and v are distinct).

We note that $\varphi(a) \neq \varphi(b)$ for $a \neq b \in 2^\omega$, since W_0 is disjoint from Δ; therefore, φ is one-to-one.

We need additional connections between *some* pairs $\langle u, v\rangle$ to ensure the correct interaction of branches in $2^{<\omega}$ and eventually establish that $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ by means of φ.

Let $u, v \in 2^n$. We say that u, v is a *critical pair* if $u = 0^{k\wedge}0^\wedge r$ and $v = 0^{k\wedge}1^\wedge r$ for some $k < n$ (0^k is the sequence of k zeros) and some $r \in 2^{n-k-1}$ (it is possible that $k = n-1$; then $r = \Lambda$).

Thus, we shall construct Σ_1^1-sets R_{uv} for all critical pairs u, v so that the following requirements will be fulfilled:

(c) $\mathrm{pr}_1\, \mathsf{R}_{uv} = X_u$, $\mathrm{pr}_2\, \mathsf{R}_{uv} = X_v$, $\mathsf{R}_{u^\wedge i, v^\wedge i} \subseteq \mathsf{R}_{uv}$, for any critical pair $u, v \in 2^n$ and $i \in \{0, 1\}$.

(d) $\mathsf{R}_{uv} \in \mathcal{P}_{n-1}^*$ for any critical pair $u, v \in 2^n$.

(e) For any k, the set $\mathsf{R}_k = \mathsf{R}_{0^{k\wedge}0, 0^{k\wedge}1}$ satisfies $\mathsf{R}_k \subseteq \mathsf{E}$.

We note that $u^\wedge i, v^\wedge i$ is a critical pair whenever u, v is a critical pair. However, the pair $u^\wedge i, v^\wedge j$ cannot be critical if $i \neq j$ (except for the case $u = v = 0^k$ for some k).

Remark 30. Condition (c) implies $X_u \,\mathsf{R}_{uv}\, X_v$, hence, by (e), also $X_u \,\mathsf{E}\, X_v$, for all critical pairs u, v. Since any pair $u, v \in 2^n$ can be joined by a finite chain of critical pairs in 2^n, we conclude that $X_u \,\mathsf{E}\, X_v$, and therefore $X_u \,\overline{\mathsf{E}}\, X_v$ for *all* pairs $u, v \in 2^n$.

We now verify that conditions (c)–(e) guarantee that $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ by means of φ.

We prove that $a \,\mathsf{E}_0\, b$ implies $\varphi(a) \,\mathsf{E}\, \varphi(b)$. It suffices to consider the case when $a = 0^{k\wedge}0^\wedge c$ and $b = 0^{k\wedge}1^\wedge c$ for some $c \in \omega$ and $k \in 2^\omega$, since any pair $u, v \in 2^n$ can be joined in 2^n by a chain of critical pairs. The intersection $\bigcap_{n\in\omega} \mathsf{R}_{0^{k\wedge}0^\wedge c\restriction n, 0^{k\wedge}1^\wedge c\restriction n}$ is non-empty by (d), but since $\mathsf{R}_{uv} \subseteq X_u \times X_v$, this intersection is none other than the point $\langle \varphi(a), \varphi(b)\rangle$. Thus, $\langle \varphi(a), \varphi(b)\rangle \in \mathsf{R}_k$, which yields $\varphi(a) \,\mathsf{E}\, \varphi(b)$ in view of (e).

We now show that $a \not\mathsf{E}_0 b$ implies $\varphi(a) \not\mathsf{E}\, \varphi(b)$. Indeed, $a \not\mathsf{E}_0 b$ means that $a(n) \neq b(n)$ for infinitely many n; then, by virtue of (b), $\langle \varphi(a), \varphi(b) \rangle \in W_n$ for infinitely many n, that is, in fact, for all n, since the sets W_n decrease. We conclude that $\varphi(a) \not\mathsf{E}\, \varphi(b)$, since E has no common points with the intersection of the sets W_n.

Thus, to prove Theorem 25 it remains to construct sets X_u and R_{uv} satisfying conditions (a)–(e). Before we start the construction we prove a useful combinatorial lemma.

Lemma 31. *Let $n \in \omega$, and let $\{X_u : u \in 2^n\}$ be a family of non-empty Σ^1_1-sets. In addition, suppose that for every critical pair $u, v \in 2^n$ there is given a Σ^1_1-set $\mathsf{R}_{uv} \subseteq \mathcal{N}^2$ satisfying $X_u \, \mathsf{R}_{uv} \, X_v$.*

1. *If $u_0 \in 2^n$, and $X' \subseteq X_{u_0}$ is a non-empty Σ^1_1-set, then there are non-empty Σ^1_1-sets $Y_u \subseteq X_u$ ($u \in 2^n$) such that $Y_u \, \mathsf{R}_{uv} \, Y_v$ continues to hold for all critical pairs u, v and $Y_{u_0} = X'$.*
2. *If $u_0, v_0 \in 2^n$ is a critical pair, and the non-empty Σ^1_1-sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$ satisfy $X' \, \mathsf{R}_{u_0 v_0} \, X''$, then there are non-empty Σ^1_1-sets $Y_u \subseteq X_u$ ($u \in 2^n$) such that $Y_u \, \mathsf{R}_{uv} \, Y_v$ continues to hold for all critical pairs u, v and $Y_{u_0} = X'$, $Y_{v_0} = X''$.*

Proof. We observe that the assertion 1 follows from 2: take any v_0 such that one of the pairs $\langle u_0, v_0 \rangle$, $\langle v_0, u_0 \rangle$ is critical and put, respectively,

$$X'' = \{ y \in X_{v_0} : \exists\, x \in X' \ (x \, \mathsf{R}_{u_0 v_0} \, y) \} \text{ or } X'' = \{ y \in X_{v_0} : \exists\, x \in X' \ (y \, \mathsf{R}_{v_0 u_0} \, x) \}.$$

The assertion 2 is proved by induction on n.

If $n = 1$, then $u_0 = \langle 0 \rangle$ and $v_0 = \langle 1 \rangle$, and we put $Y_{u_0} = Y'$ and $Y_{v_0} = Y''$.

Induction step. We prove the lemma for $n+1$ assuming that it has already been proved for n, $n \geqslant 1$. We partition the set 2^{n+1} into two parts, $U_0 = \{ s^\wedge 0 : s \in 2^n \}$ and $U_1 = \{ s^\wedge 1 : s \in 2^n \}$, joined by the unique critical pair of sequences $\widehat{u} = 0^{n\wedge} 0$ and $\widehat{v} = 0^{n\wedge} 1$.

Suppose that $u_0 = \widehat{u}$ and $v_0 = \widehat{v}$. We apply the induction hypothesis (version 1) separately to the system $\{X_u : u \in U_0\}$ and the set $X' \subseteq X_{u_0}$, on the one hand, and the system $\{X_u : u \in U_1\}$ and the set $X'' \subseteq X_{v_0}$, on the other. Combining the results, we obtain a system of Σ^1_1-sets $Y_u \subseteq X_u$ ($u \in 2^{n+1}$) such that $Y_u \, \mathsf{R}_{uv} \, Y_v$ for all critical pairs $\langle u, v \rangle$, with the possible exception of the pair $u = u_0 = \widehat{u}$, $v = v_0 = \widehat{v}$, and in addition $Y_{u_0} = X'$ and $Y_{v_0} = X''$. Finally, we note that $Y_{\widehat{u}} \, \mathsf{R}_{\widehat{u}\widehat{v}} \, Y_{\widehat{v}}$ by the choice of X' and Y'.

Consider the second case: u_0 and v_0 belong to the same part, say, U_0. Applying the induction hypothesis (version 2) to the system $\{X_u : u \in U_0\}$ and the sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$ we obtain a system of non-empty Σ^1_1-sets $Y_u \subseteq X_u$ ($u \in U_0$); in particular, the Σ^1_1-set $Y_{\widehat{u}} \subseteq X_{\widehat{u}}$. Now we put $Y_{\widehat{v}} = \{ y \in X_{\widehat{v}} : \exists\, x \in Y_{\widehat{u}} \ (x \, \mathsf{R}_{\widehat{u}\widehat{v}} \, y) \}$, so that $Y_{\widehat{u}} \, \mathsf{R}_{\widehat{u}\widehat{v}} \, Y_{\widehat{v}}$, and apply the induction hypothesis (version 1) to the family $\{X_v : v \in U_1\}$ and the set $Y_{\widehat{v}} \subseteq X_{\widehat{v}}$.

4.6. Construction of the splitting system. Beginning our construction of the sets X_u and R_{uv}, we put $X_\Lambda = H$.

We assume that a system of sets X_s ($s \in 2^n$) and relations R_{st} for critical pairs $s, t \in 2^k$, $k \leqslant n$, has already been defined, and continue the construction to the level $n+1$.

First, we define $A_{s^\wedge i} = X_s$ for all $s \in 2^n$ and $i \in \{0,1\}$, and also $\mathsf{Q}_{uv} = \mathsf{R}_{st}$ for every critical pair $u = s^\wedge i$, $v = t^\wedge i$ in 2^{n+1} except for the pair $\widehat{u} = 0^{n\wedge}0$, $\widehat{v} = 0^{n\wedge}1$; for the latter pair (observe that $A_{\widehat{u}} = A_{\widehat{v}} = X_{0^n}$), we define $\mathsf{Q}_{\widehat{u}\widehat{v}} = \overline{\mathsf{E}}$, so that $A_u \, \mathsf{Q}_{uv} \, A_v$ holds for all critical pairs $u, v \in 2^{n+1}$.

The sets A_u and Q_{uv} will be reduced (in several steps) to satisfy conditions (a)–(e).

After 2^{n+1} steps of using Lemma 31 (version 1) we have a system of non-empty Σ^1_1-sets $B_u \subseteq A_u$, $B_u \in \mathfrak{X}^*_n$ ($u \in 2^{n+1}$), such that $B_u \, \mathsf{Q}_{uv} \, B_v$ is fulfilled for every critical pair u, v in 2^{n+1}. This guarantees (a).

To guarantee (b) we consider an arbitrary pair $u_0 = s_0^\wedge 0$, $v_0 = t_0^\wedge 1$, where $s_0, t_0 \in 2^n$. According to Remark 30 and the $\mathfrak{T}^2$-density of the $\mathfrak{T}^2$-open set W_n in $H^2 \cap \overline{\mathsf{E}}$, there are non-empty Σ^1_1-sets $B' \subseteq B_{u_0}$ and $B'' \subseteq B_{v_0}$ such that $B' \times B'' \subseteq W_n$ and $P = (B' \times B'') \cap \overline{\mathsf{E}}$ is non-empty. We can assume that $\mathrm{pr}_1 P = B'$ and $\mathrm{pr}_2 P = B''$ (if this is not so, we put $B' = \mathrm{pr}_1 P$ and $B'' = \mathrm{pr}_2 P$). Under this assumption, $B' \, \overline{\mathsf{E}} \, B''$. Now we apply Lemma 31 (version 1) separately to the families $\{B_{s^\wedge 0} : s \in 2^n\}$ and $\{B_{t^\wedge 1} : t \in 2^n\}$ (compare with the proof of Lemma 31) and the sets $B' \subseteq B_{s_0^\wedge 0}$, $B'' \subseteq B_{t_0^\wedge 1}$, respectively. Combining the results, we obtain a system of non-empty Σ^1_1-sets $B'_u \subseteq B_u$ ($u \in 2^{n+1}$) such that $B'_{u_0} = B'$, $B'_{v_0} = B''$, that is, $B'_{u_0} \times B'_{v_0} \subseteq W_n$, and $B'_u \, \mathsf{Q}_{uv} \, B'_v$ continues to hold for all critical pairs u, v in 2^{n+1} except, possibly, the pair $\widehat{u} = 0^{n\wedge}0$, $\widehat{v} = 0^{n\wedge}1$, the only one that joins the two regions. We remark that for this special pair we have $B'_{\widehat{u}} \, \overline{\mathsf{E}} \, B'_{u_0}$ and $B'_{\widehat{v}} \, \overline{\mathsf{E}} \, B'_{v_0}$ (Remark 30 works in each of the two regions), so that $B'_{\widehat{u}} \, \overline{\mathsf{E}} \, B'_{\widehat{v}}$, since $B' \, \overline{\mathsf{E}} \, B''$. Finally, at this moment, $\mathsf{Q}_{\widehat{u}\widehat{v}}$ is $\overline{\mathsf{E}}$ by definition, whence $B'_{\widehat{u}} \, \mathsf{Q}_{\widehat{u}\widehat{v}} \, B'_{\widehat{v}}$.

After 2^{n+1} steps (the number of pairs u_0, v_0 which need to be considered) we obtain a system of non-empty Σ^1_1-sets $C_u \subseteq B_u$ ($u \in 2^{n+1}$) such that $C_u \times C_v \subseteq W_n$ whenever $u(n) \neq v(n)$, and $C_u \, \mathsf{Q}_{uv} \, C_v$ for all critical pairs $u, v \in 2^{n+1}$. Thus, everything is all right with the condition (b).

Now we make sure that (e) holds for the pair $\widehat{u} = 0^{n\wedge}0$, $\widehat{v} = 0^{n\wedge}1$. At the given moment, $\mathsf{Q}_{\widehat{u}\widehat{v}} = \overline{\mathsf{E}}$. Using the $\mathfrak{T}^2$-density of E in $\overline{\mathsf{E}}$ and the relation $C_{\widehat{u}} \, \overline{\mathsf{E}} \, C_{\widehat{v}}$, we find that the set $\mathsf{Q} = (C_{\widehat{u}} \times C_{\widehat{v}}) \cap \mathsf{E}$ is non-empty. Consider the Σ^1_1-sets $C' = \mathrm{pr}_1 \mathsf{Q}$ ($\subseteq C_{\widehat{u}}$) and $C'' = \mathrm{pr}_2 \mathsf{Q}$ ($\subseteq C_{\widehat{v}}$); it is clear that $C' \, \mathsf{Q} \, C''$, whence $C' \, \mathsf{Q}_{\widehat{u}\widehat{v}} \, C''$. Lemma 31 (version 2) gives a system of non-empty Σ^1_1-sets $D_u \subseteq C_u$ ($u \in 2^{n+1}$) such that $D_u \, \mathsf{Q}_{uv} \, D_v$ continues to hold for all critical pairs u, v in 2^{n+1} and $D_{\widehat{u}} = C'$, $D_{\widehat{v}} = C''$. We redefine $\mathsf{Q}_{\widehat{u}\widehat{v}}$ by $\mathsf{Q}_{\widehat{u}\widehat{v}} = \mathsf{Q}$; the relation $D_{\widehat{u}} \, \mathsf{Q}_{\widehat{u}\widehat{v}} \, D_{\widehat{v}}$ is preserved.

Finally, we ensure (c) and (d). Consider an arbitrary critical pair $u_0 = s_0^\wedge 0$, $v_0 = t_0^\wedge 1$ in 2^{n+1}. The set $\mathsf{Q}' = \mathsf{Q}_{u_0 v_0} \cap (D_{u_0} \times D_{v_0})$ is a non-empty (since $D_{u_0} \, \mathsf{Q}_{u_0 v_0} \, D_{v_0}$) Σ^1_1-subset of $\mathsf{Q}_{u_0 v_0}$. Take any non-empty Σ^1_1-set $\mathsf{Q} \subseteq \mathsf{Q}'$ that belongs to $\mathfrak{P}^*_n$. Let $D' = \mathrm{pr}_1 \mathsf{Q}$ and $D'' = \mathrm{pr}_2 \mathsf{Q}$ (then $D' \, \mathsf{Q} \, D''$, since $D' \, \mathsf{Q}_{u_0 v_0} \, D''$). We apply Lemma 31 (version 2) to the system of sets D_u ($u \in 2^{n+1}$) and the sets D' and D''. After that, we introduce the 'new' $\mathsf{Q}_{u_0 v_0}$ via $\mathsf{Q}_{u_0 v_0} = \mathsf{Q}$.

We do this successively for all critical pairs; the resulting sets—we denote them by X_u ($u \in 2^{n+1}$)—are the desired sets. The relations R_{uv} ($u, v \in 2^{n+1}$) are obtained by restricting Q_{uv} to $X_u \times X_v$.

This completes the construction and the proof of Theorem 25.

4.7. Some other results. (This brief account is not meant to be complete, but rather to demonstrate the abundance of ideas in this branch of descriptive theory.)

By Theorem 19, the equality $\Delta(\mathcal{D})$ on the Cantor discontinuum $\mathcal{D} = 2^\omega$ is $\sqsubseteq$-smallest among Borel relations with uncountably many equivalence classes. Leaving aside the smooth relations, we see that E_0 is $\sqsubseteq$-smallest among non-smooth Borel relations. One of the main problems in this field is to understand what is going on above E_0.

Hyperfinite and hypersmooth relations.

Dougherty, Jackson, and Kechris [11] considered an important special case of *hyperfinite* relations, that is, relations of the form $\mathsf{E} = \bigcup_n \mathsf{E}_n$, where $\mathsf{E}_n \subseteq \mathsf{E}_{n+1}$ and each E_n is a *finite* relation (which means that every equivalence class of E_n is a finite set). The Borel hyperfinite relations are exactly those that are induced by the action of a Borel automorphism. The following is proved in [11] for such relations. First, every non-smooth relation E of this form is similar to E_0 in the sense that $\mathsf{E} \sqsubseteq \mathsf{E}_0$ and $\mathsf{E}_0 \sqsubseteq \mathsf{E}$. (The latter follows from Theorem 25.) Second, a finer classification of Borel non-smooth hyperfinite relations is obtained within Borel isomorphism: a countable set of distinguishable isomorphism types is indicated.

The second important class of Borel relations is formed by *hypersmooth* relations, that is, relations of the form $\mathsf{E} = \bigcup_n \mathsf{E}_n$, where $\mathsf{E}_n \subseteq \mathsf{E}_{n+1}$ and all the E_n are smooth. Kechris and Louveau [17] found yet another dichotomy theorem: for any hypersmooth Borel relation E, either $\mathsf{E} \leqslant \mathsf{E}_0$ or $\mathsf{E}_1 \leqslant \mathsf{E}$, where E_1 is the relation on the space $\mathcal{D}^\omega$ of all infinite sequences of points of $\mathcal{D} = 2^\omega$ defined by

$$\langle x_n \rangle_{n\in\omega} \, \mathsf{E}_1 \langle y_n \rangle_{n\in\omega} \longleftrightarrow \exists\, n \, \forall\, m \geqslant n \ (x_m = y_m),$$

and the order $\leqslant$ (which is weaker than $\sqsubseteq$) is introduced in Definition 24. A result that has been gained from this shows that for the order $\leqslant$, Theorems 19 and 25, in principle, exhaust all the assertions of this kind. We call a Borel relation R *critical* if it is $\leqslant$-comparable with any other Borel equivalence relation E. It turns out that the equality relations $\Delta(\mathcal{D})$, $\Delta(\omega)$ and $\Delta(\{1, 2, \dots, n\})$, $n \in \omega$, are the only (aside from $\mathsf{R} \approx \mathsf{R}'$ when $\mathsf{R} \leqslant \mathsf{R}'$ and $\mathsf{R}' \leqslant \mathsf{R}$) critical Borel relations. It follows that the only instances of dicohotomy among Borel relations are the pairs

(1) $\langle \Delta(\mathcal{D}), \mathsf{E}_0 \rangle$—Theorem 25;
(2) $\langle \Delta(\omega), \Delta(\mathcal{D}) \rangle$—Silver's theorem;
(3) $\langle \Delta(n), \Delta(n+1) \rangle$ for all n—this is trivial.

Relations of the class $\mathbf{\Sigma}^1_1$.

Like the studies in connection with Silver's theorem (mentioned at the end of §3), the problem of generalizing Theorem 25 to equivalence relations of the class $\mathbf{\Sigma}^1_1$ is of great interest.

A direct statement of Theorem 25 for $\mathbf{\Sigma}^1_1$-relations fails. For let E be the relation introduced at the end of §3; in particular, E is thin, that is, it does not admit a perfect set of pairwise inequivalent points, but has uncountably many equivalence classes. Then $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ is impossible, since there is a perfect set of pairwise non-E_0-equivalent points; E is not smooth either, because a smooth thin relation may have at most countably many classes.

Quite recently, Hjorth and Kechris [15] showed that the essence of the problem is that the definition of smoothness is not adequate in the case when Σ^1_1-relations are under consideration.

We say that a relation E is *Ulm-smooth*[23] if there is a $\mathbf{\Delta}_1$-function[24] $f\colon \mathcal{N} \to 2^{<\omega_1}$ such that $x \mathrel{\mathsf{E}} y \longleftrightarrow f(x) = f(y)$ for all $x, y \in \mathcal{N}$. (We remark that $2^{<\omega_1}$ is the set of all countable (of any length $< \omega_1$) binary sequences.) It is easy to see that each relation that is smooth in the sense considered above is also Ulm-smooth.

In [15] it is shown that if all the classes of a Σ^1_1-relation E are *Borel* sets, then either $\mathsf{E}_0 \sqsubseteq \mathsf{E}$, or else E is Ulm-smooth.

So far we have not succeeded in completely avoiding the troublesome requirement for the classes to be Borel. Hjorth and Kechris [15] proved the alternative: either $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ or else the Ulm-smoothness, for an arbitrary Σ^1_1-relation E, under an assumption connected with the existence of a measurable cardinal. There is no reason to believe that a hypothesis so strong is really needed here, if only because no counterexample is known so far.[25]

§5. Decomposition of plane Borel sets

Our last application of the topology generated by effectively Suslin sets is connected with an interesting and important topic: plane Borel sets with special sections.

5.1. Decomposition theorem. Consider a set $P \subseteq \mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$; it is proper to call subsets of the 'Baire plane' $\mathcal{N}^2$ ***plane sets***. Every point $x \in \mathcal{N}$ determines a *section*

$$P/x = \{y : \langle x, y\rangle \in P\}$$

of the set P.[26] Suppose that all the sections P/x are sets of the class $\mathbf{\Sigma}^0_\lambda$, where $2 \leqslant \lambda < \omega_1$ is a fixed ordinal. Then, by the axiom of choice, we have $P = \bigcup_n P_n$, where each P_n has the property that all the sections P_n/x $(x \in \mathcal{N})$ belong to the class $\mathbf{\Pi}^0_{<\lambda} = \bigcup_{1 \leqslant \xi < \lambda} \mathbf{\Pi}^0_\xi$.

Suppose now that P is a *Borel* set; is it possible to arrange that the sets P_n are also Borel? The classical descriptive theory has given a positive answer to similar questions in some cases. For example, if each section P/x of a Borel set P is at most countable, then $P = \bigcup P_n$, where each P_n is a uniform Borel set. If each section P/x is σ-compact, then all sections P_n/x can be required to be compact and all sets P_n to be Borel. (Theorems of P. S. Novikov; see [4].)

[23] Hjorth and Kechris refer to Ulm's theorem on classification of countably Abelian p-groups, which can be interpreted in the present context.

[24] From the point of view of complexity, the class $\mathbf{\Delta}_1$ is approximately equivalent to the projective $\mathbf{\Delta}^1_2$, but is defined in the framework of set-theoretic, and not descriptive, hierarchy.

[25] The author has recently established that the alternative is true in all generic extensions of the constructible universe.

[26] We use this notation P/x to avoid confusion with indices.

Theorem 32 (Louveau [18]). *Suppose that* $2 \leqslant \lambda < \omega_1$ *and* $P \subseteq \mathcal{N} \times \mathcal{N}$ *is a Borel set such that* $P/x \in \mathbf{\Sigma}^0_\lambda$ *for any* $x \in \mathcal{N}$. *There is a sequence of Borel sets* P_n *such that* $P = \bigcup_n P_n$ *and, for any* n *and* x, P_n/x *is a set in* $\mathbf{\Pi}^0_{<\lambda}$.

Proof. Before turning to technicalities we outline the idea of the proof. Construction of a Borel set $X \subseteq \mathcal{N}$ can be formalized as a countable sequence of the operations of countable union and complementation, starting with Baire intervals; we call this a *Borel construction.* Each Borel construction has a certain *length,* the ordinal $\lambda < \omega_1$, that shows the number of elementary steps of the construction, and thereby gives an upper bound of the Borel class of the resulting set X.

However, it is possible (and we immediately come to this in our analysis of Theorem 32) that, in fact, the class of Borel set X is much lower than the one suggested by the length of a given construction of X. Thus, we may have a construction Π of length λ for X and *be aware* that there is actually a construction Π' of a known length $\xi < \lambda$.

It turns out, and this is the key point in the proof of the theorem, that in this case there is an effective (in a certain sense) procedure for finding the 'short' construction Π' from Π. For example, under the hypothesis of the theorem, we shall be able to effectively assign to each x a Borel construction of the section P/x in the form of a $\mathbf{\Sigma}^0_\lambda$-set, which automatically provides a representation of P/x as a countable union of $\mathbf{\Pi}^0_{<\lambda}$-sets; by 'integrating' the latter sets with respect to x we obtain the theorem.

Now we proceed with the details. We begin with a number of important definitions.

5.2. Coding of Borel sets. An enumeration $\{\mathcal{N}(n) : n \in \omega\}$ of all Baire intervals of $\mathcal{N}$ is assumed to be fixed; for example, $\mathcal{N}(n) = B_n[\mathcal{N}]$ in the sense of §1.3.

Let $x \in \mathcal{N}$ and $n \in \omega$. We define $(x)_n \in \mathcal{N}$ by $(x)_n(k) = x(\ulcorner n, k\urcorner + 1)$ for all k, where, we recall, $\ulcorner n, k\urcorner = 2^n(2k+1) - 1$. Thus, to each collection of a number $m \in \omega$ and a set of points $x_n \in \mathcal{N}$ $(n \in \omega)$ there corresponds, on a one-to-one way, a point $x \in \mathcal{N}$ such that $x(0) = m$ and $(x)_n = x_n$ for all n.

Finally, for $x \in \mathcal{N}$ we define $x^- \in \mathcal{N}$ by the condition $x^-(k) = x(k+1)$ for all k. The map $x \mapsto \langle x(0), x^-\rangle$ is a bijection of $\mathcal{N}$ onto $\omega \times \mathcal{N}$.

Now we introduce the collection of *Borel codes* $\mathrm{BC} \subseteq \mathcal{N}$, and for each $k \in \mathrm{BC}$, the (Borel) set $\mathbb{B}(c) \subseteq \mathcal{N}$. Namely, BC is the smallest subset of $\mathcal{N}$ such that

(1) BC contains all $c \in \mathcal{N}$ satisfying $c(0) = 0$.—*For every such* c *we put* $\mathbb{B}(c) = \bigcup_{c^-(c)=1} \mathcal{N}(c)$.

(2) If $c(0) = 1$ and $c^- \in \mathrm{BC}$, then $c \in \mathrm{BC}$.—*In this case, if* $\mathbb{B}(c^-)$ *has already been defined, then we put* $\mathbb{B}(c) = \mathcal{N} \setminus \mathbb{B}(c^-)$.

(3) If $c(0) = 2$ and $(c)_n \in \mathrm{BC}$ for all n, then $c \in \mathrm{BC}$.—*Here, if all the sets* $\mathbb{B}((c)_n)$ *have already been defined, then we put* $\mathbb{B}(c) = \bigcup_n \mathbb{B}((c)_n)$.

It is easy to see that $\{\mathbb{B}(c) : c \in \mathrm{BC}\}$ is the family of all Borel subsets of $\mathcal{N}$. By induction on λ, $1 \leqslant \lambda < \omega_1$, we define the sets of codes $\boldsymbol{\sigma}_\lambda \subseteq \mathrm{BC}$ and $\boldsymbol{\pi}_\lambda \subseteq \mathrm{BC}$, that generate specific Borel classes:

1. $\boldsymbol{\sigma}_1 = \{c \in \mathcal{N} : c(0) = 0\}$;
2. for any λ, $\boldsymbol{\pi}_\lambda = \{c : c(0) = 1 \,\&\, c^- \in \boldsymbol{\sigma}_\lambda\} \cup \boldsymbol{\pi}_{<\lambda} \cup \boldsymbol{\sigma}_{<\lambda}$;
3. for $\lambda > 1$, $\boldsymbol{\sigma}_\lambda = \{c : c(0) = 2 \,\&\, \forall n\ (c)_n \in \boldsymbol{\pi}_{<\lambda}\} \cup \boldsymbol{\pi}_{<\lambda} \cup \boldsymbol{\sigma}_{<\lambda}$.

(Here and in the sequel, $\boldsymbol{\pi}_{<\lambda} = \bigcup_{1 \leqslant \xi < \lambda} \boldsymbol{\pi}_\xi$ and so on, for similar notations.) Thus, $\{\mathbb{B}(c) : c \in \boldsymbol{\sigma}_\lambda\}$ is the collection of all $\mathbf{\Sigma}^0_\lambda$-sets $X \subseteq \mathcal{N}$, and similarly for the pair $\boldsymbol{\pi}_\lambda$ and $\mathbf{\Pi}^0_\lambda$.

Principle 33 ([24]; see also [1], Chapter 20, or [2]). *The sets* BC, $\{\langle c, x\rangle : c \in \mathrm{BC} \,\&\, x \in \mathbb{B}(c)\}$ *and* $\{\langle c, x\rangle : c \in \mathrm{BC} \,\&\, x \notin \mathbb{B}(c)\}$ *belong to the class* Π^1_1.

Definition 34 [effective Borel codes and sets]. Let $p \in \mathcal{N}$. We put $\sigma_\lambda(p) = \boldsymbol{\sigma}_\lambda \cap \Delta^1_1(p)$, $\pi_\lambda(p) = \boldsymbol{\pi}_\lambda \cap \Delta^1_1(p)$ and

$$\Sigma^0_\lambda(p) = \{\mathbb{B}(c) : c \in \sigma_\lambda(p)\}, \qquad \Pi^0_\lambda(p) = \{\mathbb{B}(c) : c \in \pi_\lambda(p)\}.$$

As usual, if p is absent, $\sigma_\lambda = \boldsymbol{\sigma}_\lambda \cap \Delta^1_1$, $\pi_\lambda = \boldsymbol{\pi}_\lambda \cap \Delta^1_1$ and $\Sigma^0_\lambda = \{\mathbb{B}(c) : c \in \sigma_\lambda\}$, $\Pi^0_\lambda = \{\mathbb{B}(c) : c \in \pi_\lambda\}$.

Thus, we have defined the *effective subclasses* Γ and $\Gamma(p)$[27] in each of the classes $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\lambda$ and $\mathbf{\Pi}^0_\lambda$. The sets of the class, say, Σ^0_λ are precisely those $\mathbf{\Sigma}^0_\lambda$-sets that admit an effective (that is, belonging to Δ^1_1) construction from Baire intervals.

5.3. Effective version of the Louveau theorem. According to Principle 33, every, say, $\Sigma^0_\lambda(p)$-set belongs to both $\mathbf{\Sigma}^0_\lambda$ and $\Delta^1_1(p)$. It is very important that under some non-restrictive conditions the converse inclusion holds as well.

Theorem 35 (Louveau [18]). *Suppose that* $p \in \mathcal{N}$ *and* $\lambda < \omega_1$ *are such that all the sets* $\boldsymbol{\sigma}_\xi$, $\boldsymbol{\pi}_\xi$, $\boldsymbol{\sigma}_{<\xi}$, $\boldsymbol{\pi}_{<\xi}$, *where* $\xi \leqslant \lambda$, *belong to* $\Delta^1_1(p)$. *Then every set* $X \subseteq \mathcal{N}$ *of* $\mathbf{\Sigma}^0_\lambda \cap \Delta^1_1(p)$ *belongs to* $\Sigma^0_\lambda(p)$.

This proves our informal assertion made before: an effective set of the class $\mathbf{\Sigma}^0_\lambda$ *effectively* belongs to this class.

We remark that, in fact, in our presentation this theorem is somewhat weaker than in Louveau [18]. First, Louveau proves a *separation theorem*, which could be stated here as follows: if, under the hypotheses of Theorem 35, disjoint $\Sigma^1_1(p)$-sets X and Y are such that the former is $\mathbf{\Pi}^0_\lambda$-separable from the latter, then there is a separating set of the class $\Pi^0_\lambda(p)$. Second, the relationship between p and λ in [18] is like this: λ is a recursive ordinal with respect to p (then it can be proved that all the sets mentioned in Theorem 35 belong, in fact, to $\Delta^1_1(p)$).

Derivation of Theorem 32 from Theorem 35. We fix a $p_0 \in \mathcal{N}$ such that a given set P and all the sets $\boldsymbol{\sigma}_\xi$, $\boldsymbol{\pi}_\xi$, $\boldsymbol{\sigma}_{<\xi}$, $\boldsymbol{\pi}_{<\xi}$, where $1 \leqslant \xi \leqslant \lambda$, belong to $\Delta^1_1(p_0)$. For $x \in \mathcal{N}$, let $p_x \in \mathcal{N}$ be defined by the equalities $p_x(2k) = p_0(k)$, $p_x(2k+1) = x(k)$ for all k. Then each section P/x belongs to $\mathbf{\Sigma}^0_\lambda$, and hence to $\Sigma^0_\lambda(p_x)$, by Theorem 35. This means that there is a code $c \in \sigma_\lambda(p_x) = \boldsymbol{\sigma}_\lambda \cap \Delta^1_1(p_x)$ such that $P/x = \mathbb{B}(c)$. We conclude that the set

$$U = \{\langle x, k\rangle : k \in \sigma_\lambda(p_x) \,\&\, \mathbb{B}(c) = P/x\}$$

satisfies $\operatorname{dom} U = \mathcal{N}$. However, $U \in \Pi^1_1(p_0)$ (Principle 33 and Corollary 6). Therefore, by the uniformization theorem (Theorem 1), there is a $\Pi^1_1(p_0)$-function F such that $U(x, F(x))$ for all $x \in \mathcal{N}$.

[27] In the special case of $\lambda = 1$ there is a little problem: the classes $\Sigma^0_1(p)$ and $\Pi^0_1(p)$ have already been defined in §1, and it can be shown that our definition here introduces classes that are much larger than those in §1. To overcome this difficulty, we simply forget in this section the definition of the classes $\Gamma^0_1(p)$ in §1.

We note that, in fact, F belongs even to $\Delta^1_1(p_0)$, since

$$F(x) = c \longleftrightarrow \forall\, c' \in \Delta^1_1(p_x)\ (c \neq c' \to F(x) \neq c),$$

and we can use Corollary 5. In particular, F is a Borel function and $F(x) \in \boldsymbol{\sigma}_\lambda$, $\mathbb{B}(F(x)) = P/x$ for any x.

By the definition of $\boldsymbol{\sigma}_\lambda$, in this case a countable sequence of Borel functions F_n ($n \in \omega$) can be given in such a way that $F_n(x) \in \boldsymbol{\pi}_{<\lambda}$ and $P/x = \mathbb{B}(F(x)) = \bigcup_{n\in\omega} \mathbb{B}(F_n(x))$ for all x. Now it remains to put $P_n = \{\langle x, y\rangle : y \in \mathbb{B}(F_n(x))\}$.

5.4. Proof of the effective theorem. First of all, let us agree to carry out the proof for the case where there is no p, that is, for the classes Δ^1_1 and Σ^0_λ; see the remark at the end of §2.1.

We use induction on λ. Even in the case $\lambda = 1$, some ingenuity is needed. Let $X \subseteq \mathcal{N}$ be open and of the class Δ^1_1; we prove that $X \in \Sigma^0_1$.[28] It suffices to find a Δ^1_1-set $N \subseteq \omega$ such that $X = \bigcup_{n\in N} \mathcal{N}(n)$; to obtain from this N a code $c \in \sigma_1$ with $X = \mathbb{B}(c)$ presents no difficulty.

Let $K = \{n \in \omega : \mathcal{N}(n) \subseteq X\}$; then $X = \bigcup_{n\in K} \mathcal{N}(n)$, but immediately we have only $K \in \Pi^1_1$. To replace K by a Δ^1_1-set with the same property we use an argument that is later referred to as the $(\Pi \to \Delta)$-*trick*. Uniformizing the Π^1_1-set

$$P = \{\langle x, n\rangle : x \in X \,\&\, x \subseteq \mathcal{N}(n) \subseteq X\}$$

by means of Corollary 2, we have a Δ^1_1-selector Q that assigns to every $x \in X$ the number $n = Q(x) \in \omega$ such that $x \in \mathcal{N}(n) \subseteq X$. Then the set $L = \{n : \exists\, x\ (n = Q(x))\}$ satisfies $X = \bigcup_{n\in L} \mathcal{N}(n)$. On the other hand, $L \subseteq K$ and L is in Σ^1_1. By the separation theorem (Corollary 3) there is a Δ^1_1-set N, $L \subseteq N \subseteq K$. This N is the desired set.

Induction step. Suppose that $2 \leqslant \lambda < \omega_1$ and Theorem 35 has already been proved below λ, that is, $\boldsymbol{\Sigma}^0_\xi \cap \Delta^1_1 \subseteq \Sigma^0_\xi$ for $\xi < \lambda$. The idea of the proof for λ itself uses the same method as for $\lambda = 1$. The role of Baire intervals is played by the sets in $\Pi^0_{<\lambda}$. The following lemma demonstrates that there are certain premises for this approach.

Lemma 36. *Suppose that $\xi \leqslant \lambda$, and $A \subseteq \sigma_\xi$ is a Δ^1_1-set, and if $\xi = \lambda$, we have $A \subseteq \sigma_{<\lambda} \cup \pi_{<\lambda}$. Then $X = \bigcup_{a\in A} \mathbb{B}(a)$ belongs to Σ^0_ξ.*

Proof. First, we see that $X \in \Delta^1_1$, since

$$[\text{for } \Pi^1_1] \quad x \in X \longleftrightarrow \exists\, a \in \Delta^1_1\ [a \in A \,\&\, x \in \mathbb{B}(a)]$$

and

$$[\text{for } \Sigma^1_1] \quad x \in X \longleftrightarrow \exists\, a\ [a \in A \,\&\, \neg(x \notin \mathbb{B}(a))]$$

(Corollary 5 and Principle 33 have been used). The induction hypothesis of the theorem closes the case $\xi < \lambda$. It remains to consider the case $\xi = \lambda$, where the code for X must be given by direct construction.

[28]See footnote 27. Now the Σ^0_1-sets are open sets with Δ^1_1-code.

It is necessary to find a Δ^1_1-enumeration of the elements of A. To this end, we note that Principle 4 (part 2), in the special case when $p = \omega \times \{0\}$ (identically zero), gives a set $K \subseteq \omega$ and a point $\mathbf{d}_n \in \mathcal{N}$ for all $n \in K$ such that $\Delta^1_1 \cap \mathcal{N} = \{\mathbf{d}_n : n \in K\}$, and the sets

$$K, \qquad \{\langle n, \mathbf{d}_n\rangle : n \in K\}, \qquad \{\langle n, x\rangle : n \in K \,\&\, x \neq \mathbf{d}_n\}$$

belong to Π^1_1. Now we apply the $(\Pi \to \Delta)$-trick.

The set $P = \{\langle a, n\rangle : a \in A \,\&\, n \in K \,\&\, \mathbf{d}_n = a\}$ also belongs to Π^1_1, and $\operatorname{dom} P = A$ is a Δ^1_1-set. By Corollary 2 there is a uniformizing Δ^1_1-set $Q \subseteq P$. Then $L = \{n : \exists\, a\, Q(a, n)\}$ is a Σ^1_1-subset of K, that is, by virtue of separability, there is a Δ^1_1-set N, $L \subseteq N \subseteq K$.

There are two principal points. First, $A = \{\mathbf{d}_n : n \in N\}$. Second, $F = \{\langle n, \mathbf{d}_n\rangle : n \in N\}$ belongs to Δ^1_1, by what was said above. The latter shows that the code $a \in \mathrm{BC}$ defined by the conditions $a(0) = 1$, $(a^-)_n = \mathbf{d}_n$ for $n \in N$, and $(a^-)_n = e$ (where e is a fixed code of the empty set) for $n \notin N$, belongs to Δ^1_1, and hence to σ_λ. Because of the former circumstance, $\mathbb{B}(a) = X$.

It seems appropriate to give here the proof of another lemma, also based on the $(\Pi \to \Delta)$-trick. The meaning of this lemma is that in certain cases, intersections over Π^1_1-sets of indices can be approximated by intersections over Δ^1_1-sets.

Lemma 37. *Suppose that $A \subseteq \mathrm{BC} \cap \Delta^1_1$ is a Π^1_1-set, and $X = \bigcap_{a \in A} \mathbb{B}(a)$ does not intersect a Σ^1_1-set Z. Then there is a Δ^1_1-set $A' \subseteq A$ such that $X' = \bigcap_{a \in A'} \mathbb{B}(a)$ does not intersect Z either.*

Proof. The set $P = \{\langle x, c\rangle : c \in A \,\&\, x \notin \mathbb{B}(c)\}$ belongs to Π^1_1 by Principle 33. Consequently, by the uniformization theorem (Principle 1), there is a Π^1_1-set $Q \subseteq P$ that uniformizes P. The set $A'' = \{c : \exists\, x \in Z\, Q(x, c)\}$ is in Σ^1_1; in fact,

$$c \in A'' \longleftrightarrow \exists\, x \in Z\, \forall\, c' \in \Delta^1_1 \; [c' \neq c \to \neg Q(x, c')]$$

and it remains to use Corollary 5. The separability theorem (Principle 3) provides a Δ^1_1-set A' such that $A'' \subseteq A' \subseteq A$. Then $X' = \bigcap_{c \in A'} \mathbb{B}(c)$ has no common points with Z. (If $x \in Z$, then there is a c such that $Q(x, c)$; hence $c \in A''$, and so $c \in A'$. But according to the choice of Q and the definition of P, we have $x \notin \mathbb{B}(c)$, that is, $x \notin X'$.)

Continuing the proof of Theorem 35, we introduce Σ^*_ξ, for $\xi \leqslant \lambda$, as the family of all sets of the form $Y = \bigcup_{a \in A} \mathbb{B}(a)$, where $A \subseteq \sigma_\xi$ is a Π^1_1-set. Similarly, Π^*_ξ denotes the family of all sets $\bigcap_{a \in A} \mathbb{B}(a)$, where $A \subseteq \pi_\xi$ belongs again to the class Π^1_1.

Fact 38. *$\Sigma^*_\xi \subseteq \Pi^1_1$ and $\Pi^*_\xi \subseteq \Sigma^1_1$.*

Proof. See the beginning of the proof of Lemma 36.

After these preparations, we prove a lemma that justifies the induction step of Theorem 35. In a certain sense, the induction step of the theorem is simply reduced to that of the lemma.

Lemma 39. *If $2 \leqslant \rho \leqslant \lambda$, then every $\mathbf{\Pi}^0_\rho$-set is $\mathcal{T}$-almost equal*[29] *to a (countable) intersection of sets of the class $\Sigma^*_{<\rho} = \bigcup_{1 \leqslant \xi < \rho} \Sigma^*_\xi$.*

Proof. We use induction on ρ. Thus, let $2 \leqslant \rho \leqslant \lambda$.

Consider a $\mathbf{\Pi}^0_\rho$-set $X \subseteq \mathcal{N}$. Then X is a countable intersection of sets in $\mathbf{\Sigma}^0_{<\rho}$, that is, we can assume that X itself belongs to $\mathbf{\Sigma}^0_\xi$ for some $\xi < \rho$. By the induction hypothesis, X is $\mathcal{T}$-almost equal to a countable union of sets in $\Pi^*_{<\xi}$. Consequently, we can assume that X itself is a countable union of sets in $\Pi^*_{<\xi}$ (or of Baire intervals—for $\xi = 1$).

We denote by X^* the intersection of all Σ^*_ξ-sets that contain X. Since Σ^*_ξ is a countable class, to finish the proof of the lemma it suffices to verify that the difference $X^* \setminus X$ is of the $\mathcal{T}$-first category. In fact, we prove that $X^* \setminus X$ is $\mathcal{T}$-nowhere dense.

Thus, for an arbitrary non-empty Σ^1_1-set Z we find a non-empty Σ^1_1-set $Z' \subseteq Z$ that has no common points with $X^* \setminus X$.

Case 1. $Z \cap X$ is non-empty. Then, by the assumption on X, there is a $\Pi^*_{<\xi}$-set (or a Baire interval—for $\xi = 1$) $Y \subseteq X$ whose intersection with Z is also non-empty. However, Y is in Σ^1_1 (Fact 38). It remains to take $Z' = Z \cap Y$.

Case 2. $Z \cap X = \varnothing$. We show that in this case $Z \cap X^* = \varnothing$, that is, we can simply take $Z' = Z$. First, consider the subcase $\xi > 1$. In view of Lemmas 37 and 36 (in the dual form), each $\Pi^*_{<\xi}$-set $Y \subseteq X$ is separable from Z by a set of the class $\Pi^0_{<\xi}$. Thus, $X \subseteq X'$, where X' is the union of all $\Pi^0_{<\xi}$-sets that do not intersect Z.

We show that $X' \in \Sigma^*_\xi$; then $X^* \subseteq X'$, and, further, $X^* \cap Z = \varnothing$. It suffices to verify that the set $C = \{c \in \pi_{<\xi} : \mathbb{B}(c) \cap Z = \varnothing\}$ belongs to Π^1_1. We have

$$c \in C \longleftrightarrow c \in \Delta^1_1 \,\&\, c \in \pi_{<\xi} \,\&\, \forall z\ \big(z \notin Z \vee z \notin \mathbb{B}(c)\big).$$

The relation $c \in \Delta^1_1$ is expressed by the Π^1_1-formula $\exists\, a \in \Delta^1_1\ (c = a)$ (we refer to Corollary 5). The second conjunction term is Π^1_1 by the hypothesis of the theorem. Finally, the last term is of the same class by the choice of Z and Principle 33.

Now consider the subcase $\xi = 1$. Here in the role of $\Pi^*_{<1}$- and $\Pi^0_{<1}$-sets we have Baire intervals. Thus, let X' be the union of all Baire intervals that do not intersect Z. We need to verify that $X' \in \Sigma^*_1$. We recall that $\mathcal{N}(n)$ denotes the nth Baire interval in the sense of some effective enumeration of these intervals. We have $X' = \bigcup_{n \in N} \mathcal{N}(n)$, where $N = \{n : \mathcal{N}(n) \cap Z = \varnothing\}$ is a Π^1_1-set.

We define $c_n \in \mathcal{N}$ by the conditions $c_n(0) = 0$, $c_n^-(n) = 1$ and $c_n^-(k) = 0$ for $k \neq n$, so that $\mathbb{B}(c_n) = \mathcal{N}(n)$. Then $C = \{c_n : n \in N\}$ belongs to Π^1_1 together with N. On the other hand, $X' = \bigcup_{c \in C} \mathbb{B}(c)$, that is, $X' \in \Sigma^*_1$.

We return to the proof of Theorem 35 (the induction step). Consider an arbitrary set $X \subseteq \mathcal{N}$ in $\mathbf{\Pi}^0_\lambda \cap \Delta^1_1$ and prove that X belongs to Π^0_λ. By the lemma, X is $\mathcal{T}$-almost equal to a countable intersection of sets in $\Sigma^*_{<\lambda}$. Let U be one of these

[29] We say that two sets X and Y are τ-almost equal, where τ is a topology, if their symmetric difference $X \triangle Y$ is a first category set in the sense of τ.

$\Sigma^*_{<\lambda}$-sets, that is, the difference $D = X \setminus U$ is of the $\mathcal{T}$-first category. But $U \in \Pi^1_1$ (Fact 38), whence $D \in \Sigma^1_1$. We recall that $\mathcal{T}$ is a Baire topology by Lemma 13, which means that an open set of the first category must be empty. Thus, in fact, $X \subseteq U$. Using Lemmas 36 and 37 we see that there is a $\Sigma^0_{<\lambda}$-set V such that $X \subseteq V \subseteq U$.

The result: the intersection X^* of all $\Sigma^0_{<\lambda}$-sets $V \supseteq X$ is $\mathcal{T}$-almost equal to X. We have $X^* = \bigcap_{a \in A} \mathbb{B}(a)$, where $A = \{a \in \sigma_{<\lambda} : X \subseteq \mathbb{B}(a)\}$. It is easy to verify that $A \in \Pi^1_1$, that is, $X^* \in \Pi^*_{<\lambda}$ and $X^* \in \Sigma^1_1$ (Fact 38). Lemma 13 gives $X = X^*$. Applying Lemma 37 to X^* and the complement of X, and then Lemma 36, we obtain $X \in \Pi^0_\lambda$, which ends the proof.

5.5. Concluding remark. Theorem 35 expresses undoubtedly a more fundamental fact than does Theorem 32, though the latter appears more classical. Here is yet another application.

Corollary 40. *If $1 \leqslant \lambda < \omega_1$, then the set $\sigma'_\lambda = \{c \in \mathrm{BC} : \mathbb{B}(c) \in \mathbf{\Sigma}^0_\lambda\}$ of the Borel codes of the $\mathbf{\Sigma}^0_\lambda$-sets belongs to $\mathbf{\Pi}^1_1$.*

Proof. We choose a p_0 and define p_x in the same way as in the argument immediately after the statement of Theorem 35 (but without the requirement concerning P). For any $c \in \mathrm{BC}$ the set $\mathbb{B}(c)$ belongs to $\Delta^1_1(c)$, by Principle 33. Consequently, if $\mathbb{B}(c) \in \mathbf{\Sigma}^0_\lambda$, then, by Theorem 35, $\mathbb{B}(c) = \mathbb{B}(c')$ for some $c' \in \sigma_\lambda(p_c) = \boldsymbol{\sigma}_\lambda \cap \Delta^1_1(p_c)$. Therefore,

$$c \in \boldsymbol{\sigma}'_\lambda \longleftrightarrow \exists c' \in \Delta^1_1(p_c)\ \big(c' \in \boldsymbol{\sigma}_\lambda \,\&\, \mathbb{B}(c) = \mathbb{B}(c')\big).$$

The equality $\mathbb{B}(c) = \mathbb{B}(c')$ can be expressed by a Π^1_1-relation in view of Principle 33; it remains to use Corollary 5.

A direct verification yields only $\boldsymbol{\sigma}'_\lambda \in \mathbf{\Sigma}^1_2$, which is, of course, much worse. Even the use of a forceful tool such as Martin's theorem on Borel determinacy leads to $\boldsymbol{\sigma}'_\lambda \in \mathbf{\Delta}^1_2$ only, and this demonstrates the power of Theorem 35. Among applications of results of the type of Corollary 40 (we speak about the effective version of it in the same sense as Theorem 35 can be called the effective version of Theorem 32), we mention one of the results of [3]: if among the constituents of a $\mathbf{\Sigma}^1_1$-set given by means of a Borel sieve there are uncountably many non-empty ones (for $\mathbf{\Sigma}^1_1$-sets this does not imply non-Borelness!), then these constituents do not form a family of bounded Borel rank. ·

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