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Dedicated to the centenary of P. S. Novikov's birth

On some classical problems of descriptive set theory

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Abstract. The centenary of P. S. Novikov's birth provides an inspiring motivation to present, with full proofs and from a modern standpoint, the presumably definitive solutions of some classical problems in descriptive set theory which were formulated by Luzin [Lusin] and, to some extent, even earlier by Hadamard, Borel, and Lebesgue and relate to regularity properties of point sets. The solutions of these problems began in the pioneering works of Aleksandrov [Alexandroff], Suslin [Souslin], and Luzin (1916–17) and evolved in the fundamental studies of Gödel, Novikov, Cohen, and their successors. Main features of this branch of mathematics are that, on the one hand, it is an ordinary mathematical theory studying natural properties of point sets and functions and rather distant from general set theory or intrinsic problems of mathematical logic like consistency or Gödel's theorems, and on the other hand, it has become a subject of applications of the most subtle tools of modern mathematical logic.

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Introduction

An international mathematical conference dedicated to the 100th anniversary of the birth of P. S. Novikov was held in Moscow in August 2001. The authors of this paper were among the participants and speakers at the conference.

The beginning of the 21st century also marks the centennial of the descriptive theory of sets and functions,[1] an area in which many fundamental achievements are due to P. S. Novikov.

The *regularity properties*, that is, the perfect kernel property, Lebesgue measurability, and the Baire property, were among Novikov's favourite topics in descriptive set theory. In what follows, let $\mathbf{K}$ denote any class of point sets. (For the moment, we mean by a *point set* a subset of the real line $\mathbb{R}$ or of a space of the form $\mathbb{R}^n$, as was customary in the early 'Luzin' era of development of descriptive set theory. A somewhat wider modern understanding of the notion of point set is given below in 1A.) The expression $\mathbf{K}$-*set* means a set belonging to the class $\mathbf{K}$. The principal question here is formulated as follows:

For a given class $\mathbf{K}$ *of point sets and for each of the three regularity properties mentioned above, is it true that every* $\mathbf{K}$-*set has this property*?

To facilitate the discussion to follow, let us explicitly formulate three possible answers, or three hypotheses concerning the principal regularity properties:[2]

- PK($\mathbf{K}$): every $\mathbf{K}$-set X has *the perfect kernel property*, that is, X is either finite or countable or contains a perfect subset (a non-empty closed set having no isolated points);
- LM($\mathbf{K}$): every $\mathbf{K}$-set X is *Lebesgue measurable*;
- BP($\mathbf{K}$): every $\mathbf{K}$-set X has *the Baire property*, that is, X coincides with a Borel set modulo a meagre set.[3]

More general formulations applicable to sets in any uncountable Polish spaces[4] and not just sets in $\mathbb{R}$ will be given in 1B below.

The perfect kernel property arose as a possible approach to the problem of cardinality of point sets; indeed, any perfect subset of the real line is in fact of cardinality the continuum $\mathfrak{c} = 2^{\aleph_0}$. Measurability and the Baire property are basic mathematical notions. Examples of non-regular sets, for instance, non-measurable point sets, have been known since the beginning of the 20th century (Vitali, Bernstein, and others), but all of them are obtained by using the axiom of choice, and hence they are not individual unambiguously defined sets. The desire to obtain more explicit, 'effective' counterexamples led to heightened interest in *explicitly definable* point sets studied in descriptive set theory. In the mid-1920s the family of such sets included the *Borel* sets (they form the smallest σ-algebra containing all open sets in the given space), the *A sets* (the closure of the Borel σ-algebra under the operation A), and the *projective* sets (the closure of the Borel σ-algebra under

[1] The origins of descriptive theory can be traced back either to the introduction of Borel sets and Baire functions in the works of Borel [13] (1898) and Baire [6] (1899) or to Lebesgue's memoir [47] (1905), which systematized the very first results on such sets and functions.

[2] The abbreviations mean: perfect kernel, Lebesgue measurability, Baire property. Those three properties are known as the *principal* regularity properties. Other properties of this kind are also considered; some of them are discussed in 9C.

[3] The *meagre* sets, or *first-category* sets, are the countable unions of nowhere dense sets. The sets complementary to meagre sets are said to be co-meagre.

[4] Separable complete metric spaces are called Polish spaces.

the operations of taking continuous images and complements). The sets of the last kind are arranged in a hierarchy of classes of sets Σ^1_n, Π^1_n, Δ^1_n (or A_n, CA_n, B_n in the classical system of notation) which increase as the index n increases; the class Π^1_n consists of the complements of the Σ^1_n sets, and the class Δ^1_n coincides with the intersection of Σ^1_n and Π^1_n. (For the exact definitions, see 1A below.)

Since measure and category are σ-additive notions, all Borel subsets of the real line are Lebesgue measurable and have the Baire property, of course. The perfect kernel property of Borel sets is not so simple, because one cannot use direct induction when constructing sets by using Borel operations. The solution was found by P. S. Aleksandrov [3] and F. Hausdorff [23] independently and by different methods. The proof given by Aleksandrov led Suslin [91] to the discovery of the *A operation* and *A sets* (now known as Σ^1_1 sets). Luzin [51] and Suslin established the three regularity properties for all *A*-sets, that is, in our notation, one has $\mathrm{PK}(\Sigma^1_1)$, $\mathrm{LM}(\Sigma^1_1)$, and $\mathrm{BP}(\Sigma^1_1)$. Measurability and the Baire property carry over to the class Π^1_1 of complementary sets (classically known as *CA-sets*), of course.

All attempts made in classical descriptive set theory to extend these results to higher projective classes, in other words, to prove or disprove at least the assertions $\mathrm{PK}(\Pi^1_1)$, $\mathrm{LM}(\Sigma^1_2)$, $\mathrm{BP}(\Sigma^1_2)$, were fruitless. The problems were very quickly recognized to be difficult and probably central problems of descriptive set theory. Moreover, Luzin expressed in [53] (1925) the opinion that those problems were undecidable in general, that is, admitted no definite answer, and he discussed in [52] the undecidability of several more general problems related to the perfect kernel property, measurability, and the Baire property for all projective sets.[5] The perfect kernel problem for Π^1_1 sets and the measurability problem for Σ^1_2 sets were later characterized by Novikov in [78] (1951) as two of the three main problems of the descriptive theory of functions.[6]

Thus, the development of classical descriptive set theory led to problems concerning the regularity properties of point sets, both in their 'minimal' form (from the point of view of the projective class involved)[7] and in the most direct form, that is, for the special classes,

$$\mathrm{PK}(\Pi^1_1), \quad \mathrm{LM}(\Delta^1_2), \quad \mathrm{BP}(\Delta^1_2), \quad \mathrm{LM}(\Sigma^1_2), \quad \mathrm{BP}(\Sigma^1_2), \qquad (*)$$

[5] "*It is not known and will never be known whether or not every set in this family* [that is, the family of projective sets] *is of cardinality the continuum, is a set of third category* [that is, a set that does not have the Baire property], *is measurable*." This was written in [52], dozens of years before the discovery of methods which enabled one to actually establish the undecidability of the problems. Moreover, at that time, the prevailing opinion was that every mathematical problem is soluble.

[6] The third problem in Novikov's list is the separation property in higher projective classes, which became especially interesting after his own paper [76], where it was shown that the laws of separation at the second projective level are opposite to the laws of separation at the first projective level. (See Luzin's comments in [58], § 23.) The uniformization problem, which was first considered in the context of descriptive set theory in [56], and several problems on uncountable sequences of Borel sets (see, for example, [58], § 23, or [59]) can be added to the list. For these problems, which we do not discuss here, see the surveys and papers [36], [38], [40], [93], [97].

[7] Except, of course, for the provable claims for the class Σ^1_1.

and for the class of all projective sets as a whole.[8] The absence of any reasonable approach to those problems, in any direction, rapidly led researchers to the idea of *undecidability* of the problems.

Let us say a few words on the very notion of undecidability.

Undecidability problems, that is, problems on the impossibility of constructing a certain mathematical object (including the construction of a proof or a refutation of some hypothesis) by using certain tools, had been known in mathematics long before the appearance of descriptive set theory; we can point to Abel's theorem on the impossibility of solving higher-degree algebraic equations in radicals or the three famous age-old geometric problems on trisecting an angle, squaring the circle, and doubling the cube. However, a rigorous statement of the undecidability problem for set-theoretic problems arose after the creation of the Zermelo–Fraenkel axiomatic set theory **ZFC**. This axiomatic system was developed (1908–1925), in particular, in connection with the aim of making mathematical proofs more precise and codifying the axioms involved in the proofs. It is of interest that there were in general no axioms subject to controversy as to whether or not they should be included in this axiomatic set theory.[9] After long study of the foundations of all branches of mathematics, it is regarded as an established fact that any mathematical argument can be converted into a proof based on the **ZFC** axioms, or, briefly, a *derivation in* **ZFC**, and, in this sense, the fact that a statement **P** cannot be proved in **ZFC** means that **P** cannot be proved in mathematics. Thus, the *undecidability* of a mathematical problem, that is, the impossibility of giving a positive or negative answer to the corresponding question, is equated to its undecidability in **ZFC**. The latter means that neither the formula **P** expressing the problem nor its negation $\neg$**P** have a proof in **ZFC**.

By the Gödel completeness theorem, the deducibility of some formula **P** in **ZFC** is equivalent to the condition that **P** holds in every model of **ZFC** theory. Therefore, a typical proof of the undecidability of some problem **P** is the construction of a model M of (all) axioms of **ZFC** in which **P** is true (that is, **P** has an affirmative solution in M) and of another model N of the **ZFC** axioms in which **P** is false (that is, **P** is solved in N in the negative). The first part of such an argument shows the consistency of adjoining the formula **P** to the axioms of **ZFC** and the second

[8]Strictly speaking, the Σ^1_2 case of the problems of measurability and the Baire property is not minimal, because the class Δ^1_2 is a proper subclass of Σ^1_2, and the problems for Δ^1_2 are also undecidable. However, this class deserves to be included in the list for at least two reasons. First, the class $\Sigma^1_2 = \mathrm{A}_2$ generally attracted more attention in classical descriptive set theory than the class $\Delta^1_2 = \mathrm{B}_2$. Second, the methods of solving the problems for Σ^1_2 and Δ^1_2 are the same in essence.

[9]If one does not consider the discussion relating to the axiom of choice, which was rather in the framework of a controversy about the admissibility of certain mathematical tools like the law of the excluded middle, non-effective constructions, 'really' infinite sets, and so on, where the negation of the axiom of choice meant rather the negation of any axiomatics at all. The famous 'Cinq lettres' [8], the letters among Hadamard, Baire, Borel, and Lebesgue, were largely devoted to this topic. Luzin dwells on it in [58], Part III, [55], pp. 31, 60, 64, and [57], Chapter 1 and the Conclusion.

the consistency of adjoining the formula $\neg\mathbf{P}$ to the axioms of **ZFC**.[10] Of course, there are proofs in which one or both parts (relating to $\mathbf{P}$ or to $\neg\mathbf{P}$) are obtained by reduction to known consistency results.

The first results concerning proofs of consistency and undecidability for set-theoretic statements were obtained by Gödel [20] (1938–1940). He defined the class $\mathbf{L}$ of all *constructible sets* and proved that $\mathbf{L}$, when regarded as a set-theoretic structure with the ordinary membership relation $\in$ for two sets, is the smallest model of the axioms of **ZFC** that contains all ordinals (ordinal numbers). Gödel also proved that the generalized continuum hypothesis **GCH** holds in $\mathbf{L}$, and thus established the consistency of **GCH**. The study of the class $\mathbf{L}$ was continued in Novikov's paper [78] (1951), where he solved the problems $\mathrm{PK}(\Pi^1_1)$ and $\mathrm{LM}(\Delta^1_2)$ (and hence the problem $\mathrm{LM}(\Sigma^1_2)$ as well, because Δ^1_2 is a subclass of Σ^1_2, and the problems $\mathrm{BP}(\Delta^1_2)$ and $\mathrm{BP}(\Sigma^1_2)$ for similar reasons) in the class $\mathbf{L}$ in the negative, that is, appropriate counterexamples can be explicitly defined in $\mathbf{L}$. Thus, the consistency of *negative* solutions of the regularity problems was established for those projective classes.

Another method, opposite in a sense to the method of constructing models of **ZFC**, is the method of *forcing* discovered by P. Cohen (1963). This is a general tool for extending any given model M of **ZFC** theory (for instance, the class $\mathbf{L}$) by adjoining some set $a \notin M$ (and of course all 'derived' sets, that is, sets whose existence in the model $M[a]$ follows from the existence in it of a) in such a way that the extended structure $M[a]$ also satisfies all the axioms of **ZFC**. Cohen himself used this method to define a model in which **GCH** (and even **CH**) fails (1962–1963; see the book [15]). Somewhat later, Solovay [89] (1970) presented a model of **ZFC** in which every projective set satisfies the three regularity properties. Together with the above results of Novikov, this result proved that the regularity problems of point sets are undecidable, both in their 'minimal' forms $\mathrm{PK}(\Pi^1_1)$, $\mathrm{LM}(\Delta^1_2)$, $\mathrm{BP}(\Delta^1_2)$, $\mathrm{LM}(\Sigma^1_2)$, $\mathrm{BP}(\Sigma^1_2)$, and for the class of all projective sets as a whole, that is, Luzin's undecidability conjecture was confirmed for the five problems. We note that, in contrast to Gödel's incompleteness theorems or, say, to the continuum hypothesis, we speak here of the undecidability of specific mathematically meaningful properties of quite individually defined and rather simple point sets. For example, there is an explicitly defined Π^1_1 set $X \subseteq \mathbb{R}$ such that the assertion $\mathrm{PK}(X)$ that X has the perfect kernel property is undecidable.

Along with undecidability theorems, the investigations of many authors in the late 1960s and early 1970s established striking connections between the problems under consideration. It turned out that $\mathrm{PK}(\Pi^1_1)$ implies both $\mathrm{LM}(\Sigma^1_2)$ and $\mathrm{BP}(\Sigma^1_2)$, and somewhat later it became clear that $\mathrm{LM}(\Sigma^1_2)$ implies $\mathrm{BP}(\Sigma^1_2)$. These results are shown in the diagram. At the end of the 1980s, the forcing method was used to

[10] All arguments on undecidability in **ZFC** and consistency with respect to **ZFC** certainly assume that **ZFC** theory itself is consistent, that is, one cannot derive in this system both some formula Φ and its negation $\neg\Phi$ (or, equivalently, some formula cannot be derived in **ZFC**.) It is impossible to *prove* the consistency of **ZFC** by the tools of contemporary mathematics, because such a proof would mean a proof of consistency of **ZFC** by using its own axioms, which is excluded by Gödel's second incompleteness theorem. Nevertheless, the long development of a mathematics free of contradictions, explicitly or implicitly on the basis of precisely the axioms of **ZFC**, has made the consistency hypothesis for **ZFC** generally accepted.

construct models which established the *completeness* of the diagram, in the sense that no other implication between its elements is derivable in **ZFC**. These results all together completed the cycle of the most principal studies of the regularity problems for the projective classes Π^1_1 (the perfect kernel property), Σ^1_2, and Δ^1_2 (measurability and the Baire property), and for the class of projective sets as a whole.[11]

$$\begin{tikzcd}
 & \boxed{\mathrm{LM}(\Sigma^1_2)} \arrow[r, dashed] & \boxed{\mathrm{LM}(\Delta^1_2)} \\
\boxed{\mathrm{PK}(\Pi^1_1)} \arrow[ur] \arrow[dr] & & \\
 & \boxed{\mathrm{BP}(\Sigma^1_2)} \arrow[r, dashed] & \boxed{\mathrm{BP}(\Delta^1_2)}
\end{tikzcd}$$

Diagram. All five hypotheses of the above list $(*)$ are undecidable in **ZFC**;
the arrows show provable implications between the hypotheses;
the dashed arrows show trivial implications (because $\Delta^1_2 \subseteq \Sigma^1_2$);
the 'derived' implication $\mathrm{PK}(\Pi^1_1) \Longrightarrow \mathrm{BP}(\Sigma^1_2)$ is distinguished because it was obtained together with $\mathrm{PK}(\Pi^1_1) \Longrightarrow \mathrm{LM}(\Sigma^1_2)$ and much earlier than the implication $\mathrm{LM}(\Sigma^1_2) \Longrightarrow \mathrm{BP}(\Sigma^1_2)$.

The aim of this paper is to present all the principal results obtained in these studies with sufficiently complete proofs (and at the same time substantially simplified and modernized as compared with those in the original works) and also to systematically present the corresponding methods. The paper is intended for the reader who is interested in set-theoretic problems, has some experience in this area and in the area of mathematical logic, and, at least for some parts of the paper, is acquainted (at least minimally) with the method of forcing. As far as the technique (the proofs) is concerned, we tried to make the exposition self-contained, although the paper certainly cannot be regarded as a textbook on Gödel constructibility and forcing theory.

The structure of the paper is as follows. Section 1 is a general introduction. Section 2 is an introduction to the theory of Gödel constructible sets (in the framework needed for this paper). In particular, we consider the definability of the structure of constructible sets within the continuum.

The non-trivial implications in the diagram are derived in Sections 3–5. To this end, corresponding to each of the five propositions in the diagram, that is, to $\mathrm{PK}(\Pi^1_1)$, $\mathrm{LM}(\Sigma^1_2)$, $\mathrm{BP}(\Sigma^1_2)$, $\mathrm{LM}(\Delta^1_2)$, and $\mathrm{BP}(\Delta^1_2)$, is a *resolvent* (Luzin's term), which for a given proposition means an (equivalent) reformulation of significantly simpler nature. Corollary 3.4 presents all five reformulations, that is, resolvents, and each is in a natural way an equivalence. More 'effective' versions of these equivalences are given in Theorem 3.3. The proofs of the equivalences in one direction are given in § 3 by using constructibility theory, and in the other direction in § 4 by

[11] Here we do not touch upon studies in the framework of the theory of set operations founded by Kolmogorov and Hausdorff in which large classes of point sets were discovered within the class Δ^1_2 that still preserve measurability and the Baire property. For this topic, see [39].

using the method of forcing. Finally, in § 5, the problems of measurability and the Baire property are related to the properties of eventual domination (for a natural partial order relation on the set $\mathbb{N}^\omega$ of all self-maps of the natural numbers), which enables us to derive the implication $\mathrm{LM}(\Sigma^1_2) \Longrightarrow \mathrm{BP}(\Sigma^1_2)$. This completes the proof of all the implications of the diagram.

It is a special feature of the standard proofs of the non-trivial implications in the diagram that, to derive statements with sufficiently elementary formulations on the nature of point sets in low projective classes, one uses methods (like constructibility and forcing) which explicitly involve the class of all sets. In Section 6 we consider a problem which is of importance in our opinion, namely, whether or not one can prove the same results without using arbitrary (Cantorian, so to speak) sets while remaining within the framework of notions used in the setting of problems in descriptive set theory, that is, using only objects of the type of point sets. By the example of the implication $\mathrm{PK}(\Pi^1_1) \Longrightarrow \mathrm{LM}(\Sigma^1_2)$, we answer this question in the positive and call these proofs 'elementary', in contrast to proofs involving arbitrary 'Cantorian' sets.

In the next section, § 7, we present a remarkable result: even if non-regular projective sets exist, for instance, if there is an uncountable projective set without perfect subsets (or a non-measurable set, or a set for which the Baire property fails), this does not imply the existence of an **explicitly definable** non-regular projective set, that is, the existence of a specific, unambiguously defined counterexample! The difference between definable and 'arbitrary' sets (for example, those given by the axiom of choice) was given great significance at earlier stages of the development of descriptive set theory (see the footnote 9). In § 7 we also prove Solovay's theorem claiming that positive solutions of the three regularity problems for all projective sets (in fact, for a somewhat wider class of point sets) are consistent with the axioms of **ZFC**.

The *completeness* of the above diagram is established in § 8. In particular, we prove by constructing appropriate models that none of the implications in the diagram can be reversed.

The final section, § 9, contains a short survey of results on regularity properties that could not be presented with their proofs in a single paper for lack of space. We consider there the role of inaccessible cardinals in some consistency proofs, regularity properties for the third projective level, and some other related issues.

§ 1. Borel sets and projective sets

This is an introductory section. We give definitions of the projective hierarchy of point sets in Baire spaces and of a finer effective hierarchy using the definability of point sets by analytic formulae, consider the sieve operation, and prove that problems on the regularity properties are independent of the choice of base space in the category of perfect Polish spaces, and that the measurability problem is independent of the choice of a measure in the category of Borel measures vanishing at points. The section ends with an exposition of encoding for Borel sets and countable ordinals which plays an important role in the proofs and in the understanding of the meaning of theorems.

1A. Real line, Polish spaces, and the Baire space. Descriptive set theory, which originated from the theory of functions at the beginning of the 20th century, developed for a long time mainly as a theory of point sets on the real line $\mathbb{R}$ and in the spaces $\mathbb{R}^n$. However, it had already become clear by the mid-1920s that the principal results in this area could be generalized to any uncountable Polish space. (By a *Polish* space one means a complete separable metric space.) Hausdorff's book [25] contained the first systematic exposition of descriptive set theory for general Polish spaces. Somewhat later, Kuratowski [45] (see also his monograph [46], § 37.II) proved the following theorem, which showed that descriptive set theory has little dependence on the choice of an uncountable Polish space.

Theorem 1.1 (Kuratowski). *Any two uncountable Polish spaces* $\mathbb{X}$ *and* $\mathbb{Y}$ *are Borel isomorphic, that is, there exists a bijection* $F\colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ (a so-called *Borel isomorphism*) *such that the* F*-images and the* F*-pre-images of Borel sets are again Borel sets.*

Thus, at least in the study of problems of descriptive set theory which are invariant (directly or modulo appropriate corrections) under Borel isomorphisms, one can restrict attention to point sets in a fixed uncountable Polish space. For this space, descriptive set theory tends now to use the Baire space $\mathbb{N}^\omega$.[12] Our paper follows this practice. Concerning the forms taken by the problems of regularity properties in the case of $\mathbb{N}^\omega$ and arbitrary uncountable Polish spaces, see the next subsection.

Let us now proceed to the technical details.

Let $2^{<\omega} \subsetneqq \mathbb{N}^{<\omega}$ be the set of all finite sequences of the numbers 0, 1 and the set of all finite sequences of natural numbers, respectively. Let $\operatorname{lh} s$ be the *length* of $s \in \mathbb{N}^{<\omega}$. If $s \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$, then $s^\wedge n \in \mathbb{N}^{<\omega}$ is the sequence obtained by adjoining n to s as the rightmost term. The notation $s \subset t$ means that the sequence t is a proper extension of s.

The *Baire space* $\mathbb{N}^\omega$ (also denoted by $\mathcal{N}$) is formed by all infinite sequences of natural numbers. Its topology is generated by the *Baire intervals*, that is, the sets of the form $\mathcal{N}_s = \{x \in \mathbb{N}^\omega : s \subset x\}$, $s \in \mathbb{N}^{<\omega}$. We note that the spaces $\mathcal{N}^\omega$ and $\mathcal{N}^{(k,\ell)} = \mathcal{N}^k \times \mathbb{N}^\ell$ ($k \geqslant 1$) are homeomorphic to $\mathbb{N}^\omega$; for example, the map $\mathbb{N}^\omega \xrightarrow{\text{onto}} \mathcal{N}^\omega$, $x \mapsto \{(x)_n\}_{n\in\mathbb{N}}$, where a point $(x)_n \in \mathbb{N}^\omega$ is defined for $x \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$ by the equality $(x)_n(k) = x(2^n(2k+1)-1)$ for any k, is a homeomorphism.

The closed subsets of $\mathbb{N}^\omega$ admit the following useful representation. A non-empty set $T \subseteq \mathbb{N}^{<\omega}$ is called a *tree* if $t \in T$ when $s \in T$, $t \in \mathbb{N}^{<\omega}$, and $t \subset s$. (It follows that the *empty sequence* Λ belongs to T.) The sets of the form $[T] = \{x \in \mathbb{N}^\omega : \forall\, m\ (x \restriction m \in T)\}$, where $T \subseteq \mathbb{N}^{<\omega}$ is a tree, are exactly the non-empty closed

[12] The reasons for preferring the space $\mathbb{N}^\omega$ (for example, to the real line $\mathbb{R}$) are certain topological properties of $\mathbb{N}^\omega$, in particular, its 0-dimensionality (see Comment 2 in [60], p. 725), but mainly the fact that the structure of $\mathbb{N}^\omega$ enables us to describe point sets and their properties by means of a simple language of analytic formulae (see subsection 1C). This enables one to greatly simplify the technical aspects of the exposition. At the same time, the space $\mathbb{N}^\omega$ is not much inferior to the real line in geometric clearness, because it is isomorphic to the *Baire line*, that is, to the set of all irrational numbers of $\mathbb{R}$, by means of the known correspondence $x \mapsto \frac{1\,|}{|\,x(0)+1} + \frac{1\,|}{|\,x(1)+1} + \frac{1\,|}{|\,x(2)+1} + \cdots$, $x \in \mathbb{N}^\omega$, in terms of continued fractions, or using the more direct construction given in [4], Russian p. 155. The Baire line is considered in Luzin's *Leçons* [57], for instance.

subsets of $\mathbb{N}^\omega$; moreover, it suffices to consider only trees T without $\subseteq$-maximal elements.

By a *branching node* of a tree T we mean any element $s \in T$ such that $s^\wedge j \in T$ for at least two different numbers $j \in \mathbb{N}$. A tree T is said to be *perfect* if for every $t \in T$ there exists a branching node $s \in T$ with $t \subset s$. The sets of the form $[T]$, where $T \subseteq \mathbb{N}^{<\omega}$ is a perfect tree, are precisely the perfect subsets of $\mathbb{N}^\omega$.

One can view the *Cantor discontinuum* 2^ω either as a closed compact subset of $\mathbb{N}^\omega$ or as an independent space equipped with topology generated by the *Cantor intervals* $\mathcal{C}_s = \{x \in 2^\omega : s \subset x\}$, $s \in 2^{<\omega}$.

The *projective classes* Σ^1_n, Π^1_n, and Δ^1_n (A_n, CA_n, and B_n in the earlier system of notation) for spaces of the form $\mathcal{N}^{(k,\ell)}$ are defined by induction on n as follows:

Σ^1_0 consists of all open sets in spaces of the form $\mathcal{N}^{(k,\ell)}$;

Π^1_n consists of all complements of sets in Σ^1_n;

Σ^1_{n+1} consists of all projections of sets in Π^1_n, where the *projection* of a set $P \subseteq \mathcal{N}^{(k+1,\ell)}$ is the set $\{\langle \vec{x}, \vec{k}\rangle : \exists\, y \in \mathbb{N}^\omega\ (\langle \vec{x}^\wedge y, \vec{k}\rangle \in P)\}$;

Δ^1_n consists of all sets that belong to both Σ^1_n and Π^1_n;

Σ^1_∞ denotes the class $\bigcup_n \Sigma^1_n = \bigcup_n \Pi^1_n$ of all projective sets.

One can equivalently define Σ^1_{n+1} as the class of all continuous images of the Π^1_n sets belonging to *the same* space $\mathcal{N}^{(k,\ell)}$. In such a form the definition of projective classes can be extended to every Polish space $\mathbb{X}$ with the following specification of the initial inductive step: by definition, the class Σ^1_1 consists of all continuous images of the $\mathbf{G}_\delta$ sets in this space. (The class Π^1_0 of all closed sets, which is used in the definition of $\mathcal{N}^{(k,\ell)}$, is insufficient in general because, for example, for σ-compact spaces (in particular, for $\mathbb{R}$) all projections (and even the continuous images of closed sets) belong to the class of $\mathbf{F}_\sigma$ sets, which is a proper subclass of Σ^1_1.)

1B. Principal regularity properties in Polish spaces. It is quite clear that the problems of a perfect kernel and the Baire property (see the Introduction) have an obvious and precise meaning for any topological space, whereas the measurability problem depends on the choice of a measure. To consider this issue in detail, we specify some notions relating to measures.

By a *Borel measure* on a space $\mathbb{X}$ we mean any measure μ defined on the σ-algebra of all Borel subsets of $\mathbb{X}$ such that μ is σ-additive and σ-finite (that is, $\mathbb{X}$ is a countable union of Borel sets of finite measure) and satisfies the conditions $\mu(\mathbb{X}) > 0$ and $\mu(\{x\}) = 0$ for any point $x \in \mathbb{X}$.[13]

A measure μ is said to be *finite* if the value $\mu(\mathbb{X})$ is finite and *σ-finite* if $\mathbb{X}$ is a countable union of sets of finite measure.

The Lebesgue extension of a measure μ on $\mathbb{X}$, which will be denoted by the same symbol μ, is defined on the σ-algebra of all sets $X \subseteq \mathbb{X}$ such that there exist Borel sets U, V with $U \subseteq X \subseteq V$ and $\mu(V \smallsetminus U) = 0$; the sets X of this kind are said to be *μ-measurable*, and we let $\mu(X) = \mu(U) = \mu(V)$.

[13] The last condition means that μ is *continuous*, a condition tacitly assumed throughout the paper for all Borel measures. It can readily be seen that this condition leads to no loss of generality in the treatment of problems considered here, at least in the class of σ-finite measures.

Since the projective classes are closed with respect to the operation of intersection with Borel sets, it follows that the measurability of all projective sets with respect to σ-finite Borel measures is a consequence of their measurability with respect to finite Borel measures. Using this fact, we can reformulate the hypotheses PK, LM, and BP (see the Introduction) for any given Polish space $\mathbb{X}$ and any given projective class $\mathbf{K}$ as follows:

$\mathrm{PK}_{\mathbb{X}}(\mathbf{K})$: every $\mathbf{K}$-set $X \subseteq \mathbb{X}$ has the perfect kernel property;

$\mathrm{LM}^{\mu}_{\mathbb{X}}(\mathbf{K})$: every $\mathbf{K}$-set $X \subseteq \mathbb{X}$ is measurable with respect to a *given* finite Borel measure μ on $\mathbb{X}$;

$\mathrm{BP}_{\mathbb{X}}(\mathbf{K})$: every $\mathbf{K}$-set $X \subseteq \mathbb{X}$ has the Baire property.

Remark 1.2. For any fixed projective class $\mathbf{K}$ each of these three statements does not depend on the choice of the uncountable Polish space $\mathbb{X}$, nor, as far as LM is concerned, on the choice of the measure μ in the family of finite Borel measures. For measurability this fact follows immediately from Theorem 1.1 and Lemma 1.3 below, because every projective class is invariant under Borel isomorphisms. In the case of PK we can apply the fact that any uncountable Borel set in a Polish space contains a perfect subset. As for the Baire property, the desired result follows from another theorem claiming that if $\mathbb{X}$ is a perfect Polish space, then there exists a set $Y \subseteq \mathbb{X}$ homeomorphic to $\mathbb{N}^\omega$ and such that $\mathbb{X} \smallsetminus Y$ is a meagre set in $\mathbb{X}$.[14]

Lemma 1.3. *Suppose that μ and ν are finite Borel measures on Polish spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively, and $\mathbf{K}$ is a projective class. Then there are Borel sets $X \subseteq \mathbb{X}$ of full μ-measure and $Y \subseteq \mathbb{Y}$ of full ν-measure and a Borel bijection $h\colon X \xrightarrow{\text{onto}} Y$ taking μ to ν. Therefore, $\mathrm{LM}^{\mu}_{\mathbb{X}}(\mathbf{K}) \Longleftrightarrow \mathrm{LM}^{\nu}_{\mathbb{Y}}(\mathbf{K})$.*

Proof. We can assume that $\mu(\mathbb{X}) = \nu(\mathbb{Y}) = 1$, and, by Remark 1.2, that $\mathbb{X} = \mathbb{Y} = [0,1]$ (a closed interval of the real line). Let us remove all intervals (a,b) of μ-measure zero from $[0,1]$; we obtain a perfect set $X \subseteq [0,1]$ satisfying $\mu(X) = 1$, and $\mu(I \cap X) > 0$ whenever the interval $I = (a,b)$ intersects X. Similarly, there is a perfect set $Y \subseteq [0,1]$ such that $\nu(Y) = 1$ and $\nu(I \cap Y) > 0$ whenever the open interval I intersects Y.

Let $f(x) = \mu(X \cap [0,x))$ for any $x \in X$. One can readily see that f is an order-preserving continuous map from X onto $[0,1]$. Moreover, f takes the measure μ on X to the ordinary Lebesgue measure on $[0,1]$. Finally, f is 'almost' one-to-one, in the sense that the equality $f(x) = f(y)$ holds for $x \neq y$ if and only if x and y are the endpoints of some maximal open interval disjoint from X. In other words, if one removes from X, say, all left endpoints of those intervals, then f becomes a bijection on the set X' thus obtained, and we still have $\mu(X') = 1$.

In the same way, beginning with ν, we construct a function $g\colon Y \xrightarrow{\text{onto}} [0,1]$ and a set $Y' \subseteq Y$ with $\nu(Y') = 1$. Then $h(x) = g^{-1}(f(x))\colon X' \xrightarrow{\text{onto}} Y'$ is a

[14] To prove this claim, we note that if $\varnothing \neq U \subseteq \mathbb{X}$ is an open set, then for any $\varepsilon > 0$ there exist countably many pairwise disjoint non-empty neighbourhoods $U' \subseteq U$ of diameter $< \varepsilon$ with union dense in U such that the closure of each U' is contained in U. This enables us to define a system $\{U_s\}_{s \in \mathbb{N}^{<\omega}}$ of neighbourhoods such that $U_\Lambda = \mathbb{X}$, the diameter of U_s does not exceed $(\operatorname{lh} s)^{-1}$ for $\operatorname{lh} s \geqslant 1$, the closure of $U_{s^\wedge n}$ is included in U_s, and $\bigcup_n U_{s^\wedge n}$ is a dense subset of U_s for any s. The set $Y = \bigcap_n \bigcup_{n = \operatorname{lh} s} U_s = \bigcup_{a \in \mathbb{N}^\omega} \bigcap_n U_{a \restriction n}$ is homeomorphic to $\mathbb{N}^\omega$ under the map sending each point $a \in \mathbb{N}^\omega$ to a unique point of the intersection $\bigcap_n U_{a \restriction n}$.

Borel isomorphism taking μ to ν. This proves the lemma, because Borel bijections preserve the projective class. $\square$

It follows that, without any loss of generality, our hypotheses can be considered just for point sets in a fixed uncountable Polish space $\mathbb{X}$ and, concerning measurability, just for a single finite Borel measure on $\mathbb{X}$. It is convenient to take for $\mathbb{X}$ the Baire space $\mathbb{N}^\omega$ for PK and BP, and the Cantor space 2^ω equipped with the probability measure $\boldsymbol{\lambda}$ assigning the value $\boldsymbol{\lambda}(\mathcal{C}_s) = 2^{-\operatorname{lh} s}$ to any Cantor interval $\mathcal{C}_s$, $s \in 2^{<\omega}$, for the problem LM.[15] However, the measure $\boldsymbol{\lambda}$ can be formally extended to the entire space $\mathbb{N}^\omega$ by the condition $\boldsymbol{\lambda}(\mathbb{N}^\omega \smallsetminus 2^\omega) = 0$, that is, $\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(X \cap 2^\omega)$ for any $X \subseteq \mathbb{N}^\omega$. Then the assertion $\mathrm{LM}^{\boldsymbol{\lambda}}_{\mathbb{N}^\omega}(\mathbf{K})$ is equivalent to $\mathrm{LM}^{\boldsymbol{\lambda}}_{2^\omega}(\mathbf{K})$ for any projective class $\mathbf{K}$.

These arguments reduce the hypotheses of a perfect kernel, measurability, and the Baire property to the following final forms, which are equivalent to the original formulations (for the real line) given in the Introduction:

PK($\mathbf{K}$): every $\mathbf{K}$-set $X \subseteq \mathbb{N}^\omega$ has the perfect kernel property;

LM($\mathbf{K}$): every $\mathbf{K}$-set $X \subseteq 2^\omega$ (equivalently, every $\mathbf{K}$-set $X \subseteq \mathbb{N}^\omega$) is $\boldsymbol{\lambda}$-measurable;

BP($\mathbf{K}$): every $\mathbf{K}$-set $X \subseteq \mathbb{N}^\omega$ has the Baire property.

The next theorem sums up several classical results.

Theorem 1.4. *The assertions* PK(Σ^1_1), LM(Σ^1_1), BP(Σ^1_1) *hold, that is, every* Σ^1_1 *set* $X \subseteq \mathbb{N}^\omega$ *has the perfect kernel property, is* $\boldsymbol{\lambda}$ *measurable, and has the Baire property. The results for* LM *and* BP *are equally true for the class* Π^1_1.

Proof. The claims relating to measurability and the Baire property are established in a more general form below (Theorem 3.7). Thus, we dwell on PK. It is known that all Σ^1_1 sets in $\mathbb{N}^\omega$ are continuous images of closed subsets of $\mathbb{N}^\omega$. Suppose that $X = F``P$, where $P \subseteq \mathbb{N}^\omega$ is closed and $F\colon P \to \mathbb{N}^\omega$ is a continuous function. We say that a Baire interval $U \subseteq \mathbb{N}^\omega$ is '*good*' if the $F$-image $F``(U \cap P)$ is uncountable. For example, the interval $\mathbb{N}^\omega$ itself is 'good' because X is uncountable. Moreover, if U is a 'good' set, then there exist 'good' sets $U_1, U_2 \subseteq U$ of diameter not exceeding half the diameter of U and such that $F``(U_1 \cap P)$ is disjoint from $F``(U_2 \cap P)$; in particular, the sets U_1 and U_2 themselves are disjoint. This enables us to define a split system $\{U_s\}_{s \in 2^{<\omega}}$ of 'good' sets $U_s \subseteq U$ such that

(1) $U_{s^\wedge i} \subseteq U_s$ and the diameter of $U_{s^\wedge i}$ is at most half the diameter of U_s;

(2) $F``(U_{s^\wedge 0} \cap P)$ is disjoint from $F``(U_{s^\wedge 1} \cap P)$.

The set $C = \bigcap_m \bigcup_{m = \operatorname{lh} s} U_s \subseteq P$ is perfect and compact and F is a bijection on C, and hence $Y = F``C$ is a perfect subset of X. $\square$

1C. Projective hierarchy and the language of analytic formulae. The structure of the Baire space $\mathcal{N} = \mathbb{N}^\omega$ enables one to use a simple language to describe sets in spaces of the form $\mathcal{N}^{(k,\ell)}$. This language has two types of variables,

[15] The preference given to 2^ω (rather than $\mathbb{N}^\omega$) is primarily explained by the compactness of 2^ω, which is useful in some arguments, and by the non-existence of a Borel measure on $\mathbb{N}^\omega$ taking equal values at all Baire intervals of equal rank. We note in addition that the map sending each point $a \in 2^\omega$ to the real number $r_a = \sum_{n=0}^\infty a(n) 2^{-n-1}$ takes the measure $\boldsymbol{\lambda}$ to the ordinary Lebesgue measure on $[0,1]$.

namely, *type* 0 with domain $\mathbb{N}$ (here one uses letters k, l, m, n, and so on) and *type* 1 with domain $\mathbb{N}^\omega$ (here one uses letters x, y, z, a, b, c, and so on). One is allowed to form *terms* by using substitutions of a term or a variable of type 0 into a variable of type 1 and by using recursive (that is, computable) functions from $\mathbb{N}^\ell$ to $\mathbb{N}$; for instance, $x(2^k + y(3n))$ is a term. Terms are obviously objects of type 0. The following classes of formulae of this language are distinguished:

- *elementary* formulae, which are of the form $t = t'$, $t < t'$, $t \leqslant t'$, where t and t' are terms (for example, variables of type 0);
- *analytic* formulae, that is, formulae obtained from elementary formulae by means of propositional connectives and quantifiers;
- *arithmetic* formulae, that is, analytic formulae which do not include quantifiers of type 1 (over $\mathbb{N}^\omega$);
- *bounded* formulae, that is, arithmetic formulae which include quantifiers only of the form $\exists\, k < t$ and $\forall\, k < t$, where k is a variable of type 0 and t is a term (in particular, all quantifier-free formulae are bounded);
- $\varSigma^0_n$ and $\varPi^0_n$, that is, arithmetic formulae of the respective forms

  $$\exists\, k_1\ \forall\, k_2\ \exists\, k_3\ \ldots\ \exists\,(\forall)\, k_n\, \varphi \quad \text{and} \quad \forall\, k_1\ \exists\, k_2\ \forall\, k_3\ \ldots\ \forall\,(\exists)\, k_n\, \varphi,$$

  where φ is a bounded formula;
- $\varSigma^1_n$ and $\varPi^1_n$, that is, analytic formulae of the respective forms

  $$\exists\, x_1\ \forall\, x_2\ \exists\, x_3\ \ldots\ \exists\,(\forall)\, x_n\, \varphi \quad \text{and} \quad \forall\, x_1\ \exists\, x_2\ \forall\, x_3\ \ldots\ \forall\,(\exists)\, x_n\, \varphi\,,$$

  where φ is an arithmetic formula.

Effective hierarchy. The free variables of analytic formulae can be replaced by particular elements of $\mathbb{N}$ (type 0) or $\mathbb{N}^\omega$ (type 1), which are called *parameters* in this case. (Such a replacement is essential only for type 1, since any natural number is definable by a quantifier-free formula.) This yields a classification of point sets from the point of view of the parameters entering the definition rather than just from the point of view of the defining formula.

For a given 'vector' $\vec{a} = \langle a_1, \ldots, a_j \rangle \in (\mathbb{N}^\omega)^j$ we denote by $\varSigma^i_n(\vec{a})$ the class of all point sets in spaces of the form $\mathcal{N}^{(k,\ell)}$ that are definable by $\varSigma^i_n$-formulae with parameters from the list $a_1, \ldots, a_j$. The class $\varPi^i_n(\vec{a})$ is defined similarly, and we write $\varDelta^i_n(\vec{a}) = \varSigma^i_n(\vec{a}) \cap \varPi^i_n(\vec{a})$. In the special cases $j = 1$ and $j = 0$ we write $\varSigma^i_n(a)$ and $\varSigma^i_n$ and do the same for $\varPi$ and $\varDelta$. In addition, we define $\varSigma^i_n(P) = \bigcup_{a_1,\ldots,a_j \in P \cap \mathbb{N}^\omega} \varSigma^i_n(a_1, \ldots, a_j)$ for any set P and we define $\varPi^i_n(P)$ and $\varDelta^i_n(P)$ similarly. If a set $P \cap \mathbb{N}^\omega$ is recursively closed, then $\varSigma^i_n(P)$ coincides with $\bigcup_{a \in P \cap \mathbb{N}^\omega} \varSigma^i_n(a)$, and a similar assertion holds for $\varPi$ and $\varDelta$.

The classes whose notation includes the letters $\varSigma$, $\varPi$, and $\varDelta$ are said to be *effective*, in contrast to the projective classes Σ^1_n, Π^1_n, and Δ^1_n. However, the projective hierarchy is only a particular case of the effective hierarchy.

Proposition 1.5. $\Sigma^i_n = \varSigma^i_n(\mathbb{N}^\omega)$, *that is, $X \in \Sigma^i_n$ if and only if $X \in \varSigma^i_n(a)$ for some $a \in \mathbb{N}^\omega$, and similar assertions hold for Π and Δ.*

Proof. All sets definable by $\varSigma^0_1$ formulae in the spaces of the form $\mathcal{N}^{(k,\ell)}$ are obviously open, and hence $\varSigma^0_1(\mathbb{N}^\omega) \subseteq \Sigma^0_1$. Conversely, if X is open, say, in $\mathbb{N}^\omega$, then

$X = \bigcup_n \mathcal{N}_{s_n}$, where $s_n \in \mathbb{N}^{<\omega}$, and hence X is definable by the Σ^0_1 formula $\exists\, n$ $\forall\, m < a(n)$ $(x(m) = b(2^n(2m+1)-1))$ (with the parameters a and b), where $a(n) = \operatorname{lh} s_n$ (the length of a finite sequence s_n), and $b(2^n(2m+1)-1)$ is equal to $s_n(m)$ for $m < \operatorname{lh} s_n$ and to 0 for $m \geqslant \operatorname{lh} s_n$. Thus, $\Sigma^0_1(\mathbb{N}^\omega) = \Sigma^0_1$. This implies the desired result for closed sets, for $\mathbf{F}_\sigma$, and for $\mathbf{G}_\delta$, as well as the fact that arithmetic formulae (with parameters) define Borel sets *of finite rank*. Concerning projective classes, the quantifier $\exists\, x$ (according to our conventions, $\exists\, x \in \mathbb{N}^\omega$) corresponds to a projection, whereas the quantifier $\forall\, x$ corresponds to the combination complement–projection–complement. $\square$

$(\exists^0\,\exists^0)$	$\exists\, i\, \exists\, j\ \ \varphi(i,j) \iff \exists\, n\ \ \varphi((n)_1,(n)_2)$
$(\forall^0\,\forall^0)$	$\forall\, i\, \forall\, j\ \ \varphi(i,j) \iff \forall\, n\ \ \varphi((n)_1,(n)_2)$
$(\exists^1\,\exists^1)$	$\exists\, x\, \exists\, y\ \ \varphi(x,y) \iff \exists\, z\ \ \varphi((z)_1,(z)_2)$
$(\forall^1\,\forall^1)$	$\forall\, x\, \forall\, y\ \ \varphi(x,y) \iff \forall\, z\ \ \varphi((z)_1,(z)_2)$
$(\forall^0\,\exists^0)$	$\forall\, i\, \exists\, j\ \ \varphi(i,j) \iff \exists\, x\, \forall\, i\ \ \varphi(i,x(i))$
$(\exists^0\,\forall^0)$	$\exists\, i\, \forall\, j\ \ \varphi(i,j) \iff \forall\, x\, \exists\, i\ \ \varphi(i,x(i))$
$(\exists^1\,\exists^0)$	$\exists\, x\, \exists\, j\ \ \varphi(x,j) \iff \exists\, y\ \ \varphi((y)_0,(y)_1(0))$
$(\forall^1\,\forall^0)$	$\forall\, x\, \forall\, j\ \ \varphi(x,j) \iff \forall\, y\ \ \varphi((y)_0,(y)_1(0))$
$(\forall^0\,\exists^1)$	$\forall\, i\, \exists\, x\ \ \varphi(i,x) \iff \exists\, x\, \forall\, i\ \ \varphi(i,(x)_i)$
$(\exists^0\,\forall^1)$	$\exists\, i\, \forall\, x\ \ \varphi(i,x) \iff \forall\, x\, \exists\, i\ \ \varphi(i,(x)_i)$

Table. The map $\langle i,j\rangle \mapsto 2^i(2j+1)-1$ is a bijection of $\mathbb{N}^2$ onto $\mathbb{N}$; $(n)_1$ and $(n)_2$ denote the inverse functions, that is, $(n)_1 = i$ and $(n)_2 = j$ whenever $2^i(2j+1)-1 = n$. If $z \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$, then $(z)_n \in \mathbb{N}^\omega$ is defined in subsection 1A.

Transformation of analytic formulae. Equivalences in the table enable one to reduce complicated analytic formulae, by simplifying the quantifier prefix, to a form which permits one to directly evaluate the type of the set defined by the formula. We note that the next-to-last equivalence $(\forall^0\,\exists^1)$ expresses the countable axiom of choice, and $(\exists^0\,\forall^1)$ expresses the dual statement.

As an elementary example, we note that, *if* $\varphi(x,y,i,j)$ *is a* Σ^1_n *formula and* $n \geqslant 1$, *then, say, the formulae* $\exists\, x\ \varphi(x,y,i,j)$, $\exists\, i\ \varphi(x,y,i,j)$, *and* $\forall\, i\ \varphi(x,y,i,j)$ *belong to the same type*, in the sense that they can be converted to a Σ^1_n form (with the same parameters) by using the rules $(\exists^1\,\exists^1)$, $(\exists^1\,\exists^0)$, and $(\forall^0\,\exists^1)$.

1D. Elements of the theory of $\boldsymbol{\Pi}^1_1$ sets. We recall that by a *sieve* over a space $\mathbb{X}$ one means any set $R \subseteq \mathbb{X} \times \mathbb{Q}$ (where $\mathbb{Q}$ is the set of all rational numbers) or, equivalently, a system $\{R_q\}_{q\in\mathbb{Q}}$ of sets $R_q = \{x \in \mathbb{X} : \langle x,q\rangle \in R\} \subseteq \mathbb{X}$, which are called the *elements* of the sieve R. In this case we assign to any point $x \in \mathbb{X}$ the set $R^{(x)} = \{q : x \in R_q\} \subseteq \mathbb{Q}$. This leads to the following partition of the space $\mathbb{X}$ into two sets:

$$E(R) = \{x : R^{(x)} \text{ is well ordered}\}, \qquad \text{the so-called } \textit{outer} \text{ set,}$$

$$\mathcal{E}(R) = \{x : R^{(x)} \text{ is not well ordered}\}, \quad \text{the so-called } \textit{inner} \text{ set}$$

(the well ordering is understood in the sense of the natural order of the rational numbers). Each of these sets admits a further partition into *constituents*,

$$E_\xi(R) = \{x \in E : \operatorname{otp} R^{(x)} = \xi\} \quad \text{and} \quad \mathcal{E}_\xi(R) = \{x \in \mathcal{E} : \operatorname{otp} \operatorname{MIS}(R^{(x)}) = \xi\}$$

($\xi < \omega_1$), where $\operatorname{otp} S$ is the order type of a set $S \subseteq \mathbb{Q}$ (if S is well ordered), and $\operatorname{MIS}(S)$ is the largest well-ordered initial segment of S.

We fix once and for all a recursive bijection $n \mapsto q_n : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{Q}$; let $q \mapsto n_q$ be the inverse bijection. This induces a homeomorphism of $\mathbb{N}^\omega \times \mathbb{Q}$ onto $\mathbb{N}^\omega \times \mathbb{N}$, which enables us to classify sieves in terms of the projective hierarchy, so that a *sieve of class K* over $\mathbb{X} = \mathcal{N}^{(k,\ell)}$ is any set $R \subseteq \mathbb{X} \times \mathbb{Q}$ of class K. For Borel and projective classes $\mathbf{K} = \Sigma^i_n$, Π^i_n, and Δ^i_n this is equivalent to the condition that every element R_q of the sieve R belongs to $\mathbf{K}$.

Theorem 1.6 (the sifting theorem).
Suppose that $\mathbb{X} = \mathcal{N}^{(k,\ell)}$.

(a) *If $R \subseteq \mathbb{X} \times \mathbb{Q}$ is a Borel sieve, then the set $E(R)$ belongs to Π^1_1, and all the constituents $E_\xi(R)$ and $\mathcal{E}_\xi(R)$ are Borel sets. Conversely, for any Π^1_1 set $X \subseteq \mathbb{X}$ there is a sieve $R \subseteq \mathbb{X} \times \mathbb{Q}$ of class Δ^0_1 (that is, with open-closed elements) such that $X = E(R)$.*

(b) (**effective version of the theorem**) *If $a \in \mathbb{N}^\omega$ and $R \subseteq \mathbb{X} \times \mathbb{Q}$ is a sieve of the class $\varDelta^1_1(a)$, then $E(R)$ is a $\varPi^1_1(a)$ set. Conversely, for any $\varPi^1_1(a)$ set $X \subseteq \mathbb{X}$ there is a $\varDelta^0_1(a)$ sieve $R \subseteq \mathbb{X} \times \mathbb{Q}$ such that $X = E(R)$.*

Proof. For any sieve $R \subseteq \mathbb{N}^\omega \times \mathbb{Q}$ we have

$$x \in E(R) \iff \forall f \colon\ \mathbb{N} \to \mathbb{Q}\ \big(f \text{ monotone decreasing } \implies \exists\, n\ (\langle x, f(n)\rangle \notin R)\big).$$

Thus, if R is a sieve of class $\varSigma^1_1(a)$, then the right-hand side can readily be reduced to the $\varPi^1_1$ form with a as a parameter by means of transformations in the table in subsection 1C (and the recursive bijection $n \mapsto q_n$ introduced above to replace $\mathbb{Q}$ by $\mathbb{N}$).

To prove the converse, consider a $\varPi^1_1(a)$ set $X = \{x \in \mathbb{N}^\omega : \forall\, y\ \varphi(x,y,a)\}$, where φ is an arithmetic formula of the form $\Pi\ \psi(x,y,a,\dots)$ for some block (prefix) Π of quantifiers of the form $\exists\, n$, $\forall\, n$ (of type 0), and ψ is a bounded formula. For example, let φ be $\exists\, i\ \forall\, j\ \exists\, k\ \psi(x,y,a,i,j,k)$. The rule $(\exists^0\,\forall^1)$ of the table in subsection 1C reduces this definition to the form $\forall\, z\ \exists\, i\ \exists\, k\ \psi(x,y,a,i,z(i),k)$, and other rules of the table reduce the definition of X to the form $X = \{x \in \mathbb{N}^\omega : \forall\, y\ \exists\, m\ \Phi(x,y,a,m)\}$, where Φ is a bounded formula.

Since Φ is bounded, the determination of the truth of $\Phi(x,y,a,m)$ can be represented as a computation by a computer program (depending only on the structure of Φ and not on the values of x,y,a,m), where x,y,a are loaded as infinite strings of numerical values $x(n)$, $y(n)$, $a(n)$, $n \in \mathbb{N}$, and the result is obtained after finitely many steps. If $s \in \mathbb{N}^{<\omega}$, then we write $s^\wedge\mathbf{0} \in \mathbb{N}^\omega$ for the extension of s by zeros. Let

$S^{xa} =$ the set of all $s \in \mathbb{N}^{<\omega}$ such that if the computation of $\Phi(x, s^\wedge\mathbf{0}, a, m)$ for every $m < \operatorname{lh} s$ is carried out without using the values $(s^\wedge\mathbf{0})(k)$, $k \geqslant \operatorname{lh} s$, then the result of the computation is 'false' (that is, $\neg\,\Phi(x, s^\wedge\mathbf{0}, a, m)$).

For any $s,\, t \in \mathbb{N}^{<\omega}$ we set $s <_{\mathrm{LS}} t$ if either $t \subset s$ or $s < t$ lexicographically (the *Luzin–Sierpiński order*). One can readily see that

$$\begin{aligned} x \in X &\iff S^{xa} \text{ has no infinite branches} \\ &\iff S^{xa} \text{ is well ordered in the sense of } <_{\mathrm{LS}}. \end{aligned}$$

However, the ordering $<_{\mathrm{LS}}$ is countable and dense in itself; it admits a maximal element and has no minimal element. Fix a recursive bijection $f\colon \mathbb{N}^{<\omega} \xrightarrow{\text{onto}} \mathbb{Q}_{\leqslant 1} = \{q \in \mathbb{Q} : q \leqslant 1\}$ which is order preserving, that is, $s <_{\mathrm{LS}} t \iff f(s) < f(t)$. According to what was proved above, the sieve $R = \{\langle x, f(s)\rangle : s \in S^{xa}\}$ satisfies the condition $X = E(R)$. It remains to note that R is a sieve of class $\varDelta^0_1(a)$; indeed, the right-hand side of the definition of S^{xa}, viewed as a formula with the variables x and s, can be reduced both to the $\varSigma^0_1(a)$ form (with the external quantifier "there is a computation") and to the $\varPi^0_1(a)$ form ("for any computation").

The Borel property of the constituents is a classical result. $\square$

Corollary 1.7. *Every Σ^1_2 set is a union of $\aleph_1$ Borel sets.*

Proof. By the sifting theorem, every Π^1_1 set is a union of $\aleph_1$ Borel sets, and therefore every Σ^1_2 set is a union of $\aleph_1$ *projections* of Borel sets, that is, of $\aleph_1$ sets of class Σ^1_1. However, any set of the last type is again a union of $\aleph_1$ Borel sets by Theorem 1.6. $\square$

Theorem 1.8. *Suppose that R is a Borel sieve over a space $\mathcal{N}^{(k,\ell)}$.*

(i) *If a Σ^1_1 set A is included in $E(R)$, then there is an ordinal $\vartheta < \omega_1$ such that $A \subseteq \bigcup_{\xi<\vartheta} E_\xi(R)$.*
(ii) *The set $E(R)$ is Borel if and only if there is an ordinal $\vartheta < \omega_1$ such that $E_\xi(R) = \varnothing$ for any $\xi > \vartheta$.*
(iii) *The class Δ^1_1 coincides with the class of all Borel sets.*

Uniformization. Let $\mathbb{X}$ and $\mathbb{Y}$ be arbitrary spaces. If $P \subseteq \mathbb{X} \times \mathbb{Y}$, then one often writes $P(x,y)$ instead of $\langle x,y\rangle \in P$. We recall that the set $\operatorname{dom} P = \{x : \exists\, y\ P(x,y)\} \subseteq \mathbb{X}$ is called the *projection* of P (on $\mathbb{X}$).

A set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is said to be *uniform* if and only if for any $x \in \mathbb{X}$ there is at most one $y \in Y$ such that $P(x,y)$. In fact, this means that P is the graph of a partial function $\mathbb{X} \to \mathbb{Y}$. If $P \subseteq Q \subseteq \mathbb{X} \times \mathbb{Y}$, P is uniform, and $\operatorname{dom} P = \operatorname{dom} Q$, then one says that P *uniformizes* Q.

Theorem 1.9 (Novikov–Kondô–Addison, [62], [43], or [86], § 7.11). *Every set $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ of class Π^1_1 can be uniformized by a set of the same class. Every set $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ of class $\varPi^1_1(a)$ $(a \in \mathbb{N}^\omega)$ can be uniformized by a set of the same class.*

1E. Digression: encoding by points of the Baire space. Elements of the space $\mathbb{N}^\omega$ can be used to encode mathematical objects of very different nature provided that, informally speaking, these objects contain only countably many 'bits' of information.

Example 1.10.1 (encoding of countable ordinals).

We recall that $q \mapsto n_q$ is a bijection of $\mathbb{Q}$ onto $\mathbb{N}$. By $\mathbf{WO}$ we denote the set of all $w \in \mathbb{N}^\omega$ such that $Q_w = \{q \in \mathbb{Q} : w(n_q) = 0\}$ is well ordered by the order of $\mathbb{Q}$. In this case let $|w| = \operatorname{otp} Q_w$ be the order type of Q_w; this means that $|w| < \omega_1$.

We set $\mathbf{WO}_\gamma = \{w \in \mathbf{WO} : |w| = \gamma\}$.

This useful encoding of countable ordinals admits a simple interpretation in terms of sieves. We write $\mathcal{L}_q = \{x \in \mathbb{N}^\omega : x(n_q) = 0\}$. This defines the *Lebesgue binary sieve* $\mathcal{L} = \{\mathcal{L}_q\}_{q \in \mathbb{Q}}$. In this case the Π^1_1 set $E(\mathcal{L})$ obviously coincides with $\mathbf{WO}$, and $\mathbf{WO}_\gamma = E_\gamma(\mathcal{L})$ for any $\gamma < \omega_1$.

Suppose that $w \in \mathbf{WO}$ and $k \in \mathbb{N}$, $w(k) = 0$. We define $w\restriction_k \in \mathbf{WO}$ as follows: $w\restriction_k(n) = 0$ for $w(n) = 0$ and $q_n < q_k$, and $w\restriction_k(n) = 1$ otherwise. It is clear that $\{\xi : \xi < |w|\} = \{|w\restriction_k| : w(k) = 0\}$. □

Example 1.10.2 (encoding of Borel sets).

Let us fix, once and for all, a recursive enumeration without repetition, $\mathbb{N}^{<\omega} = \{s_k : k \in \mathbb{N}\}$.

We set $\mathbf{BC}_0 = \{\mathbf{k} : k \in \mathbb{N}\}$, where $\mathbf{k} \in \mathbb{N}^\omega$ is a constant function (that is, $\mathbf{k}(i) = k$ for all i). For $\mathbf{k} \in \mathbf{BC}_0$ we write $\mathbf{B_k} = \mathcal{N}_{s_{k-1}}$ (a *Baire interval* in $\mathbb{N}^\omega$; see subsection 1A) for $k \geqslant 1$ and $\mathbf{B_k} = \varnothing$ for $k = 0$. If $\xi > 0$, then we define $\mathbf{BC}_\xi$ as the family of all points $c \in \mathbb{N}^\omega \smallsetminus \bigcup_{\eta<\xi} \mathbf{BC}_\eta$ such that $\{(c)_n : n \in \mathbb{N}\} \subseteq \bigcup_{\eta<\xi} \mathbf{BC}_\eta$, and we let $\mathbf{B}_c = \mathbb{N}^\omega \smallsetminus \bigcup_n \mathbf{B}_{(c)_n}$ for any c of this kind. (For the definition of $(c)_n$, see subsection 1A.) This introduces the set $\mathbf{BC} = \bigcup_{\xi<\omega_1} \mathbf{BC}_\xi \subseteq \mathbb{N}^\omega$ of *Borel codes* and a Borel set $\mathbf{B}_c \subseteq \mathbb{N}^\omega$ for each code $c \in \mathbf{BC}$.[16] □

Remark 1.10.3. The construction of $\mathbf{BC}$ and $\mathbf{B}_c$ admits another approach. We set $(c)_s = (\dots((c)_{s(0)})_{s(1)}\dots)_{s(n-1)}$ for any $c \in \mathbb{N}^\omega$, $s \in \mathbb{N}^{<\omega}$, $n = \operatorname{lh} s$ (and, separately, $(c)_\Lambda = c$ for the empty sequence Λ). One can readily see that

$$T_c = \{s \in \mathbb{N}^{<\omega} : \forall\, n < \operatorname{lh} s\ ((c)_{s\restriction n} \text{ is not a constant})\}$$

is a tree in $\mathbb{N}^{<\omega}$, and we have $c \in \mathbf{BC}$ if and only if T_c is a *well-founded* tree, that is, T_c has no infinite branches. Moreover, $s \in T_c$ is a maximal element of T_c if and only if $(c)_s$ is a constant function. Further, if $c \in \mathbf{BC}$, then we can define a Borel set $B_c(s) \subseteq \mathbb{N}^\omega$ for each sequence $s \in T_c$ as follows: $B_c(s) = \mathcal{N}_{s_{k-1}}$ for $(c)_s = \mathbf{k}$ (the constant k) and $k \geqslant 1$, $B_c(s) = \varnothing$ for $(c)_s = \mathbf{k}$ and $k = 0$, and $B_c(s) = \mathbb{N}^\omega \smallsetminus \bigcup_n B_c(s^\wedge n)$ if $(c)_s$ is not a constant, that is, if s is not a maximal element of T_c. This easily implies that $B_c(\Lambda) = \mathbf{B}_c$ and, in general, $B_c(s) = \mathbf{B}_{(c)_s}$ for any $s \in T_c$. □

When defining an encoding of some mathematical structures by objects of simpler type, the most interesting question is as follows: what are the properties of relations on encoding objects that correspond to the main relations among the encoded structures themselves? In this case the most important question is the definability of the relations thus induced. The following proposition contains the most interesting results in this direction.

[16]This encoding uses the operation of complementing countable unions. In terms of this operation one can easily express the complement itself, and hence the operations of countable union and countable intersection as well. (For the code of the complement of $\mathbf{B}_c$ one can take $c' \in \mathbb{N}^\omega$ defined by $(c')_n = c\ \forall\, n$.) Thus, every Borel set $X \subseteq \mathbb{N}^\omega$ has the form $\mathbf{B}_c$ for a suitable $c \in \mathbf{BC}$.

Proposition 1.11.

(i) *The sets* $\mathbf{WO}$ *and* $\mathbf{BC}$ *belong to* $\varPi^1_1$; *more precisely, each of them is definable in* $\mathbf{ZFC}$ *by an explicitly given* $\varPi^1_1$ *formula.*

(ii) *There exist a* $\varSigma^1_1$ *formula* $\sigma(\,\cdot\,,\cdot\,)$ *and a* $\varPi^1_1$ *formula* $\pi(\,\cdot\,,\cdot\,)$ *such that* $|w| \leqslant |z| \iff \sigma(w,z) \iff \pi(w,z)$ *for any* $w,\, z \in \mathbf{WO}$. *The same holds for* $<$ *and* $=$.

(iii) *If* R *is a sieve of class* $\varDelta^1_1(a)$, $a \in \mathbb{N}^\omega$, *then there exist* $\varSigma^1_1(a)$ *formulae* $\sigma(\,\cdot\,,\cdot\,)$ *and* $\sigma'(\,\cdot\,,\cdot\,)$ (that is, $\varSigma^1_1$ formulae with the single parameter a) *and* $\varPi^1_1(a)$ *formulae* $\pi(\,\cdot\,,\cdot\,)$ *and* $\pi'(\,\cdot\,,\cdot\,)$ *such that*

$$x \in E_{|w|}(R) \iff \sigma(w,x) \iff \pi(w,x),$$
$$x \in \mathcal{E}_{|w|}(R) \iff \sigma'(w,x) \iff \pi'(w,x),$$

for any $w \in \mathbf{WO}$ *and* $x \in \mathbb{N}^\omega$.

(iv) *There exist a* $\varSigma^1_1$ *formula* $\sigma(\,\cdot\,,\cdot\,)$ *and a* $\varPi^1_1$ *formula* $\pi(\,\cdot\,,\cdot\,)$ *such that* $x \in \mathbf{B}_c \iff \sigma(c,x) \iff \pi(c,x)$ *for any* $c \in \mathbf{BC}$ *and* $x \in \mathbb{N}^\omega$.

(v) *There exist a* $\varSigma^1_1$ *formula* $\varphi_{\boldsymbol\lambda}(\,\cdot\,,\,\cdot\,,\cdot\,)$ *and a* $\varPi^1_1$ *formula* $\psi_{\boldsymbol\lambda}(\,\cdot\,,\,\cdot\,,\cdot\,)$ *such that* $\boldsymbol\lambda(\mathbf{B}_c) = m/n \iff \varphi_{\boldsymbol\lambda}(c,m,n) \iff \psi_{\boldsymbol\lambda}(c,m,n)$ *for any* $c \in \mathbf{BC}$ *and* $m,\, n \in \mathbb{N}$. (The measure $\boldsymbol\lambda$ on $\mathbb{N}^\omega$ was introduced in subsection 1B.)

(vi) *There exist a* $\varSigma^1_1$ *formula* $\varphi_{\mathrm{cat}}(\,\cdot\,,\cdot\,)$ *and a* $\varPi^1_1$ *formula* $\psi_{\mathrm{cat}}(\,\cdot\,,\cdot\,)$ *such that* $\mathbf{B}_c \cap \mathcal{N}_{s_k}$ *is meagre* $\iff \varphi_{\mathrm{cat}}(c,k) \iff \psi_{\mathrm{cat}}(c,k)$ *for any* $c \in \mathbf{BC}$ *and* $k \in \mathbb{N}$. ($\{s_k\}_{k\in\mathbb{N}}$ is a fixed recursive enumeration of $\mathbb{N}^{<\omega}$.)

Proof. (i) The relation $w \in \mathbf{WO}$ is equivalent to the non-existence of an infinite decreasing chain of elements of Q_w (in the notation of Example 1.10.1), which is expressible by a $\varPi^1_1$ formula. Further, $c \in \mathbf{BC}$ is equivalent to the condition that for any $a \in \mathbb{N}^\omega$ there is an n for which $(s)_{a\restriction n} \notin T_c$ (in terms of Remark 1.10.3).

(ii) If $w, z \in \mathbf{WO}$, then the condition $|w| \leqslant |z|$ is equivalent to the existence of an order isomorphism of the set Q_w onto an initial segment of Q_z and to the non-existence of an order isomorphism of Q_z onto a proper initial segment of Q_w. Order isomorphisms of this kind do not belong to $\mathbb{N}^\omega$, but they can readily be encoded by points of $\mathbb{N}^\omega$.

(iii) The relation $x \in E_{|w|}(R)$ is equivalent to the existence of an order isomorphism between the sets Q_w and $R^{(x)}$ and to the non-existence of an order isomorphism between Q_w and a proper initial segment of $R^{(x)}$, or vice versa. This yields the desired formulae σ and π. The formulae for $\mathcal{E}_{|w|}(R)$ can be obtained by using similar simple considerations.

(iv) If $c,\, x \in \mathbb{N}^\omega$, then by a *$c, x$-function* we mean any function $f\colon \mathbb{N}^{<\omega} \to \{0,1\}$ such that for every $s \in \mathbb{N}^{<\omega}$ if $(c)_s$ is a constant function $\mathbf{k}$ for some k, then $f(s) = 1 \iff x \in \mathcal{N}_{s_{k-1}}$ for $k \geqslant 1$, $f(s) = 0$ for $k = 0$, and if $(c)_s$ is not constant, then $f(s) = 1 \iff \forall\, m\ (f(s^\wedge m) = 0)$. Clearly, for any $c \in \mathbf{BC}$ there is a unique c, x-function $f = f_{cx}$, and the relation $x \in \mathbf{B}_c$ is equivalent to $f_{cx}(\Lambda) = 1$. It follows that $x \in \mathbf{B}_c$ is equivalent (for $c \in \mathbf{BC}$) to the existence of a c, x-function f with $f(\Lambda) = 1$ and to the non-existence of a c, x-function f with $f(\Lambda) = 0$.

(v) By a *$\boldsymbol\lambda$-approximation* of a set $X \subseteq \mathbb{N}^\omega$ we mean a pair $\langle U, V\rangle$ of $\mathbf{F}_\sigma$ sets such that $U \subseteq X$, $V \cap X = \varnothing$, and $\boldsymbol\lambda(U \cup V) = 1$. Clearly, $\boldsymbol\lambda(U) = \boldsymbol\lambda(X)$ in this case. We note that if pairs $\langle U_n, V_n\rangle$ are $\boldsymbol\lambda$-approximations of sets X_n, then the natural idea of considering the pair $\langle V=\bigcap_n V_n, U=\bigcup_n U_n\rangle$ as a $\boldsymbol{\lambda}$-approximation of the set $X=\mathbb{N}^\omega\smallsetminus\bigcup_n X_n$ does not work, because V need not be an $\mathbf{F}_\sigma$ set. However, this construction can be modified as follows. Suppose that $V_n=\bigcup_k V_{nk}$ $\forall\, n$, where all the V_{nk} are closed. For any given j we choose in each V_n a closed subset $V'_{nj}=\bigcup_{k<k(n,j)} V_{nk}\subseteq V_n$ such that $V'_{nj}\subseteq V'_{n,j+1}$ and $\boldsymbol{\lambda}(V_n\smallsetminus V'_{nj})\leqslant 2^{-(n+j)}$. Then $V'_j=\bigcap_n V'_{nj}$ is a closed subset of V with $\boldsymbol{\lambda}(V\smallsetminus V'_j)\leqslant\sum_n 2^{-(n+j)}=2^{-j+1}$. Hence, $V'=\bigcup_j V'_j$ is an $\mathbf{F}_\sigma$ set, $V'\subseteq V$, and $\boldsymbol{\lambda}(V\smallsetminus V')=0$. Thus, the pair $\langle V',U\rangle$ is a $\boldsymbol{\lambda}$-approximation of X. Let $\Phi(V',U,\{U_{nk}\}_{n,k\in\mathbb{N}},\{V_{nk}\}_{n,k\in\mathbb{N}})$ be a formula expressing the fact that an $\mathbf{F}_\sigma$ set V' is constructed from the closed sets V_{nk} in the way described above and that $U=\bigcup_{n,k\in\mathbb{N}} U_{nk}$.

Let us now use this approach to construct the formulae $\varphi_{\boldsymbol{\lambda}}$ and $\psi_{\boldsymbol{\lambda}}$. Here we need a separate encoding of $\mathbf{F}_\sigma$ sets. We write $F[z]=\mathbb{N}^\omega\smallsetminus\bigcup_{(z)_n(k)=1}\mathcal{N}_{s_k}$ and $F_\sigma[z]=\bigcup_n F[(z)_n]$ for each $z\in\mathbb{N}^\omega$. By an *approximation code for* $c\in\mathbb{N}^\omega$ we mean any pair of maps $\gamma,\delta\colon\mathbb{N}^{<\omega}\to\mathbb{N}^\omega$ satisfying the following conditions:

(1) if $s\in\mathbb{N}^\omega$ is such that $(c)_s=\mathbf{k}$ and $k\geqslant 1$, then, first, $(\gamma(s))_n(j)=1$ for all n and j such that $\mathcal{N}_j\cap\mathcal{N}_{k-1}=\varnothing$, whereas $(\delta(s))_n(j)=0$ for $\mathcal{N}_j\cap\mathcal{N}_{k-1}\neq\varnothing$, in which case we clearly have $F_\sigma[\delta(s)]=\mathcal{N}_{s_{k-1}}=B_c(s)$, and, second, $(\delta(s))_n(k-1)=1$ and $(\delta(s))_n(k')=0$ for $k'\neq k-1$, in which case $F_\sigma[\delta(s)]=\mathbb{N}^\omega\smallsetminus\mathcal{N}_{s_{k-1}}$;

(2) if $s\in\mathbb{N}^\omega$, $(c)_s=\mathbf{k}$, and $k=0$, then $(\gamma(s))_n(j)=1\;\forall\, j$, in which case we have $F_\sigma[\gamma(s)]=\varnothing=B_c(s)$, and $(\delta(s))_n(j)=0\;\forall\, j$, in which case $F_\sigma[\delta(s)]=\mathbb{N}^\omega$;

(3) if $(c)_s$ is non-constant, then $\Phi(V',U,\{U_{nk}\}_{n,k\in\mathbb{N}},\{V_{nk}\}_{n,k\in\mathbb{N}})$, where $V'=F_\sigma[\gamma(s)]$, $U=F_\sigma[\delta(s)]$, $U_{nk}=F[(\gamma(s^\wedge n))_k]$, and $V_{nk}=F[(\delta(s^\wedge n))_k]$.

Clearly, approximation codes $\langle\gamma,\delta\rangle$ do exist for any Borel code $c\in\mathbf{BC}$, and for each of these codes and each $s\in T_c$ the pair $\langle F_\sigma[\gamma(s)],F_\sigma[\delta(s)]\rangle$ is a $\boldsymbol{\lambda}$-approximation of the set $B_c(s)$, in particular, $\langle F_\sigma[\gamma(\Lambda)],F_\sigma[\delta(\Lambda)]\rangle$ is a $\boldsymbol{\lambda}$-approximation of the set $B_c(\Lambda)=\mathbf{B}_c$. Thus, the formulae

$$\begin{aligned}\varphi_{\boldsymbol{\lambda}}(c,m,n) &:= \exists\,\langle\gamma,\delta\rangle\ (\langle\gamma,\delta\rangle \text{ is an approximation code for}\\ &\qquad c\wedge\boldsymbol{\lambda}(F_\sigma[\gamma(\Lambda)])=m/n),\\ \psi_{\boldsymbol{\lambda}}(c,m,n) &:= \forall\,\langle\gamma,\delta\rangle\ (\langle\gamma,\delta\rangle \text{ is an approximation code for}\\ &\qquad c\implies\boldsymbol{\lambda}(F_\sigma[\gamma(\Lambda)])=m/n),\end{aligned}$$

satisfy the equivalences (v). On the other hand, it is easy to verify (though the verification is a rather lengthy procedure which we omit) that the subformulae within the outer parentheses in both formulae are arithmetic, and hence have the desired type.

(vi) We can apply the same construction; however, by an approximation of a given set $X\subseteq\mathbb{N}^\omega$ we mean here a pair of the form $\langle U,F\rangle$, where $U\subseteq\mathbb{N}^\omega$ is open, F is a meagre $\mathbf{F}_\sigma$ set, and $X\triangle U\subseteq F$. ($\triangle$ stands for the symmetric difference.) The main technical point is to construct an approximation of a set $X=\mathbb{N}^\omega\smallsetminus\bigcup_n X_n$ from given approximations of the Borel sets X_n, but this problem can be solved without difficulty. □

Historical and bibliographical remarks. The sets of class Σ^1_1 (on the real line) were introduced by Suslin [91] under the name *ensembles* (A). They were also known as analytic sets, Suslin sets, and A sets. Their prehistory can be traced back to the note [3] of Aleksandrov, and even to the memoir [47] of Lebesgue, where there is a construction (which was called later the *Lebesgue binary sieve*; see Example 1.10.1) implicitly providing an example of an non-Borel Π^1_1 set (for this topic, see [48].) The history of the discovery of the operation A and A sets has been a subject of special studies; the reader can find facts, comments, and further references concerning this topic, for example, in [95], [50].

The perfect kernel property for Σ^1_1 sets (Theorem 1.4) is usually associated in the literature with the names of Aleksandrov, Hausdorff, and Suslin (though there is no theorem of this kind in the corresponding papers [3], [23], [91]). The result first appeared in [51] with a reference to Suslin (without indicating a specific publication). (For more details, see [97], § 3.3.) The theorems concerning measurability and the Baire property for Σ^1_1 sets also first appeared in [51].

The notion of sieve was introduced by Luzin in [55]; however, it first appeared in the context of a proof in [63] (and in [61] in a somewhat different form, in connection with the operation A). The systematic application of this notion in the theory of Σ^1_1 and Π^1_1 sets, including Theorems 1.6 and 1.8, was given in *Leçons* [57]. Theorem 1.8 was proved in the classical works [5], [46], [57] (for a more modern exposition, see [35]). The assertion (iii) (Suslin's theorem) follows from (ii). The uniformization theorem, Theorem 1.9, actually claimed in its initial form [62] that any Π^1_1 set can be uniformized by a Σ^1_2 set. The improvement to Π^1_1 is due to Kondô [43], and the result for classes of the form $\mathit{\Pi}^1_1(a)$ to Addison [1], [2].

Projective sets were introduced by Luzin [52] as *Lebesgue projective sets*, though the reference to Lebesgue is not well grounded, as mentioned in [97], § 3.1.

The systematic use of analytic formulae to study projective point sets (instead of the geometric constructions typical for the style of Luzin and Novikov) was started by Addison [1], [2]. The encoding of Borel sets was used by Solovay [89] to solve the measurability problem for projective sets, but the roots of this encoding (and of the encoding of countable ordinals) can be traced back to earlier studies in recursion theory (see, for instance, [86], 7.9). The results in (v) and (vi) of Proposition 1.11 go back to work in the late 1960s concerning projective sets 'with large sections' (see the references in [41]).

The standard modern reference on classical descriptive set theory (that is, without forcing, constructibility, and the effective hierarchy) is Kechris' book [42]. Among earlier sources available in Russian, we note the survey paper [5], related chapters in the *Topology* [46] of Kuratowski, and also [35] and Chapter 8 in [11]. Problems of classical descriptive set theory are discussed in the surveys [38], [40], and [97].

§ 2. Gödel's constructibility

Classical descriptive set theory came to a halt at the questions of whether or not all Π^1_1 sets satisfy the perfect kernel property, whether or not all sets of the second projective level are measurable, and whether or not all of them have the Baire property. The subsequent study of these problems became possible only after the development of the appropriate methods of mathematical logic, such as *constructibility* and *forcing*. On the other hand, these tools were developed mainly as tools aimed at solving classical problems concerning point sets.

This section presents investigations of the aspects of constructibility that are essential in the study of regularity properties of projective sets.

2A. Set-theoretic universe. A specific feature of Gödel's constructibility theory is that its constructions and arguments, in their most general and natural form, do not confine themselves to the purely topological structure of point sets. On the contrary, they appeal to the structure of the set-theoretic universe as a whole and even to the relationships between different universes, for instance, those obtained by forcing.

Ordinals and ranks. *Ordinals*, or (finite and transfinite) ordinal numbers, play a special role in the structure of the set-theoretic universe. We assume that the reader is somewhat acquainted with these notions and recall only that in modern set theory every ordinal ξ is identified with the set of smaller ordinals, in other words, $\xi = \{\alpha : \alpha < \xi\}$. In particular, $0 = \varnothing$, $1 = \{0\}$, $2 = \{0,1\}$, $\dots$, $\omega = \mathbb{N} = \{0,1,2,\dots\}$, $\omega+1 = \omega \cup \{\omega\}$. The class of all ordinals is denoted by Ord. The *cardinals* are initial ordinals, that is, those not of cardinality equal to that of any smaller ordinal. For instance, $\omega = \mathbb{N} = \aleph_0$ is the smallest infinite ordinal, and $\omega_1 = \aleph_1$ is the smallest uncountable ordinal and the first uncountable cardinal.[17]

We recall that $\operatorname{rk} x \in \mathrm{Ord}$ denotes the *set-theoretic rank of* x, defined by $\operatorname{rk} x = \sup_{y\in x} \operatorname{rk} y$, where $\sup X$ stands for the smallest ordinal strictly bigger than all ordinals in X. (For instance, $\operatorname{rk}\varnothing = 0$.) This yields the *von Neumann hierarchy*, which consists of all sets of the form $\mathbf{V}_\xi = \{x : \operatorname{rk} x < \xi\}$, $\xi \in \mathrm{Ord}$; the universe $\mathbf{V}$ of all sets coincides with $\bigcup_{\xi\in\mathrm{Ord}} \mathbf{V}_\xi$.

Models of ZFC axioms. We assume that the reader is somewhat acquainted with the Zermelo–Fraenkel set theory **ZFC**, and we do not dwell on the presentation of its axioms (see [11], Chap. 1). As usual, by a *model* of **ZFC** one means any structure of the form $\langle \mathfrak{M}; \varepsilon\rangle$, where $\mathfrak{M}$ is any set equipped with a binary relation ε such that the axioms of **ZFC** hold in the structure (provided that the membership sign is interpreted as ε). One distinguishes *standard transitive* models, in which case $\mathfrak{M}$ is a *transitive* set (that is, $x \in y \in \mathfrak{M}$ implies that $x \in \mathfrak{M}$) and ε is simply the restriction of the 'true' membership $\in$ to $\mathfrak{M}$.

It follows from the Gödel incompleteness theorem that the existence of at least one model of **ZFC** of any kind cannot be proved in **ZFC**. On the other hand, there exist structures of the form $\langle \mathfrak{M}; \varepsilon\rangle$ satisfying all axioms of **ZFC** for which $\mathfrak{M}$ is a true class rather than a set. The most elementary example is given by $\langle \mathbf{V}; \in\rangle$, where $\mathbf{V}$ is the universe of all sets. Such structures are not models in the rigorous sense,[18] because their domains are not sets. Therefore, Gödel's theorem cannot be applied to these structures. However, they are quite useful in many applications. In particular, the existence of a model of **ZFC** (even in this improper sense) in which a proposition Φ holds means that Φ does not contradict the axioms of **ZFC**.

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[17]It is customary to use the notation $\aleph_0$, $\aleph_1$, or in general $\aleph_\xi$, $\xi \in \mathrm{Ord}$, if only cardinal characteristics are important, and the notation ω, ω_1, or in general ω_ξ, if ordinal characteristics of the cardinal and its connection with other ordinals are important.

[18]Sometimes they are informally called *models*, for instance, the class $\mathbf{L}$ is called the *constructible model*. The notion of *interpretation* is a more exact term for structures whose domains are true classes. See [86], 4.7 and 9.5 for details; in the present case we mean an interpretation of **ZFC** theory in itself.

In particular, this category of structures contains the class $\mathbf{L}$ of all constructible sets and its extensions obtained by forcing.

Hereditarily countable sets. It is sometimes useful to consider models of weaker theories, in particular, of $\mathbf{ZFC}^-$, that is, $\mathbf{ZFC}$ without the power set axiom. A natural model of $\mathbf{ZFC}^-$ is the set $\mathbf{HC}$ of all *hereditarily countable sets*, that is, of all sets x such that the *transitive closure* $\mathrm{TC}(x)$ is at most countable. ($\mathrm{TC}(x)$ is the smallest transitive set containing x.) In particular, all natural numbers and sets of natural numbers, all elements and countable subsets of $\mathbb{N}^\omega$, and all countable ordinals belong to $\mathbf{HC}$. On the other hand, it is clear that $\mathcal{P}(\mathbb{N}) \notin \mathbf{HC}$, and hence the power set axiom fails in $\mathbf{HC}$. It is an easy exercise to show that the other axioms of $\mathbf{ZFC}$ hold in $\mathbf{HC}$ (or, more precisely, in the structure $\langle \mathbf{HC}; \in \rangle$). The Skolem–Löwenheim theorem yields a variety of *countable* models of $\mathbf{ZFC}^-$.

Proposition 2.1. *If $X \subseteq \mathbf{HC}$ is a countable set, then there is a countable transitive set $M \in \mathbf{HC}$ such that $X \subseteq M$ and M is a model of* $\mathbf{ZFC}^-$.

2B. General notion of Gödel's constructibility. Among all possible sets, one can obviously distinguish sets that admit a simple definition or an explicit construction by means of simple operations; let us say that these sets are 'constructible'. This is an intuitive notion; it depends both on the choice of admissible tools of definition or construction and on the choice of the initial sets to which the definitions or constructions can be applied. Fortunately, it turns out that all reasonable versions of constructibility can be reduced to a very few really different concepts. The definition of *Gödel constructibility* is the most known and most useful.

The definition of constructible sets is known in several versions giving, however, the same result. The most convenient version for our purposes is that of [20], which is directly based on the eight *Gödel operations* (presented here in the form given in [29]):

$$\begin{gathered}
\mathfrak{F}_1(X,Y) = \{X,Y\}, \quad \mathfrak{F}_2(X,Y) = X \smallsetminus Y, \quad \mathfrak{F}_3(X,Y) = X \times Y, \\
\mathfrak{F}_4(X,Y) = \{x : \exists\, y \ \ (\langle x,y\rangle \in X)\}, \quad \mathfrak{F}_5(X,Y) = \{\langle x,y\rangle \in X : x \in y\}, \\
\mathfrak{F}_6(X,Y) = \{\langle x,y,z\rangle : \langle y,z,x\rangle \in X\}, \quad \mathfrak{F}_7(X,Y) = \{\langle x,y,z\rangle : \langle z,y,x\rangle \in X\}, \\
\mathfrak{F}_8(X,Y) = \{\langle x,y,z\rangle : \langle x,z,y\rangle \in X\}.
\end{gathered}$$

Definition 2.2 (constructible sets). We define a well ordering $\lessdot$ of the class $T = \{0,1,\dots,8\} \times \mathrm{Ord} \times \mathrm{Ord}$ as follows: $\langle i,\xi,\eta\rangle \lessdot \langle i',\xi',\eta'\rangle$ if $\max\{\xi,\eta\} < \max\{\xi',\eta'\}$, if $\max\{\xi,\eta\} = \max\{\xi',\eta'\}$ and $\langle \xi,\eta\rangle < \langle \xi',\eta'\rangle$ lexicographically, or if $\xi = \xi'$, $\eta = \eta'$, and $i < i'$. There is a unique order isomorphism $K \colon \langle T; \lessdot\rangle \xrightarrow{\text{onto}} \mathrm{Ord} \smallsetminus \{0\}$. Let K_0, K_1, K_2 be the inverse functions, that is, if $K(i,\xi,\eta) = \gamma$, then $K_0(\gamma) = i$, $K_1(\gamma) = \xi$, and $K_2(\gamma) = \eta$.

It is easy to see that for $K_0(\gamma) > 0$ one has $K_1(\gamma) < \gamma$ and $K_2(\gamma) < \gamma$. This enables one to define a sequence of sets $\mathbf{F}_\gamma$ by induction on $\gamma \in \mathrm{Ord}$ by means of the following scheme:

$$\mathbf{F}_\gamma = \begin{cases} \varnothing & \text{for } \gamma = 0; \\ \mathfrak{F}_{K_0(\gamma)}(\mathbf{F}_{K_1(\gamma)}, \mathbf{F}_{K_2(\gamma)}) & \text{for } \gamma > 0 \text{ and } K_0(\gamma) > 0; \\ \{\mathbf{F}_\nu : \nu < \gamma\} & \text{for } \gamma > 0 \text{ and } K_0(\gamma) = 0. \end{cases}$$

Finally, $\mathbf{L} = \{\mathbf{F}_\gamma : \gamma \in \mathrm{Ord}\}$ is *the class of all constructible sets.*

Definition 2.3 (relative constructibility). For any set $X \subseteq \mathbf{L}$, let $\vartheta = \vartheta(X)$ be the smallest ordinal such that $X \subseteq \{\mathbf{F}_\xi : \xi < \vartheta\}$. (For example, if $X = a \in \mathbb{N}^\omega$, then we always have $\vartheta(a) = \omega$.) Let us redefine K in Definition 2.2 to be the unique order isomorphism $K\colon \langle T; \lessdot\rangle \xrightarrow{\text{onto}} \mathrm{Ord} \smallsetminus \{0, \vartheta\}$. Thus, the inverse functions K_0, K_1, K_2 are now defined on the set $\mathrm{Ord} \smallsetminus \{0, \vartheta\}$. Let us define $\mathbf{F}_\xi[X]$ by induction on $\xi \in \mathrm{Ord}$ in accordance with the scheme of Definition 2.2 but with the additional special condition $\mathbf{F}_\vartheta[X] = X$. (We note that $\mathbf{F}_\xi[X] = \mathbf{F}_\xi$ for $\xi < \vartheta$, and hence $X = \mathbf{F}_\vartheta[X] \subseteq \{\mathbf{F}_\xi[X] : \xi < \vartheta\}$.)

The family $\mathbf{L}[X] = \{\mathbf{F}_\gamma[X] : \gamma \in \mathrm{Ord}\}$ is *the class of all sets constructible relative to* X. If $x, y \in \mathbf{L}[X]$, then $x < [X]\ y$ means that x occurs in the sequence of sets $\mathbf{F}_\gamma[X]$ earlier than y.

A generalization. Suppose that $\mathbf{s} = \{X_\xi\}_{\xi<\zeta}$ is a finite or transfinite sequence of sets $X_\xi \subseteq \mathbf{L}$ ($\zeta \in \mathrm{Ord}$). In general we have $\mathbf{s} \not\subseteq \mathbf{L}$, but we can define $\mathbf{L}[\mathbf{s}] = \mathbf{L}[X]$, where $X = \{\langle \xi, x\rangle : x \in X_\xi\} \subseteq \mathbf{L}$. In particular, this definition works for classes of the form $\mathbf{L}[a_1, \dots, a_n]$, where $a_1, \dots, a_n \subseteq \mathbf{L}$, for example, for $a_1, \dots, a_n \in \mathbb{N}^\omega$.

We note that the class $\mathbf{L}$ coincides with $\mathbf{L}[X]$ for any $X \in \mathbf{L}$. The next theorem was obtained in [20] in this special case, but the general result can be proved similarly.

Theorem 2.4. *Suppose that $X \subseteq \mathbf{L}$. The class $\mathbf{L}[X]$ satisfies all axioms of* **ZFC**. *It contains X and all ordinals, and it is the smallest class with this property satisfying all axioms of* **ZFC**.

Moreover, $< [X]$ is a well ordering of $\mathbf{L}[X]$ order isomorphic to Ord.

Gödel also proved that the axiom of choice **AC** and the generalized continuum hypothesis hold in $\mathbf{L}$ (in fact, in any class of the form $\mathbf{L}[a]$, $a \in \mathbb{N}^\omega$, but the continuum hypothesis is not necessarily true in $\mathbf{L}[X]$ for an arbitrary X).

The following simple result will be used below.

Proposition 2.5. *Let $\mathfrak{M}$ be a transitive set closed under the Gödel operations and let $a \in \mathfrak{M} \cap \mathbb{N}^\omega$. Suppose in addition that the restricted sequence $\{\mathbf{F}_\xi[a]\}_{\xi<\gamma}$ belongs to $\mathfrak{M}$ for any ordinal $\gamma \in \mathfrak{M}$. Then the construction of $\{\mathbf{F}_\xi[a]\}_{\xi \in \mathrm{Ord}\cap\mathfrak{M}}$ is absolute for $\mathfrak{M}$, that is, it gives the same result in $\mathfrak{M}$ and in the universe of all sets.*

2C. Constructibility in the domain of the Baire space. Suppose that $a \in \mathbb{N}^\omega$. For any $\xi \in \mathrm{Ord}$ let $\mathbf{f}_\xi[a] = \mathbf{F}_\xi[a]$ if $\mathbf{F}_\xi[a] \in \mathbb{N}^\omega$ and let $\mathbf{f}_\xi[a] = \mathbf{O}$ (with the function $\mathbf{O}(n) = 0\ \forall\, n$) otherwise. Thus, $\mathbf{f}_\xi[a]$ always belongs to $\mathbb{N}^\omega$.

Let $\omega_1^{\mathbf{L}[a]}$ denote the first uncountable cardinal in $\mathbf{L}[a]$.

Let $\prec_a$ denote the order $< [a]$ restricted to $\mathbf{L}[a]$.

Theorem 2.6. *The following assertions hold.*

(i) *If $a \in \mathbb{N}^\omega$, then $\mathbf{L}[a] = \{\mathbf{f}_\xi[a] : \xi < \omega_1\} = \{\mathbf{f}_\xi[a] : \xi < \omega_1^{\mathbf{L}[a]}\}$, and the relation $\prec_a$ well orders $\mathbf{L}[a]$ with the order type $\omega_1^{\mathbf{L}[a]}$.*

(ii) *One can find a Σ^1_1 formula $\boldsymbol{\varphi}$ and a Π^1_1 formula $\boldsymbol{\psi}$ such that $x = \mathbf{f}_{|w|}[a] \iff \boldsymbol{\varphi}(w, a, x) \iff \boldsymbol{\psi}(w, a, x)$ for any $w \in \mathbf{WO}$ and a, $x \in \mathbb{N}^\omega$.*

(iii) *The set $\mathbf{L}[a]$ and the relation $\prec_a$ belong to $\Sigma^1_2(a)$.*

(iv) *If a, $p \in \mathbb{N}^\omega$ and $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ is a $\varDelta^1_2(p)$ set, if*

$$X = \{x \in \mathbb{N}^\omega : \text{the set } P_x(a) = \{y \in \mathbf{L}[a] : \langle x, y\rangle \in P\} \text{ is non-empty}\},$$

and if y_x is the $\prec_a$-smallest element of P_x for any $x \in X$, then both X and $P' = \{\langle x, y_x\rangle : x \in X\}$ belong to $\varSigma^1_2(a,p)$. If, in addition, X is a $\varDelta^1_2(a,p)$ set, then P' is a $\varDelta^1_2(a,p)$ set as well.

Proof (a sketch). (i) Any point $x \in \mathbb{N}^\omega$ in $\mathbf{L}[a]$ occurs in the construction of $\mathbf{L}[a]$ at some step before the step ω_1. This is a crucial point in Gödel's proof in [20] of the continuum hypothesis in the constructible universe $\mathbf{L}$ based on the Skolem–Löwenheim method. (The generalization to $\mathbf{L}[a]$, $a \in \mathbb{N}^\omega$, involves no new problems.) This implies both the equality $\mathbf{L}[a] = \{\mathbf{f}_\xi[a] : \xi < \omega_1\}$ and the fact that the length of $\prec_a$ does not exceed ω_1. Why can ω_1 be replaced by $\omega_1^{\mathbf{L}[a]}$? The point is that the inductive construction of the sets $\mathbf{F}_\xi[a]$ is *absolute for* $\mathbf{L}[a]$, that is, it can be carried out in $\mathbf{L}[a]$ with the same final result. However, any point of $\mathbb{N}^\omega$ occurs in $\mathbf{L}[a]$ before the step $\omega_1^{\mathbf{L}[a]}$.

(ii) The idea of the proof is to encode the entire Gödel construction up to the step ω_1 (where obviously the sets appearing are at most countable) by means of points of $\mathbb{N}^\omega$. In principle this is similar to the encoding of countable ordinals and Borel sets introduced in subsection 1E, but the technical details are much more complicated. The argument was carried out in full detail by Novikov [78] and Addison [1], [2] (in somewhat different versions and under the assumption that $\mathbf{V} = \mathbf{L}$). We present a sketch of another proof to give the reader the possibility of seeing the mechanism of arguments of this kind without referring to rather difficult original papers.

We consider structures of the form $\langle \mathbb{N}; \varepsilon\rangle$, where $\varepsilon \subseteq \mathbb{N}^2$ is a binary relation viewed as the 'membership' relation in $\langle \mathbb{N}; \varepsilon\rangle$. Let E be the set of all relations $\varepsilon \subseteq \mathbb{N}^2$ satisfying the following three conditions:

(a) if $i, j \in \mathbb{N}$ and $\forall\, n\ (n\varepsilon i \iff n\varepsilon j)$, then $i = j$ (*extensionality*);
(b) all axioms of $\mathbf{ZFC}^-$ ($\mathbf{ZFC}$ without the power set axiom; see subsection 2A) hold in $\langle \mathbb{N}; \varepsilon\rangle$;
(c) the family $\mathrm{Ord}^\varepsilon = \{n \in \mathbb{N} : \text{it is true in } \langle \mathbb{N}; \varepsilon\rangle \text{ that } n \text{ is an ordinal}\}$ of all ordinals of the model $\langle \mathbb{N}; \varepsilon\rangle$ has an initial segment order isomorphic to the ordinal $\omega + 1$ (in other words, $\langle \mathbb{N}; \varepsilon\rangle$ is an ω-*model*).

In this case there exists for any $n \in \mathbb{N}$ a unique element $n^\varepsilon \in \mathbb{N}$ corresponding to n in $\langle \mathbb{N}; \varepsilon\rangle$, and there is a unique element $\omega^\varepsilon \in \mathbb{N}$ corresponding to ω in $\langle \mathbb{N}; \varepsilon\rangle$. Moreover, $\{n^\varepsilon : n \in \mathbb{N}\}$ is the set of all ε-elements of ω^ε.

Suppose that $x \in \mathbb{N}^\omega$. It can happen that there is an element $k \in \mathbb{N}$ (which is unique if it exists) for which it is true in $\langle \mathbb{N}; \varepsilon\rangle$ that $k \in \mathbb{N}^\omega$ and $k(n^\varepsilon) = m^\varepsilon$ whenever $x(n) = m$. Such an element k will be denoted by x^ε. Obviously, x^ε exists for at most countably many points $x \in \mathbb{N}^\omega$.

To prove (ii), we take the formula $\boldsymbol{\varphi}(w, a, x)$ to be

$$\exists\, \varepsilon \in E\ \ (a^\varepsilon,\ w^\varepsilon,\ x^\varepsilon \text{ do exist in } \langle \mathbb{N}; \varepsilon\rangle\ \wedge\ x^\varepsilon = \mathbf{f}_{|w^\varepsilon|}[a^\varepsilon] \text{ is true in } \langle \mathbb{N}; \varepsilon\rangle).$$

We note that each of the two conjunctive terms of the formula in parentheses can be expressed by an arithmetic formula, and thus the displayed formula is in fact a Σ^1_1 formula. (We do not dwell on the fact, say, that ε should be appropriately encoded by a point of $\mathbb{N}^\omega$, because there are no problems here.)

It remains to show that the formula $\boldsymbol{\varphi}$ thus defined satisfies (ii).

Suppose that $w \in \mathbf{WO}$, $a, x \in \mathbb{N}^\omega$, and $\boldsymbol{\varphi}(w,a,x)$ holds, that is, one can find an $\varepsilon \in E$ such that the objects a^ε, w^ε, x^ε do exist in $\langle \mathbb{N}; \varepsilon\rangle$ and $x^\varepsilon = \mathbf{f}_{|w^\varepsilon|}[a^\varepsilon]$ is true in $\langle \mathbb{N}; \varepsilon\rangle$. Then the ordinal series $\mathrm{Ord}^{\langle \mathbb{N}; \varepsilon\rangle}$ has an initial segment order isomorphic to $\xi = |w|$, and hence it has an initial segment order isomorphic to $\xi + \omega + 1$ (and also initial segments order isomorphic to $\xi + \xi$, $\xi \times \xi$, and so on). Let us show that, for this reason, the model $\langle \mathbb{N}; \varepsilon\rangle$ contains an essential part of the $\in$-structure of ordinary sets.

We recall that $\operatorname{rk} x \in \mathrm{Ord}$ is a set-theoretic rank (see subsection 2A). Let M be the set of all elements m such that $\operatorname{rk} x < |w^\varepsilon| + \omega^\varepsilon$ is true in $\langle \mathbb{N}; \varepsilon\rangle$. One can readily see that every $m \in M$ belongs to the *well-founded part* $\mathbf{WF}\varepsilon$ of the model $\langle \mathbb{N}; \varepsilon\rangle$; this part consists of all m such that the ε-transitive closure $\mathrm{TC}_\varepsilon(m)$ does not contain infinite ε-decreasing chains. ($\mathrm{TC}_\varepsilon(m)$ is the smallest set $T \subseteq \mathbb{N}$ for which $m \in T$ and $i \varepsilon j \in T \Longrightarrow i \in T$.) Therefore, we can define a true set $S(m)$ for any $m \in M$ according to the equality $S(m) = \{S(n) : n \varepsilon m\}$.

Assertion 2.7. *The set M contains a^ε, w^ε, and x^ε. The set $\mathbf{S} = \{S(n) : n \in M\}$ is transitive. The map $m \mapsto S(m)$ is an isomorphism of $\langle M; \varepsilon\rangle$ onto $\langle \mathbf{S}; \in\rangle$, with $S(w^\varepsilon) = w$, $S(x^\varepsilon) = x$, and $S(a^\varepsilon) = a$.*

We note that from the point of view of the model $\langle \mathbb{N}; \varepsilon\rangle$ the structure $\langle M; \varepsilon\rangle$ is a set of the form $\{z : \operatorname{rk} z < \xi + \omega\}$, where $\xi = |w|$, and every set of this kind obviously satisfies the conditions on $\mathfrak{M}$ in Proposition 2.5. It follows that the formula $x^\varepsilon = \mathbf{f}_{|w^\varepsilon|}[a^\varepsilon]$ is true in $\langle M; \varepsilon\rangle$, because it is true in $\langle \mathbb{N}; \varepsilon\rangle$ by the above assumption. Therefore, $x = \mathbf{f}_{|w|}[a]$ holds in $\langle \mathbf{S}; \in\rangle$ by Assertion 2.7. By Proposition 2.5 again, the formula $x = \mathbf{f}_{|w|}[a]$ is true in the universe of all sets as well. Thus, $\boldsymbol{\varphi}(w,a,x)$ implies that $x = \mathbf{f}_{|w|}[a]$.

To prove the converse, suppose that $x = \mathbf{f}_{|w|}[a]$. It follows from Proposition 2.1 that there is a countable transitive set $\mathfrak{M} \subseteq \mathbf{HC}$ which contains w, a, x and is a model of $\mathbf{ZFC}^-$. Then the formula $x = \mathbf{f}_{|w|}[a]$ is true in $\langle \mathfrak{M}; \in\rangle$. Let us consider a relation $\varepsilon \subseteq \mathbb{N}^2$ such that the model $\langle \mathbb{N}; \varepsilon\rangle$ is isomorphic to $\langle \mathfrak{M}; \in\rangle$. Clearly, $\varepsilon \in E$, and the isomorphism (which is uniquely defined) sends w, a, and x to the elements w^ε, a^ε, and x^ε, respectively, and these elements really exist in $\langle \mathbb{N}; \varepsilon\rangle$. It follows that the formula $x^\varepsilon = \mathbf{f}_{|w^\varepsilon|}[a^\varepsilon]$ is true in $\langle \mathbb{N}; \varepsilon\rangle$. This means that $\boldsymbol{\varphi}(w,a,x)$ holds.

The following formula can be taken as the desired Π^1_1 formula $\boldsymbol{\psi}$:

$$\forall \varepsilon \in E \ \ (a^\varepsilon,\, w^\varepsilon,\, x^\varepsilon \text{ exist in } \langle \mathbb{N}; \varepsilon\rangle \implies x^\varepsilon = \mathbf{f}_{|w^\varepsilon|}[a^\varepsilon] \text{ is true in } \langle \mathbb{N}; \varepsilon\rangle).$$

(iii) It follows from (i) that $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega \Longleftrightarrow \exists\, w\ (w \in \mathbf{WO} \wedge x = \mathbf{f}_{|w|}[a])$. We replace the equality $x = \mathbf{f}_{|w|}[a]$ by a Σ^1_1 formula $\boldsymbol{\varphi}(w,a,x)$ in (ii) and use the fact that $\mathbf{WO} \in \Pi^1_1$ (Proposition 1.11). Concerning the order $\prec_a$, we note that $x \prec_a y$ is equivalent by definition to

$$\exists\, \eta < \omega_1 \ \ \Big(y = \mathbf{f}_\eta[a] \wedge \underbrace{\forall\, \xi < \eta \ \ (y \neq \mathbf{f}_\xi[a])}_{\vartheta(a,y,\eta)} \wedge \exists\, \xi < \eta \ \ (x = \mathbf{f}_\xi[a])\Big). \qquad (*)$$

Moreover,

$$\vartheta(a, y, |w|) \iff \forall\, k \quad (y \neq \mathbf{f}_{w \restriction k}[a])$$

for any $w \in \mathbf{WO}$ (see Example 1.10.1). This enables us to reduce the formula $(*)$ to the desired Σ^1_2 form (with the arguments x, y, a) by using (ii) and Proposition 1.11. We note that the key point of the argument is the transformation of the quantifier $\forall\, \xi < \eta$ in the subformula ϑ to a quantifier over $\mathbb{N}$, which does not affect the class of the formula.

(iv) First of all, $x \in X \iff \exists\, y \quad (y \in \mathbf{L}[a] \wedge \langle x, y\rangle \in P)$. This implies that $X \in \Sigma^1_2(a, p)$ by the first claim of (iii). We note further that $\langle x, y\rangle \in P'$ is equivalent to

$$\langle x, y\rangle \in P \ \wedge\ \exists\, \eta < \omega_1 \ \Big(y = \mathbf{f}_\eta[a] \wedge \underbrace{\forall\, \xi < \eta\ \exists\, z\ (z = \mathbf{f}_\xi[a] \wedge \langle x, z\rangle \notin P)}_{\tau(a,x,\eta)}\Big),$$

where the formula $\tau(a, x, \eta)$ expresses the fact that $\langle x, \mathbf{f}_\xi[a]\rangle \notin P$ for every $\xi < \eta$. The reduction of the displayed formula to the Σ^1_2 form can be carried out by the same arguments as above. Finally, $\langle x, y\rangle \in P' \iff x \in X \wedge \forall\, z \neq y\ (\langle x, z\rangle \notin P')$. This implies the last claim of (iv). □

2D. Absoluteness. This notion is not directly related to constructibility. However, it becomes necessary after we learn that the universe $\mathbf{V}$ of all sets has subclasses, for example, those of the form $\mathbf{L}[X]$, which themselves satisfy the axioms of **ZFC** and hence can be regarded as valuable set-theoretic universes.

Suppose that $\mathfrak{M}$ is a transitive set or class (for example, a class of the form $\mathbf{L}[X]$) satisfying all axioms of **ZFC** (or at least a substantial part of **ZFC** like $\mathbf{ZFC}^-$). Let Φ be a closed formula with sets in $\mathfrak{M}$ as parameters. Then one can pose the question of whether or not the formula Φ is true or false in $\mathfrak{M}$ (rather than just in the universe $\mathbf{V}$), which means that the formula obtained by the *relativization* of Φ to $\mathfrak{M}$ (that is, by replacing all quantifiers $\exists\, x$ and $\forall\, x$ by $\exists\, x \in \mathfrak{M}$ and $\forall\, x \in \mathfrak{M}$) is true or false (in $\mathbf{V}$). If Φ is simultaneously true or simultaneously false both in $\mathfrak{M}$ and in the universe $\mathbf{V}$ of all sets, then Φ is said to be *absolute for* $\mathfrak{M}$.

In fact, we have already encountered the notion of absoluteness in a simple situation when the absoluteness of formulae or of the constructions defined by them was quite clear. (See, for instance, Proposition 2.5 and its use in the proof of Theorem 2.6(ii).) The following theorem is of importance because it enables one to establish the absoluteness of a formula by using only its syntactical structure with no regard to its mathematical content.

Theorem 2.8 (the absoluteness theorem). *Suppose that $\mathfrak{M}$ is a transitive set or class in which all $\mathbf{ZFC}^-$ axioms hold. In this case*

- (i) *every Σ^1_1 formula with parameters in $\mathfrak{M} \cap \mathbb{N}^\omega$ is absolute for $\mathfrak{M}$;*
- (ii) *if $\omega_1 \subseteq \mathfrak{M}$, then the same assertion holds for all Σ^1_2 formulae with parameters in $\mathfrak{M} \cap \mathbb{N}^\omega$.*

Proof (a sketch). The validity of Proposition (i) (Mostowski's theorem) uses the fact that by the sifting theorem (Theorem 1.6) the formula $\neg\Phi$ is equivalent to the assertion that a certain set $Q \in \mathfrak{M}$, $Q \subseteq \mathbb{Q}$, is well ordered, where Q depends on Φ but not on $\mathfrak{M}$. Proposition (ii) (the Shoenfield absoluteness theorem) was first proved in [85]. The proof is somewhat more complicated than the proof of (i) and can also be found in [35], p. 305. Both assertions are also given in [86], Exercise 12 in Chapter 9, as exercises with hints. $\square$

Remark 2.9. The assertion (i) of Theorem 2.8 can also be extended to Π^1_1 formulae, and (ii) to Π^1_2 formulae. Moreover, *one-way* absoluteness can be extended to the next level; for example, if $\omega_1 \subseteq \mathfrak{M}$, then any Σ^1_3 formula with parameters in $\mathfrak{M} \cap \mathbb{N}^\omega$ which is true in $\mathfrak{M}$ is also true in the universe $\mathbf{V}$ of all sets, whereas the converse assertion holds for Π^1_3 formulae. $\square$

Proposition 1.11 leads to numerous applications of the absoluteness theorem (and the same can be said for some other analogous definability results). For example, it follows from 1.11(i) that $(\mathbf{BC})^{\mathfrak{M}} = \mathbf{BC} \cap \mathfrak{M}$, that is, any $c \in \mathbb{N}^\omega \cap \mathfrak{M}$ belongs to $\mathbf{BC}$ if and only if it is true in $\mathfrak{M}$ that $c \in \mathbf{BC}$. It follows from 1.11(iv) that, for any $c \in \mathbf{BC} \cap \mathfrak{M}$, the set $(\mathbf{B}_c)^{\mathfrak{M}}$ (that is, $\mathbf{B}_c$ defined in $\mathfrak{M}$) coincides with $\mathbf{B}_c \cap \mathfrak{M}$. Further, if $\omega_1 \subseteq \mathfrak{M}$ and $p, q \in \mathbf{BC} \cap \mathfrak{M}$, then it follows again from 1.11(iv) that $\mathbf{B}_p \subseteq \mathbf{B}_q$ is equivalent to $\mathbf{B}_p \cap \mathfrak{M} \subseteq \mathbf{B}_q \cap \mathfrak{M}$, and $\mathbf{B}_p = \mathbf{B}_q$ is equivalent to $\mathbf{B}_p \cap \mathfrak{M} = \mathbf{B}_q \cap \mathfrak{M}$.

Theorem 2.8 also enables us to derive 'effective' modifications of classical theorems of descriptive set theory whose direct proofs would require tedious work with details. The following result is an example.

Theorem 2.10 (effective Suslin theorem). *If $a \in \mathbb{N}^\omega$ and if $X \subseteq \mathbb{N}^\omega$ is a $\Delta^1_1(a)$ set, then there is a code $c \in \mathbf{BC} \cap \mathbf{L}[a]$ such that $X = \mathbf{B}_c$.*

Proof. Let us consider a Σ^1_1 formula φ and a Π^1_1 formula ψ such that $X = \{x : \varphi(a,x)\} = \{x : \psi(a,x)\}$. The equivalence $\forall\, x\ (\varphi(a,x) \iff \psi(a,x))$ can be expressed by a Π^1_2 formula with a as the only parameter, and hence this formula holds in $\mathbf{L}[a]$ as well. It follows that the equivalent formulae φ and ψ define a Δ^1_1 set in $\mathbf{L}[a]$. Thus, by the classical Suslin theorem (see Theorem 1.8), there is a code $c \in \mathbf{L}[a] \cap \mathbb{N}^\omega$ such that both formulae $c \in \mathbf{BC}$ and $\forall\, x\ (x \in \mathbf{B}_c \iff \varphi(a,x) \iff \psi(a,x))$ are true in $\mathbf{L}[a]$. However, the latter formula is also of class Π^1_2 with a and c as parameters (Proposition 1.11(iv)), and therefore $c \in \mathbf{BC}$ and $X = \mathbf{B}_c$ in the universe $\mathbf{V}$ of all sets (again by Theorem 2.8). $\square$

Historical and bibliographical remarks. For the studies which led to the picture of the set-theoretic universe briefly described in subsection 2a and to **ZFC** theory, see, for instance, [19].

The absoluteness Theorem 2.8(ii) was proved by Shoenfield [85].

The class $\mathbf{L}$ of all constructible sets was defined by Gödel [20]. Specific problems of constructibility in the area of the Baire space, including the descriptive definability of the Gödel sequence restricted to the countable ordinals and the associated well ordering of the continuum, were considered by Novikov [78] and later by Addison [1], [2]. Relative constructibility (that is, classes of the form $\mathbf{L}[x]$) attracted attention already in the 1960s, in particular in connection with the development of forcing. Some references on constructibility, including applications to descriptive set theory, are given in [29], [35], and [11], Chap. 5.

A systematic survey of properties of the important set of all constructible points (that is, of the set $\mathbf{L} \cap \mathbb{N}^\omega$) will be presented in a forthcoming paper of the authors, to appear in a volume of *Proceedings of the Steklov Institute of Mathematics* dedicated to L. V. Keldysh.

§ 3. Resolvents of classical problems: Part 1

It turns out that the perfect kernel problem for Π^1_1 sets and the problems of measurability and the Baire property for sets of the second projective level are closely related to special properties of some particular sets, in the sense close to that meant by Luzin when he used the term *resolvents*.

Suppose that $\mathrm{P}(\mathbf{K})$ is the existence problem for a set satisfying some property (say, a non-measurable set or a set without the Baire property) in a given projective class $\mathbf{K}$. Then, as a rule, by using the language of analytic formulae it is not difficult to find a projective set X such that the problem $\mathrm{P}(\mathbf{K})$ admits a positive solution if and only if X is non-empty. Luzin [54] referred to sets X of this kind as resolvents (*résolvantes*) of the original problems. In a somewhat wider sense, a *resolvent* of a problem $\mathrm{P}(\mathbf{K})$ is understood to be a property $\mathbf{p}$ (of a certain set X) equivalent to a positive solution of the problem. (In Luzin's sense, $\mathbf{p}$ is the non-emptiness of X.)

In this section we begin an exposition of investigations on resolvents of classical problems on the regularity properties represented in the diagram in the Introduction; these investigations both revealed the essence of the problems and established relationships among them.

3A. Cohen points and random points. The resolvents used in the main results (Theorem 3.3 and Corollary 3.4 below) are formulated in terms of properties of sets of the form $\mathbf{L}[a] \cap \mathbb{N}^\omega$, $a \in \mathbb{N}^\omega$, and of sets introduced by the following definition.

Definition 3.1. Let $\mathfrak{M}$ be a transitive model of **ZFC**, for instance, a class of the form $\mathbf{L}[X]$. In this case we say that an $x \in \mathbb{N}^\omega$ is a *Cohen point over* $\mathfrak{M}$ and write $x \in \mathbf{Coh}\,\mathfrak{M}$ if $x \notin \mathbf{B}_c$ whenever $c \in \mathfrak{M} \cap \mathbf{BC}$ and $\mathbf{B}_c$ is a meagre subset of $\mathbb{N}^\omega$.

Suppose in addition that μ is a Borel measure on $\mathbb{N}^\omega$. We say that a point $x \in \mathbb{N}^\omega$ is *μ-random over* $\mathfrak{M}$ and write $x \in \mathbf{Rand}_\mu\,\mathfrak{M}$ if $x \notin \mathbf{B}_c$ whenever $c \in \mathfrak{M} \cap \mathbf{BC}$ and $\mu(\mathbf{B}_c) = 0$. We set $\mathbf{Rand}\,\mathfrak{M} = \mathbf{Rand}_{\boldsymbol{\lambda}}\,\mathfrak{M}$.[19] □

Thus, the Cohen points (over $\mathfrak{M}$) are the points avoiding any meagre Borel set with a code in $\mathfrak{M}$, and the random points are the points avoiding any zero-measure set. The Cohen points and the random points admit a convenient characterization in terms of forcing; see subsection 4C.

We note that $\mathbf{Rand}\,\mathfrak{M} \subseteq 2^\omega$, because $\boldsymbol{\lambda}(\mathbb{N}^\omega \smallsetminus 2^\omega) = 0$.

The following lemma will be useful below. Its assertions (1), (2) show that, due to a certain uniformity of measure and category, the non-emptiness of the sets $\mathbf{Coh}\,\mathfrak{M}$ and $\mathbf{Rand}\,\mathfrak{M}$ implies their *denseness* in a rather strong sense.

Lemma 3.2. *Let* $\mathfrak{M}$ *be a transitive model of* **ZFC**. *In this case*

(1) *if* $\mathbf{Rand}\,\mathfrak{M} \neq \varnothing$, *then any Borel set of non-zero* $\boldsymbol{\lambda}$*-measure with a code in* $\mathfrak{M}$ *intersects* $\mathbf{Rand}\,\mathfrak{M}$;

[19] We recall that the measure $\boldsymbol{\lambda}$ on 2^ω was introduced in subsection 1B, and the encoding of Borel sets and the definitions of $\mathbf{BC}$ and $\mathbf{B}_c$ in subsection 1E. The meagre sets are the first-category sets, and the co-meagre sets are their complements.

(2) *if* $\mathbf{Coh}\,\mathfrak{M} \neq \varnothing$, *then any non-meagre Borel set with a code in* $\mathfrak{M}$ *intersects* $\mathbf{Coh}\,\mathfrak{M}$;

(3) *the conditions* $\mathbf{Rand}\,\mathfrak{M} \neq \varnothing$ *and* $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathfrak{M}) = 1$ *do not depend on the choice of the measure*, *in the sense that if* μ *is a finite Borel measure on* $\mathbb{N}^\omega$ *and its code* $\mathrm{Cod}(\mu)$ (that is, the function $\mathrm{Cod}(\mu)(n) = \mu(\mathcal{N}_{s_n})$) *belongs to* $\mathfrak{M}$, *then*

$$\mathbf{Rand}\,\mathfrak{M} \neq \varnothing \iff \mathbf{Rand}_\mu\,\mathfrak{M} \neq \varnothing \quad \textit{and}$$
$$\boldsymbol{\lambda}(\mathbf{Rand}\,\mathfrak{M}) = 1 \iff \mu(\mathbf{Rand}_\mu\,\mathfrak{M}) = \mu(\mathbb{N}^\omega).$$

Proof. (1) In the case of a measure suppose that $c \in \mathbf{BC} \cap \mathfrak{M}$ and $m = \boldsymbol{\lambda}(\mathbf{B}_c) > 0$. The arguments in the proof of Lemma 1.3 yield Borel sets $X \subseteq 2^\omega$ and $Y \subseteq \mathbf{B}_c$ with $\boldsymbol{\lambda}(X) = 1$ and $\boldsymbol{\lambda}(Y) = m$ and a Borel isomorphism $F\colon X \xrightarrow{\text{onto}} Y$ preserving $\boldsymbol{\lambda}$ with the coefficient $\varkappa = \frac{m}{M}$, which means that $\boldsymbol{\lambda}(F``B) = \varkappa\,\boldsymbol{\lambda}(B)$ for any Borel $B \subseteq X$. Moreover, since $c \in \mathfrak{M}$, it follows from the absoluteness theorem (Theorem 2.8) that one can choose X, Y, F with codes in $\mathfrak{M}$. Then F takes Borel $\boldsymbol{\lambda}$-null sets $B \subseteq X$ with codes in $\mathfrak{M}$ to Borel $\boldsymbol{\lambda}$-null sets $B \subseteq Y$ with codes in $\mathfrak{M}$, and conversely. Therefore, if $x \in \mathbf{Rand}\,\mathfrak{M}$, then $F(x) \in \mathbf{Rand}\,\mathfrak{M} \cap \mathbf{B}_c$.

(2) We employ a similar argument using the fact that, if $A \subseteq \mathbb{N}^\omega$ is a non-meagre Borel set, then there exist a Baire interval $\mathcal{N}_s$ and a $\mathbf{G}_\delta$ set $Y \subseteq \mathcal{N}_s \cap A$ dense in $\mathcal{N}_s$. One can readily see that the set Y is homeomorphic to $\mathbb{N}^\omega$, and so on.

(3) This assertion follows from Lemma 1.3 in the same way that (1) does. □

3B. Main results on resolvents of the regularity properties.

Theorem 3.3. *For every point* $a \in \mathbb{N}^\omega$ *the following equivalences hold*:

(i) $\mathrm{PK}(\mathit{\Pi}^1_1(a)) \iff$ ($\mathbf{L}[a] \cap \mathbb{N}^\omega$ *is a countable set*);[20]
(ii) $\mathrm{LM}(\mathit{\Sigma}^1_2(a)) \iff$ ($\mathbf{Rand}\,\mathbf{L}[a]$ *is a set of full* $\boldsymbol{\lambda}$*-measure*);
(iii) $\mathrm{BP}(\mathit{\Sigma}^1_2(a)) \iff$ ($\mathbf{Coh}\,\mathbf{L}[a]$ *is a co-meagre set*);
(iv) $\mathrm{LM}(\mathit{\Delta}^1_2(a)) \iff$ ($\mathbf{Rand}\,\mathbf{L}[a] \neq \varnothing$);
(v) $\mathrm{BP}(\mathit{\Delta}^1_2(a)) \iff$ ($\mathbf{Coh}\,\mathbf{L}[a] \neq \varnothing$).

Corollary 3.4.

(I) $\mathrm{PK}(\Pi^1_1) \iff \forall\, a \in \mathbb{N}^\omega$ ($\mathbf{L}[a] \cap \mathbb{N}^\omega$ *is countable*);
(II) $\mathrm{LM}(\Sigma^1_2) \iff \forall\, a \in \mathbb{N}^\omega$ ($\mathbf{Rand}\,\mathbf{L}[a]$ *is a set of full* $\boldsymbol{\lambda}$*-measure*);
(III) $\mathrm{BP}(\Sigma^1_2) \iff \forall\, a \in \mathbb{N}^\omega$ ($\mathbf{Coh}\,\mathbf{L}[a]$ *is a co-meagre set*);
(IV) $\mathrm{LM}(\Delta^1_2) \iff \forall\, a \in \mathbb{N}^\omega$ ($\mathbf{Rand}\,\mathbf{L}[a] \neq \varnothing$);
(V) $\mathrm{BP}(\Delta^1_2) \iff \forall\, a \in \mathbb{N}^\omega$ ($\mathbf{Coh}\,\mathbf{L}[a] \neq \varnothing$).

Proof. $\Pi^1_1 = \bigcup_{a \in \mathbb{N}^\omega} \mathit{\Pi}^1_1(a)$, and similarly for Σ^1_2 and Δ^1_2. □

The properties of the sets $\mathbf{L}[a] \cap \mathbb{N}^\omega$, $\mathbf{Rand}\,\mathbf{L}[a]$, and $\mathbf{Coh}\,\mathbf{L}[a]$ on the right-hand sides of the equivalences can be viewed as resolvents of the problems on the

[20]The condition "$\mathbf{L}[a] \cap \mathbb{N}^\omega$ is countable" is equivalent to the inequality $\omega_1^{\mathbf{L}[a]} < \omega_1$, and hence the right-hand side of (i) is equivalent to the condition $\forall\, a \in \mathbb{N}^\omega (\omega_1^{\mathbf{L}[a]} < \omega_1)$, which occurs in the literature more frequently.

left-hand sides of the equivalences, in the sense discussed above. Theorem 3.3 and Corollary 3.4 themselves do not solve the problems on the left-hand sides of the equivalences. For example, nothing can be immediately deduced concerning the question of whether $\mathrm{PK}(\Pi^1_1)$ is true or false. However, the problems are reduced to a uniform and intuitively more clear setting. Moreover, the relationships among the problems become much more transparent. In particular, we immediately obtain the implications in the diagram in the Introduction that come out of the box $\mathrm{PK}(\Pi^1_1)$.

Corollary 3.5 (Lyubetskii [65], [66], [68]). $\mathrm{PK}(\Pi^1_1)$ *implies* $\mathrm{LM}(\Sigma^1_2)$ *and* $\mathrm{BP}(\Sigma^1_2)$.

Proof. If $\mathbf{L}[a]\cap\mathbb{N}^\omega$ is countable, then $\mathbf{Rand}\,\mathbf{L}[a]$ is the complement of a union of countably many zero-measure sets, and hence $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a])=1$; $\mathbf{Coh}\,\mathbf{L}[a]$ is a co-meagre set for similar reasons. □

We prove the implications $\Longrightarrow$ of Theorem 3.3 in this section, and the next section is devoted to the inverse implications.

3C. Uncountable $\boldsymbol{\Pi^1_1}$ sets without a perfect kernel. Here we prove the implication $\Longrightarrow$ in (i) of Theorem 3.3. (The converse implication will be obtained in subsection 4B below.) The result is proved in the 'contrapositional' form, that is, for every $a\in\mathbb{N}^\omega$,

$$\mathbf{L}[a]\cap\mathbb{N}^\omega \text{ is uncountable} \Longrightarrow \neg\,\mathrm{PK}(\mathit{\Pi}^1_1(a)).$$

Proof. Suppose that $a\in\mathbb{N}^\omega$ and that $\mathbf{L}[a]\cap\mathbb{N}^\omega$ is uncountable. The set $\Xi=\{\langle x,w\rangle : w\in\mathbf{WO}\wedge x=\mathbf{f}_{|w|}[a]\}$ belongs to $\mathit{\Pi}^1_1(a)$ by 1.11(i) and Theorem 2.6(ii). By the uniformization theorem, Theorem 1.9, there is a *uniform* $\mathit{\Pi}^1_1(a)$ set $C\subseteq\Xi$ such that $\operatorname{dom}C=\operatorname{dom}\Xi$. In other words, C is the graph of a function η such that $\operatorname{dom}\eta=\mathbf{L}[a]\cap\mathbb{N}^\omega$ and $x=\mathbf{f}_{|\eta(x)|}[a]$ for each $x\in\mathbf{L}[a]\cap\mathbb{N}^\omega$. Thus, the $\mathit{\Sigma}^1_2(a)$ set $R=\operatorname{ran}\eta=\{\eta(x): x\in\mathbf{L}[a]\cap\mathbb{N}^\omega\}\subseteq\mathbf{WO}$ is uncountable, because $\mathbf{L}[a]\cap\mathbb{N}^\omega$ is.[21]

Obviously, R contains at most one point in common with any constituent $\mathbf{WO}_\xi=E_\xi(\mathfrak{L})$ of the Lebesgue sieve $\mathfrak{L}$ (see Example 1.10.1), and hence R has no perfect subsets. (If $X\subseteq R$ is a perfect set, then, by Theorem 1.8, there exists an ordinal $\vartheta<\omega_1$ such that $X\subseteq\bigcup_{\xi<\vartheta}E_\xi(\mathfrak{L})$, a contradiction.) Thus, R is an uncountable $\mathit{\Sigma}^1_2(a)$ set without a perfect kernel.

To find a $\mathit{\Pi}^1_1(a)$ set with the same properties, we note that, by Theorem 1.9, there is a uniform $\mathit{\Pi}^1_1(a)$ set $C\subseteq\mathbb{N}^\omega\times\mathbb{N}^\omega$ such that $R=\operatorname{dom}C=\{x:\exists\,y\,(\langle x,y\rangle\in C)\}$. (The uniformity means that for any x' there is at most one y such that $\langle x',y\rangle\in C$.) If $P\subseteq C$ is a perfect subset, then its projection $A=\operatorname{dom}P\subseteq R$ is uncountable, because C and P are uniform; moreover, A is a Σ^1_1 set. It follows from Theorem 1.4 that there is a perfect set $Y\subseteq A\subseteq R$. However, R has no perfect subsets, a contradiction. □

Another proof of the existence of Π^1_1 sets without the perfect kernel property (under the assumption that $\mathbf{L}[a]\cap\mathbb{N}^\omega$ is uncountable for some $a\in\mathbb{N}^\omega$) is given in [37]. It is more elementary, in the sense that it uses neither sieves nor constituents.

[21] The same result holds if we define R as the set formed by the $\prec_a$-smallest elements w_ξ in the sets $\mathbf{WO}_\xi$. We note that the proof of the lemma does not directly use Gödel's well ordering $\prec_a$ of the set $\mathbf{L}[a]\cap\mathbb{N}^\omega$ explicitly (though it uses the sequence $\{\mathbf{f}_\xi[a]\}_{\xi<\omega_1}$, of course).

3D. Non-measurable sets of second projective level. In this subsection we prove the implications $\Longrightarrow$ in (ii)–(v) of Theorem 3.3. They are proved in the following 'contrapositional' form:

(ii′) $\mathbf{Rand}\,\mathbf{L}[a]$ is not a set of full $\boldsymbol{\lambda}$-measure $\Longrightarrow \neg\,\mathrm{LM}(\Sigma^1_2(a))$;
(iii′) $\mathbf{Coh}\,\mathbf{L}[a]$ is not a co-meagre set $\Longrightarrow \neg\,\mathrm{BP}(\Sigma^1_2(a))$;
(iv′) $\mathbf{Rand}\,\mathbf{L}[a] = \varnothing \Longrightarrow \neg\,\mathrm{LM}(\Delta^1_2(a))$;
(v′) $\mathbf{Coh}\,\mathbf{L}[a] = \varnothing \Longrightarrow \neg\,\mathrm{BP}(\Delta^1_2(a))$.

Proof. (iii′) Suppose that $a \in \mathbb{N}^\omega$ and $\mathbf{Coh}\,\mathbf{L}[a]$ is not a co-meagre set. Then $C = \operatorname{cod}\mathfrak{I}_{\mathrm{cat}} = \{c \in \mathbf{BC} : \mathbf{B}_c \in \mathfrak{I}_{\mathrm{cat}}\}$ is a Π^1_1 set by 1.11(vi). However, to present the proof in a form applicable to a more general case below, we proceed in what follows with the weaker assumption that $C \in \Sigma^1_2$. Then, by Theorem 1.9, there is a Π^1_1 set $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ such that $c \in C \Longleftrightarrow \exists\, y\ P(c,y)$.

We recall that $\mathbf{f}_\xi[a]$ is the ξth element of $\mathbf{L}[a] \cap \mathbb{N}^\omega$ in the sense of the well ordering $\prec_a$ of $\mathbf{L}[a] \cap \mathbb{N}^\omega$. For any $\xi < \omega_1$, if $\mathbf{f}_\xi[a] = h$ satisfies $\langle (h)_0, (h)_1\rangle \in P$ (in which case $(h)_0 \in C$), then we define $S_\xi = \mathbf{B}_{(h)_0}$ (the Borel set encoded by $(h)_0$). Otherwise we set $S_\xi = \varnothing$. Then $S_\xi \in \mathfrak{I}_{\mathrm{cat}}$ in any case. Let $X_\xi = S_\xi \smallsetminus \bigcup_{\eta<\xi} S_\eta$. We claim that the sets

$$S = \{\langle w, x\rangle : w \in \mathrm{Ord} \wedge x \in S_{|w|}\} \quad \text{and} \quad W = \{\langle w, x\rangle : w \in \mathrm{Ord} \wedge x \in X_{|w|}\}$$

belong to $\Sigma^1_2(a)$. Indeed, for S this follows from the equivalence

$$\langle w, x\rangle \in S \iff w \in \mathbf{WO} \wedge \exists\, h\ \bigl(h = \mathbf{f}_{|w|}[a] \wedge \langle (h)_0, (h)_1\rangle \in P \wedge x \in \mathbf{B}_{(h)_0}\bigr),$$

where $\mathbf{WO} \in \Pi^1_1$ and $\{\langle w, \mathbf{f}_{|w|}[a]\rangle : w \in \mathbf{WO} \wedge a \in \mathbb{N}^\omega\}$ is a Π^1_1 set by Theorem 2.6(ii). As for W, the relation $x \in X_{|w|}$ is equivalent to the formula $x \in S_{|w|} \wedge \forall\, k\ (w(k) = 0 \Longrightarrow x \notin S_{|w\restriction k|})$ in the notation of Example 1.10.1. Then

$$Z = \bigcup_{\eta \leqslant \xi < \omega_1} (X_\eta \times X_\xi) = \bigcup_{\eta \leqslant \xi < \omega_1} \{\langle x, y\rangle : x \in X_\eta \wedge y \in X_\xi\}$$

is a $\Sigma^1_2(a)$ set as well, because

$$\langle x, y\rangle \in Z \iff \exists\, w, w' \in \mathbf{WO}\ \bigl(|w| \leqslant |w'| \wedge \langle w, x\rangle \in W \wedge \langle w', y\rangle \in W\bigr),$$

where both $\mathbf{WO}$ and the relation $|w| < |w'|$ belong to Π^1_1 (see Proposition 1.11).

We claim that *the set Z does not have the Baire property in* $\mathbb{N}^\omega \times \mathbb{N}^\omega$. Indeed, otherwise Z is a meagre set by the Ulam–Kuratowski theorem, because every cross-section $Z^y = \{x : \langle x, y\rangle \in Z\}$ is a union of countably many sets of the form X_ξ, and hence a meagre set. On the other hand, the projection $X = \{x : \exists\, y\ (\langle x, y\rangle \in Z)\} = \bigcup_{\xi<\omega_1} X_\xi$ of Z obviously coincides with the complement D of the set $\mathbf{Coh}\,\mathbf{L}[a]$. Therefore, D is a non-meagre set in the case under consideration. However, every cross-section $Z_x = \{y : \langle x, y\rangle \in Z\}$, $x \in X$, also differs from D by a union of countably many sets of the form X_ξ, and hence Z_x is a non-meagre set as well. Thus, Z itself is not a meagre set by the Ulam–Kuratowski theorem, which contradicts what was said above. We conclude that the $\Sigma^1_2(a)$ set $Z \subseteq (\mathbb{N}^\omega)^2$ does not have the Baire property. Using some recursive homeomorphism $(\mathbb{N}^\omega)^2 \xrightarrow{\text{onto}} \mathbb{N}^\omega$, we can transfer this counterexample to $\mathbb{N}^\omega$.

(v′) Here $\mathbf{Coh}\,\mathbf{L}[a] = \varnothing$, and hence the set Z (defined in the above argument) belongs to $\varDelta^1_2(a)$. Indeed, Z is the complement of $Z' = \bigcup_{\xi<\eta<\omega_1}(X_\eta \times X_\xi)$, which belongs to $\varSigma^1_2(a)$ for the same reasons as for Z.

(ii′) and (iv′). Here the arguments are similar to those used above; however, one must consider the ideal $\mathcal{I}_{\boldsymbol{\lambda}}$ instead of $\mathcal{I}_{\mathrm{cat}}$ and use Lemma 1.3 (for the effective class $\varDelta^1_2(a)$) to transfer the counterexample to $\mathbb{N}^\omega$. In general, here we must consider sets in the space 2^ω (because the measure $\boldsymbol{\lambda}$ is defined on 2^ω), for example, to define $S_\xi = \mathbf{B}_{\mathbf{f}_\xi[a]} \cap 2^\omega$ for $\mathbf{f}_\xi[a] \in \mathbf{BC}$ (see the beginning of the proof of (v′)). The Ulam–Kuratowski theorem is replaced by the Fubini theorem, of course. However, this does not change the arguments substantially.

$\square$ (*the implications* $\Longrightarrow$ *in* (ii)–(v) *of Theorem* 3.3)

3E. Generalization: measurability with respect to an ideal. Obvious similarities between the assertions (ii) and (iv) of Theorem 3.4 on the one hand and the assertions (iii) and (v) on the other hand lead us to the following general approach to equivalences of this kind. Let $\mathcal{I}$ be an ideal in the family of all Borel subsets of $\mathbb{N}^\omega$. A set $X \subseteq \mathbb{N}^\omega$ is called

an $\mathcal{I}$-*null* set if it is covered by a set in $\mathcal{I}$,

an $\mathcal{I}$-*measurable* set if there is a Borel set U such that the symmetric difference $X \triangle U$ is an $\mathcal{I}$-null set, and

an $\mathcal{I}$-*full* set if $\mathbb{N}^\omega \smallsetminus X$ is an $\mathcal{I}$-null set.

Definition 3.6. Let $\mathfrak{M}$ be a transitive model of **ZFC**. We say that a point $x \in \mathbb{N}^\omega$ is $\mathcal{I}$-*random over* $\mathfrak{M}$ and write $x \in \mathbf{Rand}_{\mathcal{I}}\,\mathfrak{M}$ if $x \notin \mathbf{B}_c$ whenever $c \in \mathbf{BC} \cap \mathfrak{M}$ and $\mathbf{B}_c \in \mathcal{I}$.

By a σ-CAC *ideal* we mean any ideal $\mathcal{I}$ of Borel subsets in $\mathbb{N}^\omega$ satisfying the following two conditions.

1°. $\mathcal{I}$ is closed under countable unions (σ-*additivity*).

2°. Every family of pairwise $\mathcal{I}$-disjoint Borel sets (that is, all pairwise intersections of these sets belong to $\mathcal{I}$) that do <u>not</u> belong to $\mathcal{I}$ is at most countable. (This is called the σ-*saturation* condition, or the *countable antichain condition*, briefly, CAC.)

The next theorem generalizes the classical results on measurability and the Baire property for the classes Σ^1_1 and Π^1_1 (see Theorem 1.4).

Theorem 3.7. *If* $\mathcal{I}$ *is a* σ-*CAC ideal in the algebra of Borel subsets of* $\mathbb{N}^\omega$, *then every* Σ^1_1 *or* Π^1_1 *set* $X \subseteq \mathbb{N}^\omega$ *is* $\mathcal{I}$-*measurable.*

Proof. For any set $Z \subseteq \mathbb{Q}$ we write $Z' = Z \smallsetminus \{q\}$ if q is the smallest element of Z, and simply take $Z' = Z$ if Z has no smallest element. If $R \subseteq \mathbb{N}^\omega \times \mathbb{Q}$, then we define $R' \subseteq R$ in such a way that $R'^{(x)} = (R^{(x)})'$ for each $x \in \mathbb{N}^\omega$. (We recall that $R^{(x)} = \{q : \langle x, q\rangle \in R\}$.)

Let us consider a Π^1_1 set $X = E(R)$, where $R \subseteq \mathbb{N}^\omega \times \mathbb{Q}$ is a Borel sieve. We define $R(\xi) \subseteq R$ by induction on $\xi < \omega_1$ in such a way that $R(\xi+1) = R(\xi)'$ and $R(\vartheta) = \bigcap_{\xi<\vartheta} R(\xi)$ for any limit ordinal ϑ (and $R(0) = R$). Then it is obvious that $R(\xi)$ and $R_q(\xi) = \{x : \langle x, q\rangle \in R(\xi)\}$ ($q \in \mathbb{Q}$) are Borel sets. In addition,

$R_q(\xi) \subseteq R_q(\eta)$ for $\eta < \xi$ and any q. Therefore, by the choice of $\mathcal{I}$, there is an ordinal $\xi < \omega_1$ such that $R_q(\xi) \smallsetminus R_q(\eta) \in \mathcal{I}$ for any $\xi \leqslant \eta$ and $q \in \mathbb{Q}$.

On the other hand, if x belongs to the set $D = X \smallsetminus E_{\leqslant\xi}(R)$, then $\langle x, q\rangle \in R(\xi) \smallsetminus R(\xi+1)$ for at least one q, and hence $D \subseteq \bigcup_{q\in\mathbb{Q}} R_q(\xi) \smallsetminus R_q(\xi+1)$. It follows that D is a $\mathcal{I}$-null set (because the ideal $\mathcal{I}$ is σ-additive). It remains to note that $E_{\leqslant\xi}(R) = \bigcup_{\zeta\leqslant\xi} E_\zeta(R)$ is a Borel set.

The following definition expresses the fact that the properties 1° and 2° of a given ideal $\mathcal{I}$ with respect to a given transitive model are absolute. For clear reasons, it is simpler to express the absoluteness by using codes of Borel sets rather than the sets themselves. In this connection we write $\operatorname{cod}\mathcal{X} = \{c \in \mathbf{BC} : \mathbf{B}_c \in \mathcal{X}\}$ for any family $\mathcal{X}$ of Borel subsets of $\mathbb{N}^\omega$. We say that $\mathcal{X}$ is $\mathbf{K}$-*encoded* (where $\mathbf{K}$ is a class of point sets) if $\operatorname{cod}\mathcal{X}$ belongs to $\mathbf{K}$.

Let $\mathfrak{M}$ be a transitive model of **ZFC** (for instance, a class of the form $\mathbf{L}[a]$, $a \in \mathbb{N}^\omega$). A σ-CAC ideal $\mathcal{I}$ is said to be $\mathfrak{M}$-*absolute* if

3° the set $C' = \operatorname{cod}\mathcal{I} \cap \mathfrak{M}$ belongs to $\mathfrak{M}$ and the properties 1° and 2° are absolute for $\mathfrak{M}$, that is, it is true in $\mathfrak{M}$ that $\mathcal{I}' = \{\mathbf{B}_c : c \in C'\}$ is a σ-CAC ideal.

Lemma 3.8. *Let $\mathfrak{M}$ be any transitive model of* **ZFC**. *The following σ-CAC ideals are $\mathfrak{M}$-absolute*:

- *the ideal $\mathcal{I}_{\mathrm{cat}}$ of all meagre Borel sets $X \subseteq \mathbb{N}^\omega$*;
- *the ideal $\mathcal{I}_\mu$ of all Borel sets $X \subseteq \mathbb{N}^\omega$ of μ-measure zero if μ is a Borel measure on $\mathbb{N}^\omega$ whose code* (*that is, the function* $\operatorname{cod}(\mu)(n) = \mu(\mathcal{N}_{s_n})$) *belongs to $\mathfrak{M}$*;
- *in particular, the ideal $\mathcal{I}_{\boldsymbol\lambda}$ of all Borel sets $X \subseteq \mathbb{N}^\omega$ such that* $\boldsymbol\lambda(X) = 0$.

The ideals $\mathcal{I}_{\mathrm{cat}}$ and $\mathcal{I}_{\boldsymbol\lambda}$ are $\varPi^1_1$-encoded, and $\mathcal{I}_\mu$ is $\varPi^1_1(\operatorname{cod}(\mu))$-*encoded.*

Proof. For the ideal $\mathcal{I}_{\mathrm{cat}}$ the set $C = \operatorname{cod}\mathcal{I}_{\mathrm{cat}}$ is $\varPi^1_1$ by 1.11(vi), and hence $\mathcal{I}_{\mathrm{cat}}$ is $\varPi^1_1$-*encoded*. Thus, by the absoluteness theorem, Theorem 2.8, the set $C' = C \cap \mathfrak{M}$ belongs to $\mathfrak{M}$, and it is true in $\mathfrak{M}$ that $C' = \{c \in \mathbf{BC} : \mathbf{B}_c \text{ is meagre}\} = \operatorname{cod}\mathcal{I}_{\mathrm{cat}}$. This implies 3°. Indeed, the properties 1° and 2° for $\mathcal{I}_{\mathrm{cat}}$ are theorems of **ZFC**, and hence they hold in $\mathfrak{M}'$.

The ideals $\mathcal{I}_{\boldsymbol\lambda}$ and $\mathcal{I}_\mu$ can be treated similarly, except for the fact that one must use 1.11(v) (and its analogue for an arbitrary measure μ) instead of 1.11(vi). □

Let us consider the following $\mathcal{I}$-*measurability hypothesis* for any given class $\mathbf{K}$ (for instance, for some projective class) and for any σ-CAC ideal $\mathcal{I}$:

$\mathrm{M}_\mathcal{I}(\mathbf{K})$: all sets $X \subseteq \mathbb{N}^\omega$ in $\mathbf{K}$ are $\mathcal{I}$-measurable.

For instance, we have $\mathrm{M}_\mathcal{I}(\Sigma^1_1)\ \forall\,\mathcal{I}$ by Theorem 3.7.

3F. Sets that are non-measurable with respect to an ideal. The following theorem will be proved below.

Theorem 3.9. *Let $a \in \mathbb{N}^\omega$ and let $\mathcal{I}$ be an $\mathbf{L}[a]$-absolute σ-CAC ideal that is $\varSigma^1_2(\mathbf{L}[a])$-encoded. Then the following equivalences hold*:

(a) $\mathrm{M}_\mathcal{I}(\varSigma^1_2(\mathbf{L}[a])) \iff \mathbf{Rand}_\mathcal{I}\,\mathbf{L}[a]$ *is an $\mathcal{I}$-full set*;

(b) $\mathrm{M}_\mathcal{I}(\varDelta^1_2(\mathbf{L}[a])) \iff \mathbf{Rand}_\mathcal{I}\,\mathbf{L}[a] \cap \mathbf{B}_c \neq \varnothing$ *whenever $\mathbf{B}_c \notin \mathcal{I}$ is a Borel set with a code $c \in \mathbf{L}[a] \cap \mathbf{BC}$*.

Remark 3.10. Let us say a few words concerning the relationships between this theorem and Theorem 3.3 and Corollary 3.4. The right-hand side of (a) is obviously equivalent to the right-hand sides of (ii) and (iii) in Theorem 3.3 for $\mathcal{I} = \mathcal{I}_{\boldsymbol{\lambda}}$ and $\mathcal{I} = \mathcal{I}_{\mathrm{cat}}$, respectively. (We note that both ideals satisfy the conditions in Theorem 3.9 by Lemma 3.8.) The right-hand side of (b) looks stronger than the right-hand sides of (iv), (v) in Theorem 3.3 (for the ideals $\mathcal{I}_{\boldsymbol{\lambda}}$ and $\mathcal{I}_{\mathrm{cat}}$), but in fact it is equivalent to these two ideals by Lemma 3.2(1), (2). The left-hand sides of (a) and (b) obviously include the left-hand sides of (ii), (iii) and of (iv), (v), respectively.

Thus, since the implications $\Longrightarrow$ in the equivalences (ii)–(v) of Theorem 3.3 have already been established in subsection 3D, it follows that Theorem 3.9 yields nothing new in the direction $\Longrightarrow$ for the ideals $\mathcal{I}_{\boldsymbol{\lambda}}$ and $\mathcal{I}_{\mathrm{cat}}$. However, the scope of Theorem 3.9 is much wider, because the σ-CAC ideals satisfying the properties listed in the theorem are not exhausted by the above two ideals. The implications $\Longrightarrow$ in Theorem 3.9 will be proved below in this subsection.

On the other hand, for the same reasons the implications $\Longleftarrow$ of Theorem 3.9 are sufficient to derive the implications $\Longleftarrow$ in the equivalences (ii)–(v) in Theorem 3.3. The implications in this direction will be established in § 4 by using forcing.

We note finally that Theorem 3.9 (for the ideals $\mathcal{I}_{\boldsymbol{\lambda}}$, $\mathcal{I}_{\mathrm{cat}}$) implies Corollary 3.4 (II)–(V) for obvious reasons. $\square$

To prove the implications $\Longrightarrow$ in Theorem 3.9, we suppose that $\mathcal{I}$ is a σ-CAC ideal and the set $C = \operatorname{cod}\mathcal{I}$ belongs to $\varSigma^1_2(\mathbf{L}[a])$. It is more convenient to present the desired result in the contrapositional form,

(a′) $\mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a]$ is not an $\mathcal{I}$-full set $\Longrightarrow \neg\mathrm{M}_{\mathcal{I}}(\varSigma^1_2(\mathbf{L}[a]))$;

(b′) $\mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a] \cap \mathbf{B}_c = \varnothing$ for some Borel set $\mathbf{B}_c \notin \mathcal{I}$ with a code $c \in \mathbf{L}[a] \cap \mathbf{BC}$ $\Longrightarrow \neg\mathrm{M}_{\mathcal{I}}(\varDelta^1_2(\mathbf{L}[a]))$.

We first note that the last part of the argument in subsection 3D, in which an $\mathcal{I}$-non-measurable set in $\mathbb{N}^\omega$ is derived from an $\mathcal{I}^2$-non-measurable subset of $\mathbb{N}^\omega \times \mathbb{N}^\omega$, does not work here ($\mathcal{I}^2$ stands for the Fubini product). The possibility of going from $\mathbb{N}^\omega \times \mathbb{N}^\omega$ to $\mathbb{N}^\omega$ is clear for the ideals $\mathcal{I}_{\boldsymbol{\lambda}}$ and $\mathcal{I}_{\mathrm{cat}}$, but this is hardly the case for an arbitrary σ-CAC ideal $\mathcal{I}$. Fortunately, there is another argument producing counterexamples directly in $\mathbb{N}^\omega$. However, the result is not as precise as the construction in subsection 3D, because the counterexamples will be obtained in the class $\varSigma^1_2(\mathbf{L}[a])$ even if C is a $\varSigma^1_2(a)$ set.

(a′) Since $C \in \varSigma^1_2(\mathbf{L}[a])$, there is a $\varPi^1_1(\mathbf{L}[a])$ set $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ such that $c \in C \Longleftrightarrow \exists\, y\; P(c,y)$. Starting from this set C, we define $S_\xi \in \mathcal{I}$, X_ξ, S, and W as in subsection 3D. Of course, the sets S and W are now $\varSigma^1_2(\mathbf{L}[a])$ sets by the very choice of C.

Further, we have $\omega_1^{\mathbf{L}[a]} = \omega_1$. (Otherwise the set $\mathbf{L}[a] \cap \mathbb{N}^\omega$ is countable; hence, $\mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a]$ is a countable intersection of $\mathcal{I}$-full sets, and therefore itself an $\mathcal{I}$-full set, which contradicts our assumptions.) It follows that for any $\xi < \omega_1 = \omega_1^{\mathbf{L}[a]}$ there is a $b \in \mathbf{L}[a]$ such that

$$\{(b)_k : k \in \mathbb{N}\} \subseteq \mathbf{WO} \quad \text{and} \quad \{|(b)_k| : k \in \mathbb{N}\} = \xi \cup \{\xi\},$$

and if $\xi \geqslant \omega$, then $|(b)_k| \neq |(b)_{k'}|$ for $k \neq k'$. (Such an element b encodes an enumeration of all ordinals $\leqslant \xi$ which is without repetitions if $\xi \geqslant \omega$.) Let b_ξ be the $\prec_a$-smallest element among all elements b of this kind. Finally, we write

$$\Xi_{\eta k} = \{\xi < \omega_1 : |(b_\xi)_k| = \eta\}, \quad Y_{\eta k} = \bigcup_{\xi \in \Xi_{\eta k}} X_\xi, \quad Y_\eta = \bigcup_k Y_{\eta k}.$$

It is clear that $Y_\eta = \bigcup_{\xi \geqslant \eta} X_\xi$. In particular, the set $Y_0 = \bigcup_{\xi < \omega_1} X_\xi$ is exactly the complement $\mathbb{N}^\omega \smallsetminus \mathbf{Rand}_{\mathfrak{I}}\, \mathbf{L}[a]$ of the set $\mathbf{Rand}_{\mathfrak{I}}\, \mathbf{L}[a]$. It follows that Y_0 is *not* an $\mathfrak{I}$-null set under our assumptions. Suppose (to arrive at a contradiction) that *all sets* $Y_{\eta k}$ (and then all sets Y_η as well) *are* $\mathfrak{I}$-*measurable*. None of the sets Y_η can be $\mathfrak{I}$-null, since $Y_0 \smallsetminus Y_\eta = \bigcup_{\zeta \leqslant \eta} X_\zeta$ is a countable union of $\mathfrak{I}$-null sets (by the property 1° of the ideal $\mathfrak{I}$.) In this case (again by the property 1°!) there exists for any $\eta < \omega_1$ an integer k_η such that $Y_{\eta\, k_\eta}$ is not an $\mathfrak{I}$-null set, and hence there is an integer k such that the set $H_k = \{\eta : k_\eta = k\}$ is uncountable. We conclude that $Y_{\eta k}$ is not an $\mathfrak{I}$-null set for any $\eta \in H_k$. To obtain a contradiction to the property 2° of the ideal $\mathfrak{I}$, it remains to note that $Y_{\eta k} \cap Y_{\eta' k} = \varnothing$ for $\eta' \neq \eta$. (If $x \in Y_{\eta k} \cap Y_{\eta' k}$, then there are ordinals $\xi \neq \xi'$ such that $\eta = (b_\xi)_k$, $\eta' = (b_{\xi'})_k$, and $x \in X_\xi \cap X_{\xi'}$, which is impossible because the sets X_ξ are pairwise disjoint.)

Thus, some sets $Y_{\eta k}$ are not $\mathfrak{I}$-measurable. It remains to show that each set $Y_{\eta k}$ belongs to $\varSigma^1_2(\mathbf{L}[a])$. Indeed, the sets

$$B = \{\langle w, b_{|w|}\rangle : w \in \mathrm{Ord}\} \quad \text{and} \quad \Omega = \{\langle t, w, k\rangle : t, w \in \mathbf{WO} \wedge |t| \in \Xi_{|w|\, k}\}$$

are $\varSigma^1_2(a)$ sets by Theorem 2.6(iv). Let us take any $\eta < \omega_1$ and k and choose a $w \in \mathbf{WO}_\eta \cap \mathbf{L}[a]$. Then $T = \{t \in \mathbf{WO} : \langle t, w, k\rangle \in \Omega\}$ is a $\varSigma^1_2(a, w)$ set. Therefore, $Y_{\eta k} = \{x : \exists\, t \in T\ (\langle t, x\rangle \in W)\}$ is a $\varSigma^1_2(\mathbf{L}[a])$ set together with W.

(b′) Here we assume that $c \in \mathbf{BC} \cap \mathbf{L}[a]$, $\mathbf{B}_c \notin \mathfrak{I}$, and $\mathbf{Rand}_{\mathfrak{I}}\, \mathbf{L}[a] \cap \mathbf{B}_c = \varnothing$. Let us repeat the construction in the proof of (a), and let us begin this construction by setting $S_\xi = \mathbf{B}_{(h)_0} \cap \mathbf{B}_c$ if $\langle (h)_0, (h)_1\rangle \in P$ and $S_\xi = \varnothing$ otherwise. Then $Y_{\eta k} \subseteq \mathbf{B}_c$, and some sets $Y_{\eta k}$ are not $\mathfrak{I}$-measurable. In addition, every set $Y_{\eta k}$ belongs to $\varSigma^1_2(\mathbf{L}[a])$. It remains to show that any set $Y_{\eta k}$ is a $\varPi^1_2(\mathbf{L}[a])$ set as well. In this case we have $Y_0 = \mathbf{B}_c$, and hence every $Y_\eta = Y_0 \smallsetminus \bigcup_{\xi < \eta} X_\xi$ is a Borel set. Moreover, for $\eta \geqslant \omega$ we have $Y_{\eta k} \cap Y_{\eta k'} = \varnothing$ if $k \neq k'$, because $|(b_\xi)_k| \neq |(b_\xi)_{k'}|$ by definition if $k \neq k'$, and the sets X_ξ are pairwise disjoint. Therefore, $Y_{\eta k} = Y_\eta \smallsetminus \bigcup_{k' \neq k} Y_{\eta k'}$ is a Π^1_2 set. A more careful analysis based on the results for the sets Ω and W enables us to conclude that $Y_{\eta k} \in \varPi^1_2(\mathbf{L}[a])$ (by the same argument as in the last part of the proof in (a)). The values $\eta < \omega$ can be disregarded here.

□ (*the implications* $\Longrightarrow$ *in Theorem* 3.9)

3G. Consistency of the existence of counterexamples.

Theorem 3.11. *If* $\mathbb{N}^\omega \subseteq \mathbf{L}[a]$ *holds for some* $a \in \mathbb{N}^\omega$, *then each of the assertions* $\mathrm{PK}(\varPi^1_1(a))$, $\mathrm{BP}(\varDelta^1_2(a))$, *and* $\mathrm{LM}(\varDelta^1_2(a))$ *fails. Thus, the conjunction of the negations of* $\mathrm{PK}(\varPi^1_1)$, $\mathrm{BP}(\varDelta^1_2)$, *and* $\mathrm{LM}(\varDelta^1_2)$ (*in other words, the assertion that there are counterexamples in the corresponding classes*) *is consistent with the axioms of* **ZFC**.

Proof. Under our assumptions, the set $\mathbf{L}[a] \cap \mathbb{N}^\omega = \mathbb{N}^\omega$ is uncountable. Moreover, every singleton $\{x\}$, $x \in \mathbb{N}^\omega$, is a Borel set with a code in $\mathbf{L}[a]$, and this set is meagre and of measure zero. Thus, we have $\mathbf{Coh}\,\mathbf{L}[a] = \mathbf{Rand}\,\mathbf{L}[a] = \varnothing$. The assertions $\mathrm{LM}(\varDelta^1_2(a))$ and $\mathrm{BP}(\varDelta^1_2(a))$ are now violated, by results in subsection 3D, and $\mathrm{PK}(\varPi^1_1(a))$ is violated, by results in subsection 3C.

To prove the second part of the theorem, we note that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies that $\mathbb{N}^\omega \subseteq \mathbf{L}[a]$ for *any* a, and on the other hand, the axiom of constructibility is consistent with **ZFC** by Gödel's result. □

The theorem proved above can be refined in connection with the property PK. Let us consider the following modification of $\mathrm{PK}(\mathbf{K})$:

$\mathrm{PK}^-(\mathbf{K})$: every $\mathbf{K}$-set $P \subseteq \mathbb{N}^\omega \times \mathbb{N}^\omega$ which is the graph of an everywhere defined function from $\mathbb{N}^\omega$ to $\mathbb{N}^\omega$ (in this case P is uncountable, of course) contains a perfect subset.

Lemma 3.12. *If $a \in \mathbb{N}^\omega$ and $\mathbb{N}^\omega \subseteq \mathbf{L}[a]$, then $\mathrm{PK}^-(\varPi^1_1(a))$ fails.*

Proof. Returning to the proof in subsection 3C, we note that the function η defined in the course of the proof satisfies the condition $\mathbb{N}^\omega = \operatorname{dom}\eta$ if $\mathbb{N}^\omega \subseteq \mathbf{L}[a]$, and hence we have $\neg\mathrm{PK}^-(\varPi^1_1(a))$. □

Historical and bibliographical remarks. The notions of Cohen and random points (see Definition 3.1) are due to Solovay [89]. However, Cohen points were introduced by Cohen [15] in a different but equivalent way; in fact, they were the first and most elementary objects given by forcing. For the relationships between Cohen and random points and forcing, see subsection 4C.

The proof of Theorem 3.7 uses a method of Selivanowski [83]. Earlier separate proofs of the Lebesgue measurability and the Baire property for Σ^1_1 sets (see, for example, [57]) differ from each other and cannot be generalized to the result of 3.7.

A few words concerning Theorem 3.3 and Corollary 3.4. (We recall that the proofs of the implications $\Longleftarrow$ in 3.3 and 3.4, as well as in Theorem 3.9, were postponed to the next section.) The equivalence (I) was presented in the survey [74] with a reference to Mansfield and Solovay; their proofs appeared somewhat later in [72] and [88]. The result was obtained independently by Lyubetskii ([17], [65], [67]). Earlier Lyubetskii [64] proved the implication $\Longrightarrow$.

The equivalences (II) and (III) are absent in [74]. As far as we know, the equivalence (II) was first given by Lyubetskii [65]. (For complete proofs of (II) and (III), see [68].) These two equivalences are often referred to unpublished papers of Solovay in the 1960s. (Sometimes (for instance, in [10], p. 457) a reference is given to the paper [88], where these results are simply absent, and measurability and the Baire property are not considered at all.)

The equivalences (IV) and (V) of Corollary 3.4 were proved by Lyubetskii [68], [69] (and included in the survey paper [35]). However, these papers have never been translated from Russian and thus the results remained unknown to Western set theorists. Ihoda [Judah] and Shelah re-proved these results in [28].

A few words about some earlier results on the existence of counterexamples. Gödel announced in [20] that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ (that is, the assumption that all sets belong to $\mathbf{L}$) implies the existence of a non-measurable Δ^1_2 set and of an uncountable Π^1_1 set having no perfect subset. The proof of this result (Theorem 3.11 for $\mathbf{L}$ instead of $\mathbf{L}[a]$) first appeared in Novikov's paper [78].

Novikov's counterexample for the perfect kernel property was reproduced in subsection 3C and in Lemma 3.12 with inessential changes. (Following [78], we would have to define the set Ξ in subsection 3C as the set of all pairs $\{x, w\}$ such that $w \in \mathbf{WO}$ and $x \in \{\mathbf{f}_\xi[a] : \xi < |w|\}$. Moreover, Novikov considers 'absolute' constructibility, that is, $\mathbf{L}$ instead of $\mathbf{L}[a]$. However, this does not change the essence of the arguments.) The counterexample for $\mathrm{LM}(\Delta^1_2)$ given in [78] (again under the assumption that $\mathbf{V} = \mathbf{L}$) can be represented as follows: if a function $\eta \colon \mathbb{N}^\omega \to \mathbb{N}^\omega$ induces a counterexample for $\mathrm{PK}^-(\Pi^1_1)$, then the set $\{\langle x, y\rangle : y <_{\mathrm{lex}} \eta(x)\}$, where $<_{\mathrm{lex}}$ is the lexicographical order on $\mathbb{N}^\omega$, is a non-measurable Δ^1_2 set (if the measure of each Baire interval is strictly positive).

The construction of counterexamples for measure and category in the classes Δ^1_2 and Σ^1_2 (see subsection 3D) is known from many sources (see, for instance, [10], p. 453 or [92]). Sometimes it is given with a reference to Solovay's unpublished papers written in the 1960s. (The papers [88] and [89] contain neither the construction nor the results.) The idea of the generalized approach of subsection 3E and, in particular, of the approach used in subsection 3F is taken from [69]. We shall see below in subsection 9C that there are other ideals which differ from the ideals of meagre sets and zero-measure sets and their intersections and to which Theorem 3.9 can be applied.

§ 4. Resolvents of classical problems: Part 2

In § 3 we proved the implications $\Longrightarrow$ in Theorems 3.3 and 3.9 and in Corollary 3.4. This showed (see Theorem 3.11, with the reference to Gödel's result on the consistency of the axiom of constructibility $\mathbf{V} = \mathbf{L}$) that *negative* solutions of the problems treated in Corollary 3.4 (that is, the assertions that there are counterexamples to the left-hand sides of the five equivalences) are consistent with the **ZFC** axioms. The present section contains proofs of the implications $\Longleftarrow$ in Theorems 3.3 and 3.9 and in Corollary 3.4.

The proofs use forcing. The available experience in eliminating forcing from the proofs of statements similar to Theorems 3.3 and 3.9, that is, results *not* related to independence, hardly gives any hope of obtaining simple proofs free of forcing. However, if only the implications coming out of the block $\mathrm{PK}(\Pi^1_1)$ in the diagram in the Introduction are considered, then the role of forcing can be reduced to the single assertion that $\omega_1^{\mathbf{L}[a,x]} = \omega_1^{\mathbf{L}[a]}$ whenever $x \in \mathbf{Rand}\,\mathbf{L}[a]$ or $x \in \mathbf{Coh}\,\mathbf{L}[a]$ (Lemma 4.12). The proof of these implications is completed in subsection 4E.

The consistency of *positive* solutions of the above problems is established in § 7 in a stronger form (including the regularity properties for all **ROD** sets, and, in particular, for all projective sets).

4A. Basics of forcing. We assume that the reader is somewhat acquainted with forcing, and therefore the text below is rather a review than a self-contained exposition of the basics of this method.

In the course of this subsection we fix a transitive set or a class $\mathfrak{M}$ satisfying all axioms of **ZFC**; $\mathfrak{M}$ is called the *ground model*. For example, $\mathfrak{M}$ can be a countable model, a class of the form $\mathbf{L}[x]$, or even the universe $\mathbf{V}$ of all sets. We assume that $\mathbb{P} \in \mathfrak{M}$ is a fixed partially ordered set, properly called a *forcing*. Elements of $\mathbb{P}$ are called (*forcing*) *conditions*. If $p \leqslant q$, then p is said to be a *stronger*[22] 'condition'. The 'conditions' p and $q \in \mathbb{P}$ are said to be *compatible* if there exists an $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$; otherwise p and q are said to be *incompatible*. A set $A \subseteq \mathbb{P}$ is said to be

dense (*in* $\mathbb{P}$) if $\forall p \in \mathbb{P}\, \exists q \in A\ (q \leqslant p)$;
an antichain (*in* $\mathbb{P}$) if any two $p \neq q \in A$ are incompatible.

Finally, a set $G \subseteq \mathbb{P}$ is said to be $\mathbb{P}$-*generic over* $\mathfrak{M}$ if

(1) $G \cap D \neq \varnothing$ for any dense set $D \subseteq \mathbb{P}$, $D \in \mathfrak{M}$;
(2) if $p \in G$, $q \in \mathbb{P}$, and $p \leqslant q$, then $q \in G$.

We define $t[G] = \{s[G] : \exists p \in G\ (\langle p, s\rangle \in t)\}$ (by $\in$-induction on t). The sets t occurring in the text in expressions of the form $t[G]$ are usually called *names*. For instance, let $\breve{x} = \{\langle p, \breve{y}\rangle : p \in \mathbb{P} \wedge y \in x\}$ (again by $\in$-induction). It is clear that $\breve{x}[G] = x$ for any set x. Such a name $\breve{x}$ is said to be the *canonical* $\mathbb{P}$-*name* of x. Typically, $\breve{x}$ and x are identified if this leads to no ambiguity. We shall sometimes use this identification if x belongs to $\mathbb{N}$, $\mathbb{N}^{<\omega}$, or Ord.

Theorem 4.1. *If $\mathbb{P} \in \mathfrak{M}$ is a partially ordered set and if $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathfrak{M}$, then there is a unique transitive set or class $\mathfrak{M}[G]$* (the so-called $\mathbb{P}$-*generic extension* of the model $\mathfrak{M}$) *such that*

(i) *$\mathfrak{M} \subseteq \mathfrak{M}[G]$, $G \in \mathfrak{M}[G]$, the ordinals of $\mathfrak{M}$ and $\mathfrak{M}[G]$ coincide, and all axioms of* **ZFC** *hold in* $\mathfrak{M}[G]$;
(ii) *$\mathfrak{M}[G]$ is the smallest class satisfying* (i).

If $x \in \mathfrak{M}[G]$, then there is a name $t \in \mathfrak{M}$ such that $x = t[G]$.

Theorem 4.2 below contains several of the most important properties of generic extensions, that is, classes of the form $\mathfrak{M}[G]$, where G is a $\mathbb{P}$-generic set over $\mathfrak{M}$ for some forcing $\mathbb{P} \in \mathfrak{M}$. We first present the necessary definitions.

(1) A partially ordered set $\mathbb{P}$ satisfies the $\varkappa$-*antichain condition* (where $\varkappa$ is an infinite cardinal) if the cardinality of any antichain $A \subseteq \mathbb{P}$ is $< \varkappa$ (strictly); the *countable antichain condition* (briefly, CAC) is the $\aleph_1$-antichain condition.
(2) Partially ordered sets $\mathbb{P}$ and $\mathbb{P}'$ are said to be *weakly isomorphic* if there are dense subsets $D \subseteq \mathbb{P}$ and $D' \subseteq \mathbb{P}'$ order isomorphic to each other.

Theorem 4.2. *Let $\mathbb{P}$, $\mathbb{P}' \in \mathfrak{M}$ be partially ordered sets. In this case*

(i) *if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, $\varkappa$ is a regular cardinal in $\mathfrak{M}$, and $\mathbb{P}$ satisfies the $\varkappa$-antichain condition in $\mathfrak{M}$, then $\varkappa$ is a regular cardinal in $\mathfrak{M}[G]$, and in particular, if $\mathbb{P}$ satisfies the* CAC *in $\mathfrak{M}$, then all cardinals of $\mathfrak{M}$ remain cardinals in $\mathfrak{M}[G]$*;
(ii) *if $\mathbb{P}$, $\mathbb{P}'$ are weakly isomorphic in $\mathfrak{M}$, then any $\mathbb{P}$-generic extension of $\mathfrak{M}$ is also a $\mathbb{P}'$-generic extension of $\mathfrak{M}$*;

[22] That is, a condition forcing more properties of generic extensions (for these extensions, see below). This is a rather standard convention (though certainly not generally accepted; see, for instance, [10]) which does not always agree with intuition, because for many forcings the order relation $\leqslant$ is reversed with respect to the inclusion relation $\subseteq$. However, one has to get used to this.

(iii) *every* $\mathbb{P}\times\mathbb{P}'$*-generic set over* $\mathfrak{M}$ *is of the form* $G\times G'$, *where the set* $G\subseteq\mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $G'\subseteq\mathbb{P}'$ *is* $\mathbb{P}'$*-generic over* $\mathfrak{M}[G]$ (*and over* $\mathfrak{M}$), *and furthermore*, $\mathfrak{M}[G]\cap\mathfrak{M}[G']=\mathfrak{M}$;

(iv) *conversely, if a set* $G\subseteq\mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and a set* $G'\subseteq\mathbb{P}'$ *is* $\mathbb{P}'$*-generic over* $\mathfrak{M}[G]$, *then the set* $G'\times G''$ *is* $\mathbb{P}'\times\mathbb{P}''$*-generic over* $\mathfrak{M}$.

The assertions (iii) and (iv) of Theorem 4.2 are known as the **product theorem**.

In the theory of forcing one defines the *forcing relation*, or simply *forcing*, typically denoted by $p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi(t_1,\dots,t_n)$ (verbalizing, *p forces* $\varphi(t_1,\dots,t_n)$), where $\mathbb{P}\in\mathfrak{M}$ is a partially ordered set, $p\in\mathbb{P}$, the elements $t_1,\dots,t_n\in\mathfrak{M}$ are understood as names, and $\varphi(x_1,\dots,x_n)$ is any $\in$-formula, that is, a formula in the language of **ZFC**. This relation satisfies the following theorem.

Theorem 4.3. *Under the above assumptions, suppose that* $\varphi(x_1,\dots,x_n)$ *is an arbitrary* $\in$*-formula. In this case*

(i) *the forcing of the formula* φ *is definable in* $\mathfrak{M}$ *in the sense that the family*

$$\{\langle p,t_1,\dots,t_n\rangle : p\in\mathfrak{M}\wedge t_1,\dots,t_n\in\mathfrak{M}\wedge p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi(t_1,\dots,t_n)\}$$

is definable in $\mathfrak{M}$ *by an* $\in$*-formula with* $\mathbb{P}$ *as a parameter*;

(ii) *if* $G\subseteq\mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $t_1,\dots,t_n\in\mathfrak{M}$ *are names, then*

$$\varphi(t_1[G],\dots,t_n[G]) \text{ is true in } \mathfrak{M}[G] \iff \exists\, p\in G\ (p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi(t_1,\dots,t_n));$$

(iii) *the set* $\{p\in\mathbb{P} : p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi(t_1,\dots,t_n)\vee p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\neg\varphi(t_1,\dots,t_n)\}$ *is dense in* $\mathbb{P}$ *and belongs to* $\mathfrak{M}$ *for any names* $t_j\in\mathfrak{M}$.

The symbol $\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi$ means that $p\Vdash_{\mathbb{P}}^{\mathfrak{M}}\varphi$ for all $p\in\mathbb{P}$. The relation $\Vdash_{\mathbb{P}}$ is understood as $\Vdash_{\mathbb{P}}^{\mathbf{V}}$ (that is, the ground model is the universe $\mathbf{V}$ of all sets).

Let us now discuss the existence of generic sets. One can readily show that there are $\mathbb{P}$-generic sets over $\mathfrak{M}$ if $\mathfrak{M}$ is a countable (transitive) model of **ZFC**. However, this assumption is not always convenient. For example, in some arguments it is desirable to consider generic extensions of classes of the form $\mathbf{L}[a]$ or even extensions of the universe $\mathbf{V}$ of all sets. In this case one can use *Boolean-valued* extensions $\mathbf{V}^{\mathbb{P}}$ of the universe $\mathbf{V}$ (for Boolean-valued models, see [29], [70]) as well as the following principle (equivalent in essence).

Theorem 4.4. *Let* $\mathbb{P}$ *be a partially ordered set in the universe* $\mathbf{V}$ *of all sets. The assumption that in a 'virtual' wider universe* $\mathbf{V}^+$ (*for instance, of the form* $\mathbf{V}^{\mathbb{P}}$) *there exists for any* $p\in\mathbb{P}$ *a set* $G\subseteq\mathbb{P}$ *which is* $\mathbb{P}$*-generic over* $\mathbf{V}$ *and contains* p *does not lead to any contradictions nor ungrounded deductions.*

4B. If there are few constructible points, then the $\mathbf{\Pi}^1_1$ sets have the perfect kernel property. The implication from right to left in Theorem 3.3(i) is an elementary corollary to Lemma 4.5. Indeed, if any set of the form $\mathbf{L}[a]\cap\mathbb{N}^\omega$, $a\in\mathbb{N}^\omega$, is countable, then, by the lemma, every Π^1_1 set without a perfect kernel is countable as well.

Lemma 4.5 (Solovay [88], Lyubetskii [66]). *If $a \in \mathbb{R}$ and a $\Sigma_2^1(a)$ set $X \subseteq \mathbb{N}^\omega$ contains no perfect subsets, then $X \subseteq \mathbf{L}[a]$.*

Proof. By the uniformization theorem, Theorem 1.9, it suffices to prove the lemma for $\Pi_1^1(a)$ sets. (See the end of the proof in subsection 3C.) Let us consider a $\Pi_1^1(a)$ set $X \subseteq \mathbb{N}^\omega$. By Theorem 1.6, $X = E(R) = \bigcup_{\xi<\omega_1} E_\xi(R)$ for a suitable sieve $R = \{R_q\}_{q\in\mathbb{Q}}$ over $\mathbb{N}^\omega$ of class $\Delta_1^1(a)$ (even of class $\Delta_1^0(a)$, but we do not need this fact). Thus, it remains to show that $E_\xi(R) \subseteq \mathbf{L}[a]\ \forall\,\xi$. We note that the set $E_\xi(R)$ is at most countable; indeed, it is Borel and without a perfect kernel by assumption.

Case 1: $\xi < \omega_1^{\mathbf{L}[a]}$. Then there is a $w \in \mathbf{WO}_\xi \cap \mathbf{L}[a]$. The statement "$E_\xi(R)$ is at most countable" can be expressed by the formula

$$\exists\, z\ \ \forall\, x\ \ \exists\, n\ \ (\sigma(w,x) \Longrightarrow x = (z)_n),$$

where σ is the $\Sigma_1^1(a)$ formula given by Proposition 1.11(iii). However, the displayed formula is obviously a Σ_2^1 formula with the parameters a, $w \in \mathbf{L}[a]$, and hence it also holds in $\mathbf{L}[a]$ by the absoluteness theorem, that is, there is a $z \in \mathbf{L}[a] \cap \mathbb{N}^\omega$ such that the formula $\forall\, x\ \exists\, n\ (\sigma(w,x) \Longrightarrow x = (z)_n)$ is true in $\mathbf{L}[a]$. Again by absoluteness, the last formula also holds in the universe of all sets, which means that $E_\xi(R) \subseteq \{(z)_n : n \in \mathbb{N}\} \subseteq \mathbf{L}[a]$.

Case 2: $\omega_1^{\mathbf{L}[a]} \leqslant \xi < \omega_1$. Let us prove that $E_\xi(R) = \varnothing$. Suppose the contrary: let the constituent $E_\xi(R)$ be non-empty. As above, if $w \in \mathbf{WO}_\xi$, then the set $E_\xi(R)$ belongs to $\mathbf{L}[a,w]$ and is countable in $\mathbf{L}[a,w]$.

Let us consider the forcing $\mathbb{C}(\xi) = \mathrm{Coll}(\mathbb{N},\xi)$ designed to 'collapse' ξ. Thus, $\mathbb{C}(\xi)$ consists of all finite sequences of ordinals $< \xi$, or, which is the same, of all functions $p\colon m \to \xi$, where $m = \{0,1,\dots,m-1\}$ and $p \leqslant q$ (that is, p is 'stronger') if $q \subseteq p$, that is, p extends q as a function. Every $\mathbb{C}(\xi)$-generic set $G \subseteq \mathbb{C}(\xi)$ produces a function $f[G] = \bigcup G\colon \mathbb{N} \overset{\text{onto}}{\longrightarrow} \xi$, the so-called *collapse function* for ξ. Conversely, $G = \{f[G]\restriction n : n \in \mathbb{N}\}$.

By Theorem 4.4, one can consider a $\mathbb{C}(\xi)\times\mathbb{C}(\xi)$-generic extension $\mathbf{V}[G,G']$ of the universe $\mathbf{V}$ generated by a pair of sets G_1, $G_2 \subseteq \mathbb{C}(\xi)$ generic over $\mathbf{V}$ (and hence over $\mathbf{L}[a]$ as well). By the foregoing, such an extension admits two collapse functions $f_1 = \bigcup G_1$ and $f_2 = \bigcup G_2$ from $\mathbb{N}$ onto ξ. Accordingly, there are codes $w_1 \in \mathbf{WO}_\xi \cap \mathbf{L}[f_1]$ and $w_2 \in \mathbf{WO}_\xi \cap \mathbf{L}[f_2]$ of the ordinal ξ. As proved above, $E_\xi(R) \subseteq \mathbf{L}[a,f_1] \cap \mathbf{L}[a,f_2] = \mathbf{L}[a,G_1] \cap \mathbf{L}[a,G_2]$.

Since the sets G_1, G_2 form a generic pair over $\mathbf{L}[a]$, we have $E_\xi(R) \subseteq \mathbf{L}[a]$ by Theorem 4.2(iii). In other words, since the constituent $E_\xi(R)$ is non-empty, there exists an $x \in E_\xi(R) \cap \mathbf{L}[a]$. However, in this case it is clear that $\xi < \omega_1^{\mathbf{L}[x]} \leqslant \omega_1^{\mathbf{L}[a]}$, a contradiction.

$\square$ (*Lemma* 4.5 *and Theorem* 3.3(i))

Remark 4.6 (Mansfield [72]). Taking the contraposition of Lemma 4.5, we see that any $\Sigma_2^1(a)$ set $X \subseteq \mathbb{N}^\omega$ with $X \not\subseteq \mathbf{L}[a]$ necessarily contains a perfect subset $P \subseteq X$. It turns out that the subset can be chosen to have a code in $\mathbf{L}[a]$, in the sense that there is a perfect tree $T \in \mathbf{L}[a]$, $T \subseteq \mathbb{N}^{<\omega}$ (see the notation in subsection 1A), such that $[T] \subseteq X$. Indeed, it follows from considerations related to the uniformization theorem (Theorem 1.9) that X can be assumed to be a $\Pi^1_1(a)$ set. In this case the formula "X has a perfect subset of the form $[T]$" can be reduced to the $\Sigma^1_2(a)$ form, and hence it is absolute by Theorem 2.8; thus, there exists a perfect tree $T \in \mathbf{L}[a]$, $T \subseteq \mathbb{N}^{<\omega}$, such that $[T] \subseteq X$ holds in $\mathbf{L}[a]$. However, the formula $[T] \subseteq X$ is absolute as well.

Lemma 4.5 enables one to prove another result revealing the nature of the version $\mathrm{PK}^-(\Pi^1_1(a))$ of the perfect kernel hypothesis (subsection 3G).

Corollary 4.7. *If* $a \in \mathbb{N}^\omega$, *then* $\mathrm{PK}^-(\Pi^1_1(a)) \iff (\mathbb{N}^\omega \not\subseteq \mathbf{L}[a])$.

Proof. The implication from left to right was established in Lemma 3.12. To prove the converse implication, suppose that the assertion $\mathrm{PK}^-(\Pi^1_1(a))$ fails, that is, one can find a function $\eta\colon \mathbb{N}^\omega \to \mathbb{N}^\omega$ whose graph P is a $\Pi^1_1(a)$ set without a perfect kernel. Then $P \subseteq \mathbf{L}[a]$ by Lemma 4.5, and hence $\mathbb{N}^\omega = \operatorname{dom} P \subseteq \mathbf{L}[a]$. □

Lemma 4.5 admits other interesting versions and generalizations. We have seen above (Case 2 in the proof of Lemma 4.5) that if R is a $\Delta^1_1(a)$ sieve, $a \in \omega_1$, then all non-empty constituents $E_\xi(R)$ with $\omega_1^{\mathbf{L}[a]} \leqslant \xi < \omega_1$ are uncountable. Some other similar results on 'distant' constituents are known. For example, in the above situation the set $E_\xi(R)$ cannot belong to the class $\mathbf{F}_\sigma$. Generally, if $\omega_\alpha^{\mathbf{L}[a]} \leqslant \xi < \omega_1$ and $E_\xi(R) \neq \varnothing$, then $E_\xi(R)$ cannot be a Borel set of level α in the Borel hierarchy (see [93], [36], [40]). The proofs of these theorems are rather too laborious to be presented in this paper and, the main point, are not directly related to the problems discussed here.

4C. Forcing induced by an ideal of Borel sets. We begin the proof of the implications $\Longleftarrow$ in Theorem 3.9. The principal idea is to find a characterization of random points in terms of forcing. To avoid repetitions, we assume in this subsection that $\mathfrak{M}$ is a fixed transitive model of **ZFC** (for example, the universe $\mathbf{V}$ of all sets or a class of the form $\mathbf{L}[X]$). In the variety of forcings $\mathbb{P} \in \mathfrak{M}$ we distinguish those induced by σ-CAC ideals in a natural way.

Definition 4.8. Suppose that $\mathfrak{I}$ is an $\mathfrak{M}$-absolute σ-CAC ideal. By the *Borel forcing modulo* $\mathfrak{I}$ we mean the set

$$\mathbb{P}_{\mathfrak{I}} = \{p \in \mathbf{BC} \cap \mathfrak{M} : \mathbf{B}_p \notin \mathfrak{I}\} \text{ ordered as follows: } p \leqslant q \text{ if } \mathbf{B}_p \subseteq \mathbf{B}_q.$$

We write $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}} = \mathbb{P}_{\mathfrak{I}} \cap \mathfrak{M}$.[23]

We note that 'conditions' $p,\, q \in \mathbb{P}_{\mathfrak{I}}$ are compatible in $\mathbb{P}_{\mathfrak{I}}$ (and in $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$) if and only if $\mathbf{B}_p \cap \mathbf{B}_q \notin \mathfrak{I}$. Indeed, there is a code $c \in \mathbf{BC} \cap \mathfrak{M}$ such that $\mathbf{B}_p \cap \mathbf{B}_q = \mathbf{B}_c$. (To obtain c, we first define codes p' and q' by the rule $(p')_n = p$, $(q')_n = q\ \forall\, n$, and thus $\mathbf{B}_{p'}$ and $\mathbf{B}_{q'}$ are the complements of $\mathbf{B}_p$ and $\mathbf{B}_q$, respectively. We now define c in such a way that $(c)_0 = p'$ and $(c)_n = q'$ for $n \geqslant 1$.) In the non-trivial direction if $\mathbf{B}_p \cap \mathbf{B}_q \notin \mathfrak{I}$, then $c \in \mathbb{P}_{\mathfrak{I}}$, and of course $c \leqslant p$ and $c \leqslant q$.

The next lemma shows that $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ is a CAC forcing in $\mathfrak{M}$.

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[23]Thus, under this definition, a forcing consists of *codes* of Borel sets rather than the Borel sets themselves. If one takes the Borel sets themselves as 'conditions', which is often done, then the forcing becomes more intuitive. However, one then faces the fact that a Borel set $\mathbf{B}_p$ depends on the universe in which the operation $\mathbf{B}_p$ is carried out. For example, if $p \in \mathfrak{M} \cap \mathbf{BC}$, then $\mathbf{B}_p$ differs from $(\mathbf{B}_p)^{\mathfrak{M}}$ in general (that is, from the set $\mathbf{B}_p$ defined in $\mathfrak{M}$); in fact, $(\mathbf{B}_p)^{\mathfrak{M}} = \mathbf{B}_p \cap \mathfrak{M}$. In our opinion it is generally simpler to deal with codes.

Lemma 4.9. *Under the assumptions of Definition* 4.8, *the set* $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$ *and the order* $\leqslant$ *belong to* $\mathfrak{M}$. *The forcing* $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$ *satisfies the* CAC *in* $\mathfrak{M}$, *that is*, *every antichain* $A \in \mathfrak{M}$, $A \subseteq \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$, *is at most countable in* $\mathfrak{M}$.

Proof. The assertion $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}} \in \mathfrak{M}$ follows readily because $\mathcal{I}$ is $\mathfrak{M}$-absolute. The relation $\mathbf{B}_p \subseteq \mathbf{B}_q$ can be expressed by a Π^1_1 formula in view of 1.11(iv). Thus, the ordering $\leqslant$ on $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$ belongs to $\mathfrak{M}$ by the absoluteness theorem (Theorem 2.8). Finally, the CAC in $\mathfrak{M}$ follows from the property 3° (with respect to 2°) of the ideal $\mathcal{I}$. $\square$

Lemma 4.10. *Under the assumption of Definition* 4.8, *let* $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$, *be a dense set in* $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$. *Then there is a set* $A \in \mathfrak{M}$, $A \subseteq D$, *countable in* $\mathfrak{M}$ *and such that the set* $U = \bigcup_{c \in A} \mathbf{B}_c$ *is* $\mathcal{I}$*-full.*

Thus, *in this case if* $x \in \mathbf{Rand}_{\mathcal{I}}\, \mathfrak{M}$, *then* $x \in \bigcup_{c \in D} \mathbf{B}_c$.

Proof. Arguing in $\mathfrak{M}$, let us choose a maximal antichain $A \subseteq D$, $A \in \mathfrak{M}$. Then A is countable in $\mathfrak{M}$ by Lemma 4.9. We claim that the set $U = \bigcup_{c \in A} \mathbf{B}_c$ is $\mathcal{I}$-full. Suppose the contrary: let $V = \mathbb{N}^\omega \smallsetminus U \notin \mathcal{I}$. Since A is countable, it follows that $V = \mathbf{B}_p$ is a Borel set with a code $p \in \mathfrak{M}$, and hence $p \in \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$. Since D is dense, there is a 'condition' $q \in D$, $q \leqslant p$. Since A is maximal, it follows that the 'condition' q (and therefore the 'condition' p as well) is compatible with some $r \in A$; in particular, $\mathbf{B}_p \cap \mathbf{B}_r \notin \mathcal{I}$ and $\mathbf{B}_p \cap \mathbf{B}_r \neq \varnothing$. However, $\mathbf{B}_r \subseteq U$, a contradiction.

To prove the last part of the lemma, we note that U is a Borel set having a code in $\mathbf{BC} \cap \mathfrak{M}$ by the choice of A. $\square$

The forcing $\mathbb{P}_{\mathcal{I}}$ enables one to use another approach to $\mathcal{I}$-random points, which is based on the following lemma.

Lemma 4.11. *If a set* $G \subseteq \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$ *is generic over* $\mathfrak{M}$, *then there is a unique point* $\pi_G \in \mathbb{N}^\omega$ *satisfying the condition* $c \in G \iff \pi_G \in \mathbf{B}_c$ *for each* $c \in \mathbf{BC} \cap \mathfrak{M}$.

Proof. We set $\mathbf{B}_G = \{\mathbf{B}_p : p \in G\}$; thus, $\mathbf{B}_G$ consists of the Borel sets themselves and not their codes. We note that any sets $X, Y \in \mathbf{B}_G$ are compatible, that is, there is a $Z \in \mathbf{B}_G$ such that $Z \subseteq X \cap Y$. The set $D_n = \{p \in \mathbb{P} : \exists\, s \in {}^n2\ (\mathbf{B}_p \subseteq \mathcal{N}_s)\}$ is dense in $\mathbb{P}$ for any n by σ-additivity. It follows that for any n there is a finite sequence $s_n \in {}^n2$ (which is unique by compatibility) such that $\exists\, p \in G\ (\mathbf{B}_p \subseteq \mathcal{N}_{s_n})$. Then $s_n \subset s_{n+1}\ \forall\, n$, and thus there is a unique point $\pi_G \in \mathbb{N}^\omega$ satisfying $\pi_G \upharpoonright n = s_n\ \forall\, n$.

Let us show that $c \in G \iff \pi_G \in \mathbf{B}_c$, $\forall\, c \in \mathbf{BC} \cap \mathfrak{M}$.

This equivalence can be proved by induction on ξ, where $c \in \mathbf{BC}_\xi$.

The base of induction: $\xi = 0$. In this case $c = \mathbf{k}$, $k \in \mathbb{N}$, and $\mathbf{B}_c = \mathcal{N}_{s_k}$. Let $n = \operatorname{lh} s_k$. It follows from the pairwise compatibility that $c \in G \iff s_k = t_n = \pi_G \upharpoonright n \iff \pi_G \in \mathbf{B}_c$.

The induction step: let $0 < \xi < \omega_1$ and let the result be valid for any $c' \in \bigcup_{\eta < \xi} \mathbf{BC}_\eta$. By definition, $\mathbf{B}_c = \mathbb{N}^\omega \smallsetminus \bigcup_n \mathbf{B}_{(c)_n}$. This readily implies that the set

$$D_c = \{p \in \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}} : \mathbf{B}_p \subseteq \mathbf{B}_c \ \vee\ \exists\, n\ (\mathbf{B}_p \subseteq \mathbf{B}_{(c)_n})\}$$

is dense in $\mathbb{P}^{\mathfrak{M}}_{\mathcal{I}}$. Therefore, by the genericity condition, there is a $p \in G \cap D_c$. Thus, $c \in G \iff \forall\, n\ ((c)_n \notin G)$, and hence $c \in G \iff \forall\, n\ (\pi_G \notin \mathbf{B}_{(c)_n})$ by the induction assumption. However, here the right-hand side is equivalent to $\pi_G \in \mathbf{B}_c$. $\square$

The points of the form π_G obtained in this way from $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-generic sets G are said to be $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-*generic* (over $\mathfrak{M}$).

Let us denote by $\breve{\,}\langle x,y\rangle$ the canonical $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-name $\breve{z}$ of the pair $z=\langle x,y\rangle$. (Unfortunately, this name differs from $\langle\breve{x},\breve{y}\rangle$.) By the *canonical name for a* $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-*generic point* we mean the $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-name

$$\underline{\pi}=\{\langle p,\breve{\,}\langle n,j\rangle\rangle : p\in\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}\wedge n,\ j\in\mathbb{N}\wedge\forall\, y\in\mathbf{B}_p\ (y(n)=j)\}\in\mathfrak{M}.$$

This name certainly depends on $\mathfrak{I}$ and $\mathfrak{M}$, but we do not indicate this dependence explicitly, because both the ideal and the ground model under consideration will always be clear from the context. Then $\pi_G=\underline{\pi}[G]$ for any set $G\subseteq\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ which is generic over $\mathfrak{M}$. Thus, under the assumptions of Definition 4.8, it follows from Lemma 4.11 that π_G is the only point in the intersection $\bigcap_{p\in G}\mathbf{B}_p=\bigcap\mathbf{B}_G$, and $\mathbf{B}_G=\{c\in\mathbf{BC}\cap\mathbf{L}[a] : \pi_G\in\mathbf{B}_c\}$. This implies that $\mathfrak{M}[G]=\mathfrak{M}[\pi_G]$.

Lemma 4.12. *The points* $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-*generic over* $\mathfrak{M}$ *are exactly the elements of the family* $\mathbf{Rand}_{\mathfrak{I}}\,\mathfrak{M}$. *Hence, if* $x\in\mathbf{Rand}_{\mathfrak{I}}\,\mathfrak{M}$, *then* $\omega_1^{\mathfrak{M}[x]}=\omega_1^{\mathfrak{M}}$.

Proof. Let us consider a set $G\subseteq\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ which is generic over $\mathfrak{M}$. If $c\in\mathbf{BC}$ and $B_c\in\mathfrak{I}$, then $c\notin G$, and hence $\pi_G\notin\mathbf{B}_c$ by Lemma 4.11. Therefore, $\pi_G\in\mathbf{Rand}_{\mathfrak{I}}\,\mathfrak{M}$. Conversely, if $x\in\mathbf{Rand}_{\mathfrak{I}}\,\mathfrak{M}$, then the set $G_x=\{c\in\mathbf{BC}\cap\mathbf{L}[a] : x\in\mathbf{B}_c\}$ is $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-generic over $\mathfrak{M}$ by Lemma 4.10. (The relation $G_x\subseteq\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ follows directly from the choice of x.) The last assertion of the lemma follows from Theorem 4.2(i) and Lemma 4.9. □

We recall that forcing is connected with truth in generic extensions by Theorem 4.3(ii). In our case it turns out that forcing of quite simple formulae is connected with truth in the ground model as well.

Lemma 4.13. *Suppose that* $\varphi(x)$ *is a* Σ^1_1 *or a* Π^1_1 *formula with parameters in* $\mathfrak{M}\cap\mathbb{N}^\omega$ *and that* $\breve{\varphi}$ *is obtained from* φ *by replacing every parameter* $z\in\mathbb{N}^\omega$ *by the* $\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$-*name* $\breve{z}$. *Let* $p\in\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$. *Then* $p\Vdash^{\mathfrak{M}}_{\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}}\breve{\varphi}(\underline{\pi})$ *if and only if the set* $X=\{x\in\mathbf{B}_p : \neg\varphi(x)\}$ *is* $\mathfrak{I}$-*null in* $\mathfrak{M}$.

Proof. If X is an $\mathfrak{I}$-null set, then we assume that $c\in\mathbf{BC}\cap\mathfrak{M}$ satisfies the conditions $\mathbf{B}_c\in\mathfrak{I}$ and $X\subseteq\mathbf{B}_c$ in $\mathfrak{M}$. By the choice of φ (Proposition 1.11(iv) is also used), the relation $X\subseteq\mathbf{B}_c$ can be expressed by a formula of type not higher than Π^1_2. Therefore, $x\notin\mathbf{B}_c\implies\varphi(x)$ for any $x\in\mathbf{B}_p$ in every extension of $\mathfrak{M}$, by the absoluteness theorem (Theorem 2.8). However, $\pi_G\in\mathbf{B}_p$ for any generic set $G\subseteq\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ containing p.

To prove the converse, suppose that X is not $\mathfrak{I}$-null. Then by Theorem 3.7, there is a condition $q\in\mathbf{BC}\cap\mathfrak{M}$ such that $\mathbf{B}_q\notin\mathfrak{I}$ and $\mathbf{B}_q\subseteq X$ in $\mathfrak{M}$. In this case $q\in\mathbb{P}^{\mathfrak{M}}_{\mathfrak{I}}$ and $q\leqslant p$. Moreover, it follows from the same considerations as above that q forces $\neg\breve{\varphi}(\underline{\pi})$, and hence p cannot force $\breve{\varphi}(\underline{\pi})$. □

4D. Two examples. Let us take $\mathfrak{I}$ to be the ideal $\mathfrak{I}_{\mathrm{cat}}$ of meagre Borel sets or the ideal $\mathfrak{I}_{\boldsymbol{\lambda}}$ of Borel sets of $\boldsymbol{\lambda}$-measure zero. (Some other examples will be discussed in subsection 9C.) We obtain two special forcings connected with the problems of the Baire property and measurability:

Cohen forcing: $\mathbb{P}_{\mathrm{coh}} = \mathbb{P}_{\mathcal{I}_{\mathrm{cat}}} = \{p \in \mathbf{BC} \cap \mathfrak{M} : \mathbf{B}_p \text{ is a non-meagre set}\}$;
random forcing:[24] $\mathbb{P}_{\boldsymbol{\lambda}} = \mathbb{P}_{\mathcal{I}_{\boldsymbol{\lambda}}} = \{p \in \mathbf{BC} : \boldsymbol{\lambda}(\mathbf{B}_p) > 0\}$.

Accordingly, $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}} = \mathbb{P}_{\mathrm{coh}} \cap \mathfrak{M} = \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}_{\mathrm{cat}}}$ and $\mathbb{P}^{\mathfrak{M}}_{\boldsymbol{\lambda}} = \mathbb{P}_{\boldsymbol{\lambda}} \cap \mathfrak{M} = \mathbb{P}^{\mathfrak{M}}_{\mathcal{I}_{\boldsymbol{\lambda}}}$. We recall that $\boldsymbol{\lambda}$ is a measure on $\mathbb{N}^\omega$ (in fact, on 2^ω) introduced in subsection 1B.

Corollary 4.14. $\mathbf{Coh}\,\mathfrak{M} = \{$*all* $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$*-generic points over* $\mathfrak{M}\}$. $\mathbf{Rand}\,\mathfrak{M} = \{$*all* $\mathbb{P}^{\mathfrak{M}}_{\boldsymbol{\lambda}}$*-generic points over* $\mathfrak{M}\}$.

It should be noted that Cohen forcing is usually associated with another partially ordered set, namely, the set $\mathbf{C} = \mathbb{N}^{<\omega}$ ordered in such a way that $s \leqslant t$ (s is stronger) if $t \subseteq s$. However, let us show that $\mathbf{C}$ and $\mathbb{P}_{\mathrm{coh}}$ produce the same generic extensions.

One can readily define a recursion function $h\colon (\mathbb{N}^\omega)^2 \to \mathbb{N}^\omega$ such that $h(p,q) \in \mathbf{BC}$ and $\mathbf{B}_p \cap \mathbf{B}_q = \mathbf{B}_{h(p,q)}$ whenever $p,\ q \in \mathbf{BC}$. By Proposition 1.11(vi), this implies that if the symmetric difference $\mathbf{B}_p \triangle \mathbf{B}_q$ is a meagre set ($p,\ q \in \mathbf{BC}$), then the binary relation $p \approx q$ is a Σ^1_2 relation, and thus its restriction to $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$ belongs to $\mathfrak{M}$. On the other hand, if $p \approx q$ belongs to $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$, then the set

$$D = \{r \in \mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}} : \mathbf{B}_r \subseteq \mathbf{B}_p \cap \mathbf{B}_q \ \ \vee\ \mathbf{B}_r \cap (\mathbf{B}_p \cup \mathbf{B}_q) = \varnothing\}$$

is dense in $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$ (and belongs to $\mathfrak{M}$). We conclude that the equivalence $p \in G \iff q \in G$ holds for any $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$-generic set G over $\mathfrak{M}$. Thus, the $\mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}$-generic extensions of $\mathfrak{M}$ coincide with the $\mathbb{P}$-generic extensions, where $\mathbb{P} = \mathbb{P}^{\mathfrak{M}}_{\mathrm{coh}}/{\approx}$, with the order given by $[p]_\approx \leqslant [q]_\approx$ if $\mathbf{B}_p \smallsetminus \mathbf{B}_q$ is a meagre set.

For any $s \in 2^{<\omega} = \mathbf{C}$ we choose in $\mathfrak{M}$ a code $p_s \in \mathbf{BC} \cap \mathfrak{M}$ such that $\mathbf{B}_{p_s} = \mathcal{N}_s$. Then the map $s \mapsto [p_s]_\approx$ is an order isomorphism between $\mathbf{C}$ and a dense subset of $\mathbb{P}$. (We use the fact that every non-meagre Borel set $X \subseteq \mathbb{N}^\omega$ is co-meagre on some Baire interval $\mathcal{N}_s$.) It remains to apply Theorem 4.2(ii).

The random forcing is also usually identified with closed sets (rather than Borel sets) of positive measure. (See subsection 8D below.) The reason is quite clear, namely, each Borel set of positive measure contains a closed subset of positive measure.

We note that the well-known *Sacks forcing* [82] also can be represented in the form $\mathbb{P}_{\mathcal{I}}$, where $\mathcal{I}$ is the ideal of all at most countable sets, though $\mathcal{I}$ is not a σ-CAC ideal, of course.

4E. If there are few non-random points, then all sets of the second projective level are measurable. Here we prove the implications $\Longleftarrow$ in Theorem 3.9, and thus in the equivalences (ii)–(v) of Theorem 3.3 and (II)–(V) of Corollary 3.4 (see Remark 3.10). The results use the following assertion.

Lemma 4.15. *Suppose that $a \in \mathbb{N}^\omega$ and that $\mathcal{I}$ is an $\mathbf{L}[a]$-absolute σ-CAC ideal. For any $\Sigma^1_2(\mathbf{L}[a])$ set $X \subseteq \mathbb{N}^\omega$ there is a Borel set $U \subseteq \mathbb{N}^\omega$ with a code in $\mathbf{L}[a]$ such that $X \cap \mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a] = U \cap \mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a]$ and $U \smallsetminus X$ is an $\mathcal{I}$-null set.*

Proof. Let $X = \{x \in \mathbb{N}^\omega : \Gamma(x,z)\}$, where $z \in \mathbf{L}[a] \cap \mathbb{N}^\omega$, Γ is a Σ^1_2 formula $\exists\, y\ \vartheta(x,z,y)$, and ϑ is a Π^1_1 formula. Then

$$x \in X \iff \Gamma(x,z) \text{ is true in } \mathbf{L}[x,a] \iff \exists\, \xi < \omega_1^{\mathbf{L}[a,x]}\ (x \in X_\xi) \tag{1}$$

[24] In other terms, Solovay forcing.

by Theorems 2.8 and 2.6(i), where $X_\xi = \{x : \vartheta(x, z, \mathbf{f}_\xi[x^\frown a])\}$ and $x^\frown a$ stands for the 'junction' of x and a, that is, $(x^\frown a)(2k) = x(k)$ and $(x^\frown a)(2k+1) = a(k)$. The equality $\omega_1^{\mathbf{L}[a,x]} = \omega_1^{\mathbf{L}[a]}$ of Lemma 4.12 enables one to improve this result as follows:

$$\text{for any } x \in \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]: \quad x \in X \iff \exists\,\xi < \omega_1^{\mathbf{L}[a]}\ (x \in X_\xi). \tag{2}$$

Let $\gamma(w, x, a, z)$ be the formula $\forall\, y\ \bigl(\varphi(w, x^\frown a, y) \Longrightarrow \vartheta(x, z, y)\bigr)$, where φ is the Σ^1_1 formula provided by Theorem 2.6(ii), and thus we have $X_\xi = \{x : \gamma(w, x, a, z)\}$ whenever $w \in \mathbf{WO}_\xi$. We set $X'_\xi = X_\xi \cap \mathbf{L}[a]$.

We argue in $\mathbf{L}[a]$. Let us define C' and $\mathcal{J}'$ as in 3° in subsection 3E for $\mathfrak{M} = \mathbf{L}[a]$. Then $X'_\xi = \{x : \gamma(w, x, a, z)\}$ (in $\mathbf{L}[a]$) whenever $\xi < \omega_1$ and $w \in \mathbf{WO}_\xi$ by the absoluteness theorem (Theorem 2.8). Therefore, all the sets X'_ξ are Π^1_1 sets. Thus, each set X'_ξ is $\mathcal{J}'$-measurable by Theorem 3.7, and hence there are codes $c_\xi \in \mathbf{BC}$ and $d_\xi \in C'$ such that $X'_\xi \triangle \mathbf{B}_{c_\xi} \subseteq \mathbf{B}_{d_\xi}$; in other words, $\mathbf{B}_{c_\xi} \smallsetminus \mathbf{B}_{d_\xi} \subseteq X'_\xi \subseteq \mathbf{B}_{c_\xi} \cup \mathbf{B}_{d_\xi}$.

However, the ideal $\mathcal{J}'$ satisfies condition 2° in $\mathbf{L}[a]$ (by the property 3° of the ideal $\mathcal{J}$). It follows that there is an ordinal $\vartheta < \omega_1^{\mathbf{L}[a]}$ such that $\mathbf{B}_{c_\xi} \smallsetminus (\bigcup_{\eta<\vartheta} \mathbf{B}_{c_\eta}) \in \mathcal{J}'$ for any $\xi \geqslant \vartheta$.

We argue in the universe of all sets. We write $U_\xi = \mathbf{B}_{c_\xi}$ and $D_\xi = \mathbf{B}_{d_\xi}$. Then $D_\xi \in \mathcal{J}$. Applying the absoluteness theorem to the corresponding formulae, we see that $X_\xi \triangle U_\xi \subseteq D_\xi$ for any $\xi < \omega_1^{\mathbf{L}[a]}$, and $U_\xi \smallsetminus U \in \mathcal{J}$ for any $\xi \geqslant \vartheta$, where $U = \bigcup_{\eta<\vartheta} U_\eta$ is obviously a Borel set with a code in $\mathbf{L}[a]$. Then $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \cap X_\xi = \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \cap U_\xi$ by the definition of $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]$ for any $\xi < \omega_1^{\mathbf{L}[a]}$, and $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \cap U_\xi \subseteq U$ for $\vartheta \leqslant \xi < \omega_1^{\mathbf{L}[a]}$. The equality $(X \triangle U) \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] = \varnothing$ follows now from (2), and $U \smallsetminus X \subseteq \bigcup_{\eta<\vartheta} U_\eta \smallsetminus X_\eta \subseteq \bigcup_{\eta<\vartheta} D_\eta$, which means that U is the desired set.

We can now prove the implications $\Longleftarrow$ in Theorem 3.9. For convenience of the references we state the result as a separate corollary.

Corollary 4.16. (a) *Under the assumptions of Lemma* 4.15, *if* $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]$ *is an* $\mathcal{J}$-*full set, then all* $\Sigma^1_2(\mathbf{L}[a])$ *sets* $X \subseteq \mathbb{N}^\omega$ *are* $\mathcal{J}$-*measurable*; (b) *under the assumptions of Lemma* 4.15, *if* $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \cap \mathbf{B}_c \neq \varnothing$ *for every Borel set* $\mathbf{B}_c \notin \mathcal{J}$ *with a code in* $\mathbf{L}[a]$, *then all* $\Delta^1_2(\mathbf{L}[a])$ *sets* $X \subseteq \mathbb{N}^\omega$ *are* $\mathcal{J}$-*measurable.*

Proof. (a) It follows from Lemma 4.15 that there is a Borel set U such that $X \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] = U \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]$. If $\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]$ is $\mathcal{J}$-full, then X is $\mathcal{J}$-measurable because the Borel set U is.

(b) Let us consider a pair of mutually complementary $\Sigma^1_2(a)$ sets $X, Y \subseteq \mathbb{N}^\omega$. Lemma 4.15 gives us Borel sets $U, V \subseteq \mathbb{N}^\omega$ with codes in $\mathbf{L}[a]$ such that

$$X \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] = U \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \quad \text{and} \quad Y \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] = V \cap \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a],$$

and the sets $U \smallsetminus X$ and $V \smallsetminus Y$ are $\mathcal{J}$-null. It remains to show that the complement $C = \mathbb{N}^\omega \smallsetminus (U \cup V)$ of $U \cup V$ is an $\mathcal{J}$-null set. Suppose the contrary; let $C \notin \mathcal{J}$. We note that C is also a Borel set with a code in $\mathbf{L}[a]$, and hence under our assumptions there exists a point $x \in \mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a] \cap C$. Since the sets X and Y are mutual complements, it follows that x belongs to one of them, say X. Then $x \in U$ by the choice of U, a contradiction. □ (*Theorems* 3.9 *and* 3.3 *and Corollary* 3.4)

If we ignore the condition that $U \smallsetminus X$ is an $\mathcal{J}$-null set, then Lemma 4.15 admits a far-reaching generalization.

Theorem 4.17. *Under the assumptions of Lemma* 4.15, *for any* $\in$*-formula* $\varphi(x)$ *with parameters in* $\mathbf{L}[a]$ *there is a Borel code* $c \in \mathbf{BC} \cap \mathbf{L}[a]$ *such that the equivalence "*$\varphi(x)$ *is true in* $\mathbf{L}[a,x] \iff x \in \mathbf{B}_c$*" holds for any* $x \in \mathbf{Rand}_{\mathcal{J}}\, \mathbf{L}[a]$.

In other words, any set of the form $X = \{x \in \mathbb{N}^\omega : \varphi(x) \text{ is true in } \mathbf{L}[a,x]\}$, where φ has parameters only in $\mathbf{L}[a]$, coincides with some Borel set with a code in $\mathbf{L}[a]$, modulo points that do not belong to $\mathbf{Rand}_{\mathcal{J}}\, \mathbf{L}[a]$. For instance, this holds for any set X in $\Sigma^1_2(a)$ or $\Pi^1_2(a)$, because the corresponding formulae are absolute for $\mathbf{L}[a]$ by the absoluteness theorem (Theorem 2.8). Some other applications of this theorem will be discussed below.

Proof. We assume that φ is $\varphi(z,x)$ with the single parameter $z \in \mathbf{L}[a]$. (The case of several parameters can readily be reduced to this case.) We recall that $\underline{\pi} \in \mathbf{L}[a]$ is a $\mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]}$-name such that $\pi_G = \underline{\pi}[G]$ for any generic set $G \subseteq \mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]}$ (see subsection 4C). Let $\Vdash$ stand for the relation of $\mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]}$-forcing over $\mathbf{L}[a]$. By Theorem 4.3(iii), the set

$$D = \{p \in \mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]} : p \Vdash \varphi(\check{z}, \underline{\pi}) \text{ or } p \Vdash \neg\varphi(\check{z}, \underline{\pi})\}$$

is dense in $\mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]}$ and belongs to $\mathbf{L}[a]$. By Lemma 4.10, there is a countable set $\{p_n : n \in \mathbb{N}\} \in \mathbf{L}[a]$ of conditions $p_n \in D$ such that $\bigcup_n \mathbf{B}_{p_n}$ is an $\mathcal{J}$-full set. We write

$$u = \{n : p_n \Vdash \varphi(\check{z}, \underline{\pi})\} \quad \text{and} \quad v = \mathbb{N} \smallsetminus u = \{n : p_n \Vdash \neg\varphi(\check{z}, \underline{\pi})\}.$$

Obviously, there exist Borel codes c, $c' \in \mathbf{BC} \cap \mathbf{L}[a]$ such that $\mathbf{B}_c = \bigcup_{n\in u} \mathbf{B}_{p_n}$ and $\mathbf{B}_{c'} = \mathbb{N}^\omega \smallsetminus \mathbf{B}_c = \bigcup_{n\in v} \mathbf{B}_{p_n}$. We claim that c is a desired code. Let $x \in \mathbf{Rand}_{\mathcal{J}}\, \mathbf{L}[a]$. Then $x \in \mathbf{B}_c \cup \mathbf{B}_{c'}$. If $x \in \mathbf{B}_c$, then $x \in \mathbf{B}_{p_n}$ for some $n \in u$. On the other hand, it follows from Lemma 4.12 that $x = \pi_G$ for some set $G \subseteq \mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]}$ which is generic over $\mathbf{L}[a]$; in fact, $G = G_x = \{p \in \mathbb{P}_{\mathcal{J}}^{\mathbf{L}[a]} : x \in \mathbf{B}_p\}$. Thus, $p_n \in G = G_x$. Therefore, by the definition of forcing, the formula $\varphi(\check{z}[G], \underline{\pi}[G])$ (that is, $\varphi(z,x)$) is true in $\mathbf{L}[a,G] = \mathbf{L}[a,x]$. Similarly, if $x \notin \mathbf{B}_c$, then $x \in \mathbf{B}_{c'}$, and the formula $\varphi(z,x)$ is false in $\mathbf{L}[a,x]$. $\square$

Historical and bibliographical remarks. Cohen's method of forcing [15] is perhaps the most important tool in set theory. We recommend [87], [29], [11], Chap. 5, as well as [10], [44], [30] in English as basic references concerning forcing and Theorems 4.1–4.3.

Our references to Theorems 3.3 and 3.9 (and Corollary 3.4), whose proofs were begun in § 3 and completed in subsections 4B and 4E, were given in the historical and bibliographical remarks to § 3.

The forcing $\mathbb{P}_{\mathcal{J}}$ for the ideal of sets of measure zero and for the ideal of meagre sets was introduced by Solovay [89], together with the notions of random and Cohen points and the basic constructions in subsections 4C–4E, in particular, including Theorem 4.17. For the equivalence of the two definitions of Cohen forcing (subsection 4D), see [89]; however, the arguments there are different.

§ 5. Combinatorics of eventual domination

This short section contains several results connecting the regularity properties of sets at the second projective level with properties of a special partial order relation on $\mathbb{N}^\omega$ that belongs to the family of *eventual relations*. This family consists of relations characterized by the requirement that some property holds for almost all[25] positive integer values.

Let f and g be functions with $\operatorname{dom} f = \operatorname{dom} g = \omega$. We recall the following definitions:

$f \leqslant^* g$ means that $f(n) \leqslant g(n)$ for almost all $n \in \omega$. (Here it is assumed that f and g take values in a fixed ordered set, for example, $\mathbb{N}$ or $\mathbb{R}$.)
$f \subseteq^* g$ means that $f(n) \subseteq g(n)$ for almost all $n \in \omega$.
$f \in^* g$ means that $f(n) \in g(n)$ for almost all $n \in \omega$.

The relation $\leqslant^*$ of *eventual domination*[26] on sets of type $\mathbb{N}^\omega$ or $\mathbb{R}^\omega$ is of special interest. In particular, each $\leqslant^*$-increasing ω-sequence $\{f_n\}_{n\in\omega}$ of $f_n \in \mathbb{N}^\omega$ is $\leqslant^*$-bounded. Indeed, we set $f(k) = \sup_{n\leqslant k} f_n(k)$. Then $f_n \leqslant^* f\ \forall\, n$. (This assertion fails for ordinary domination, of course, that is, for the relation $f \leqslant g$ given by the rule $f(n) \leqslant g(n)$ for all n.)

As an application we prove Theorem 5.4 claiming that $\mathrm{LM}(\Sigma^1_2)$ implies $\mathrm{BP}(\Sigma^1_2)$; however, the proof of this theorem involves some ideas related to Theorem 3.3. The results of this section will be used in the study of regularity properties in some complicated generic models in § 8.

5A. Eventual domination and measurability. We show here how the hypothesis $\mathrm{LM}(\Sigma^1_2)$ is connected with order properties of $\leqslant^*$ on the set

$$\boldsymbol{\ell}_1 = \left\{x \in \mathbb{Q}_+^\omega : \sum_{n\in\omega} x(n) < \infty\right\}$$

(where $\mathbb{Q}_+ = \{q \in \mathbb{Q} : q \geqslant 0\}$ stands for the set of non-negative rational numbers). We also consider the family $\mathcal{S}$ of all functions φ, $\omega = \operatorname{dom}\varphi$, taking values in the family of finite sets and satisfying the condition $\sum_n \frac{\#\varphi(n)}{(n+1)^2} < \infty$. We call these functions *slow* (that is, slowly increasing); the symbol $\#s$ stands for the number of elements in a finite set s.

Theorem 5.1 ([9] or [10], § 2.3). $\mathrm{LM}(\Sigma^1_2) \Longleftrightarrow$ (i) $\Longleftrightarrow$ (ii), *where*

(i) $\forall\, a \in \mathbb{N}^\omega\ \exists\, \varphi \in \mathcal{S}\ \ (x \in^* \varphi$ *for any* $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega)$,
(ii) $\forall\, a \in \mathbb{N}^\omega\ \exists\, f \in \boldsymbol{\ell}_1\ \ (x \leqslant^* f$ *for any* $x \in \boldsymbol{\ell}_1 \cap \mathbf{L}[a])$.

Proof. $\mathrm{LM}(\Sigma^1_2) \Longrightarrow$ (i). We fix some $a \in \mathbb{N}^\omega$. Let $\langle i, j\rangle \mapsto n_{ij}$ be a recursive bijection from $\mathbb{N}^2$ onto $\mathbb{N}$. Let us consider the probability measure μ on 2^ω defined in such a way that every set $A_{ij} = \{u \in 2^\omega : u(n_{ij}) = 1\}$ satisfies the condition $\mu(A_{ij}) = (i+1)^{-2}$ and these sets are jointly independent with respect to μ (that is, for instance, $\mu(A_{ij} \cap A_{i'j'} \smallsetminus A_{i''j''}) = \mu(A_{ij})\,\mu(A_{i'j'})\,(1 - \mu(A_{i''j''}))$.) We set $G_x = \limsup_{i\to\infty} A_{i,x(i)}$ for any $x \in \mathbb{N}^\omega$.[27] Since $\sum_i (i+1)^{-2} < \infty$, it follows that $\mu(G_x) = 0$ (by the Borel–Cantelli lemma).

By Theorem 3.3, it follows from the assumption $\mathrm{LM}(\Sigma^1_2)$ that $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$. Hence, $\mu(\mathbf{Rand}_\mu\,\mathbf{L}[a]) = 1$ (Lemma 3.2(3)). Thus, there is a closed set $B = [T] \subseteq 2^\omega$

[25] "Almost all" means all but finitely many.
[26] In the Russian literature, "final domination".
[27] We recall that $\limsup_{n\in\mathbb{N}} X_n = \bigcap_n \bigcup_{m\geqslant n} X_m$.

of positive μ-measure such that $B \cap G_x = \varnothing$ for all $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega$, where $T \subseteq 2^{<\omega}$ is a perfect tree. One can assume that $\mu(\mathcal{C}_s \cap B) > 0$ for all $s \in T$ (recall that $\mathcal{C}_s = \{u \in 2^\omega : s \subset u\}$ is a Cantor interval in 2^ω).

We claim that the set $\varphi_s(i) = \{j : B \cap \mathcal{C}_s \cap A_{ij} = \varnothing\}$ is finite for all $s \in T$ and $i \in \mathbb{N}$. Indeed, if $j \in \varphi_s(i)$, then $B \cap \mathcal{C}_s \subseteq 2^\omega \smallsetminus A_{ij}$. Hence, since the sets A_{ij}, $j \in \varphi_s(i)$, are independent with respect to the measure and each of them has the same measure $(i+1)^{-2}$, we would have $\mu(B \cap \mathcal{C}_s) = 0$ if $\varphi_s(i)$ were infinite.

We now claim that every φ_s is a slow map. To prove this, we note that the set $B \cap \mathcal{C}_s$ of positive measure is disjoint from A_{ij} for any $i \in \mathbb{N}$, $j \in \varphi_s(i)$. Therefore,

$$\sum_{i \in \mathbb{N},\, j \in \varphi_s(i)} (i+1)^{-2} = \sum_i \frac{\#\varphi_s(i)}{(i+1)^2} < \infty$$

again by the Borel–Cantelli lemma.

We claim further that there is a *unique* function $\varphi \in \mathcal{S}$ such that $\varphi_s \subseteq^* \varphi$ for all $s \in T$. Indeed, for any k the function

$$\varphi_k(i) = \bigcup_{s \in T,\, \operatorname{lh} s \leqslant k} \varphi_s(i),$$

is obviously slow, together with all the functions φ_s. Therefore, there is an increasing sequence of natural numbers $m_0 < m_1 < m_2 < \cdots$ such that

$$\sum_{m_k \leqslant i < m_{k+1}} \frac{1}{\#\varphi_k(i)} \geqslant 1.$$

It remains to define $\varphi(i) = \varphi_k(i)$ for $m_k \leqslant i < m_{k+1}$.

Let us prove that φ is the desired function. Let $x \in \mathbf{L}[a]$. Since $B \cap G_x = \varnothing$, there exist $s \in T$ and i_0 such that $B \cap \mathcal{C}_s \cap \bigl(\bigcup_{i \geqslant i_0} A_{i,x(i)}\bigr) = \varnothing$. This means that $x(i) \in \varphi_s(i)$ for $i \geqslant i_0$, and hence $x \in^* \varphi_s \subseteq^* \varphi$, as was to be proved.

(i) $\Longrightarrow$ (ii). We fix an $a \in \mathbb{N}^\omega$. For any $f \in \boldsymbol{\ell}_1$ there is a function $y_f \in \mathbf{L}[a] \cap \mathbb{N}^\omega$ such that $\sum_{k \geqslant y_f(n)} f(n) \leqslant 2^{-n}$. Let us choose φ according to (i) and then set $y(n) = \sup \varphi(n)$. Thus, $y \in \mathbb{N}^\omega$ and $y_f \leqslant^* y$ for any $f \in \boldsymbol{\ell}_1 \cap \mathbf{L}[a]$.

For any $f \in \boldsymbol{\ell}_1 \cap \mathbf{L}[a]$ we now define $f'(n) = f \restriction [y(n), y(n+1))$, and therefore $f'(n)$ is a finite sequence of rational numbers. Thus, applying (i) to the set $\boldsymbol{\ell}_1 \cap \mathbf{L}[a, y]$, we find a function $\varphi \in \mathcal{S}$ satisfying $f' \in^* \varphi$ for each $f \in \boldsymbol{\ell}_1 \cap \mathbf{L}[a]$. Then $\sum_n \frac{\#\varphi(n)}{(n+1)2} < \infty$, and one can assume that $\#\varphi(n) \leqslant (n+1)^2$ for any n. Moreover, using the special form of the functions f', we can assume that $\varphi(n)$ consists of functions $s \colon [y(n), y(n+)) \to \mathbb{Q}_+$ such that $\sum_{y(n) \leqslant i < y(n+1)} s(n) \leqslant 2^{-n}$.

We set $h(i) = \sup_{s \in \varphi(n)} s(i)$ for $y(n) \leqslant i < y(n+1)$. Under our assumptions, $\sum_i h(i) \leqslant \sum_n \sum_{s \in \varphi(n)} \sum_{y(n) \leqslant i < y(n+1)} s(n) \leqslant \sum_n n^2\, 2^{-n} < \infty$, and thus $h \in \boldsymbol{\ell}_1$. On the other hand, $f \leqslant^* h$ for any $f \in \boldsymbol{\ell}_1 \cap \mathbf{L}[a]$.

(ii) $\Longrightarrow$ $\mathrm{LM}(\Sigma^1_2)$. We begin with the following lemma.

Lemma 5.2. *If* $X \subseteq 2^\omega$, $\boldsymbol{\lambda}(X) = 0$, *and* $\varepsilon_n \in \mathbb{R}$ *with* $\varepsilon_n > 0$ *for all* $n \in \mathbb{N}$, *then there is a sequence of open-closed sets* $C_n \subseteq 2^\omega$ *such that* $\boldsymbol{\lambda}(C_n) < \varepsilon_n$ *and* $X \subseteq \limsup_n C_n$.

Proof. First, there is a system of Cantor intervals $I_{kl} \subseteq 2^\omega$ with $X \subseteq \bigcap_k \bigcup_l I_{kl}$ and $\sum_l \boldsymbol{\lambda}(I_{kl}) < \varepsilon_0 2^{-k-1}\ \forall\, k$. We index the intervals: let $\{J_n : n \in \mathbb{N}\} = \{I_{kl} : k, l \in \mathbb{N}\}$. Then $\sum_n \boldsymbol{\lambda}(J_n) < \varepsilon_0$ and $X \subseteq \limsup_n J_n$.

We now set $k_n = \min k : \sum_{m \geqslant k} \boldsymbol{\lambda}(J_m) \leqslant \varepsilon_n$ and $C_n = \bigcap_{k_n \leqslant k < k_{n+1}} J_k$ for any n. This proves the lemma. □ (*Lemma* 5.2)

Continuing the proof of the theorem (the implication (ii) $\Longrightarrow$ $\mathrm{LM}(\Sigma^1_2)$), we take an arbitrary $a \in \mathbb{N}^\omega$ and show that $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$. This is sufficient by Theorem 3.3. Let $\{X_\xi : \xi < \omega_1\}$ be the family of all Borel sets of $\boldsymbol{\lambda}$-measure zero that have codes in $\mathbf{L}[a]$. We must prove that $\boldsymbol{\lambda}(\bigcup_\xi X_\xi) = 0$.

Let us fix a recursive indexing $\{C_n : n \in \mathbb{N}\}$ of all open-closed subsets of 2^ω. Using Lemma 5.2 in $\mathbf{L}[a]$, we can find for any $\xi < \omega_1$ a function $x_\xi \in \mathbf{L}[a] \cap \mathbb{N}^\omega$ such that $X_\xi \subseteq \limsup_n C_{x_\xi(n)}$ and $\boldsymbol{\lambda}(C_{x_\xi(n)}) < 2^{-n}$ for any n. We set $f_\xi(k) = \boldsymbol{\lambda}(C_k)$ if $k \in \operatorname{ran} x_\xi = \{x_\xi(n) : n \in \mathbb{N}\}$ and $f_\xi(k) = 0$ otherwise. All functions f_ξ belong to $\boldsymbol{\ell}_1 \cap \mathbf{L}[a]$, and hence, by the assumption (ii), there is a function $f \in \boldsymbol{\ell}_1$ such that $f_\xi \leqslant^* f\ \forall\, \xi$.

We set $K = \{k : f(k) \geqslant \boldsymbol{\lambda}(C_k)\}$. Suppose that $x \in \mathbb{N}^\omega$ enumerates all the elements of K in increasing order. We claim that

$$X_\xi \subseteq X = \limsup_n C_{x(n)}\ \forall\, \xi, \quad \text{and} \quad \boldsymbol{\lambda}(X) = 0.$$

First, $\sum_n \boldsymbol{\lambda}(C_{x(n)}) = \sum_{k \in K} \boldsymbol{\lambda}(C_k) \leqslant \sum_k f(k) < \infty$ (because $f \in \boldsymbol{\ell}_1$), and hence $\boldsymbol{\lambda}(X) = 0$ by the Borel–Cantelli lemma. Further, if $z \in X_\xi$, then the set $K_\xi = \{k \in \operatorname{ran} x_\xi : z \in C_k\}$ is infinite, and $\boldsymbol{\lambda}(C_k) = f_\xi(k) \leqslant f(k)$ for almost all $k \in K_\xi$. Thus, the set $\{k \in K : z \in C_k\}$ is infinite by the definition of K. This implies that $z \in X$, as was to be proved.

□ (*Theorem* 5.1)

5B. Eventual domination and the Baire property. Among several known results relating to $\mathrm{BP}(\Sigma^1_2)$ and similar to Theorem 5.1 (see the survey in [10], 2.2 and 2.4), the following theorem ([10], 9.3.3) is the most useful here. This theorem combines several separate earlier results, in particular, those in [96]. A set $X \subseteq \mathbb{N}^\omega$ is said to be $\leqslant^*$-*bounded* if there is an $h \in \mathbb{N}^\omega$ such that $\forall\, x \in X\ (x \leqslant^* h)$. (Boundedness of X is obviously equivalent to the possibility of covering X by a σ-compact set.)

Theorem 5.3. *Assume* $\mathrm{BP}(\Delta^1_2)$. *Then* $\mathrm{BP}(\Sigma^1_2)$ *is equivalent to the statement that every set of the form* $\mathbf{L}[a] \cap \mathbb{N}^\omega$, $a \in \mathbb{N}^\omega$, *is* $\leqslant^*$-*bounded in* $\mathbb{N}^\omega$.

Proof. Suppose that $\mathrm{BP}(\Sigma^1_2)$ holds and consider an arbitrary $a \in \mathbb{N}^\omega$. The set $\mathbf{Coh}\,\mathbf{L}[a]$ is co-meagre by Theorem 3.3. For $f \in \mathbb{N}^\omega$ we set $f'(k) = 1 + \max_{i \leqslant k} f(i)$ $\forall\, k$, and $\Phi(f) = \{x \in \mathbb{N}^\omega : x \leqslant^* f'\}$. The last set is meagre for any f. In addition, this is a Borel set with a code in $\mathbf{L}[a]$ for any $f \in \mathbf{L}[a] \cap \mathbb{N}^\omega$, that is, $\Phi(f) \cap \mathbf{Coh}\,\mathbf{L}[a] = \varnothing$ in this case. Hence, there is an $\mathbf{F}_\sigma$ set $Z = \bigcup_n Z_n \subseteq \mathbb{N}^\omega$ such that any Z_n is a closed nowhere dense set and $\Phi(f) \subseteq Z$ for any $f \in \mathbf{L}[a] \cap \mathbb{N}^\omega$. Using this set, we define a number $k_n \in \mathbb{N}$ and an $s_n \in \mathbb{N}^{<\omega}$ by induction on n.

We set $k_0 = 0$. If k_n has already been constructed, then there is an $s \in \mathbb{N}^{<\omega}$ for which $\mathcal{N}_{t^\wedge s} \cap Z_j = \varnothing$ whenever $j \leqslant n$, and $t \in \mathbb{N}^{<\omega}$ is such that $\operatorname{lh} t = k_n$ and $\operatorname{ran} t \subseteq [0, k_n)$. For s_n we take any number s of this kind and set

$$k_{n+1} = k_n + 1 + \operatorname{lh} s_n + \max_{i<\operatorname{lh} s_n} s_n(i),$$

which completes the induction step of constructing s_n and k_n. We now set $h(n) = \max_{i<\operatorname{lh} s_n} s_n(i)\ \forall\, n$, and claim that $x \leqslant^* h$ for any $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega$.

Suppose the contrary: let the set $A = \{n : x(n) > h(n)\}$ be infinite.

Let $A = \{a_n : n \in \mathbb{N}\}$ be an indexing in increasing order. We set

$$z = u_0{}^\wedge s_{a_0}{}^\wedge u_1{}^\wedge s_{a_1}{}^\wedge \cdots {}^\wedge u_n{}^\wedge s_{a_n} \cdots,$$

where u_n is a sequence of $k_{a_n} - \sum_{j<n} k_{a_j} - \sum_{j<n} \operatorname{lh} s_{a_j}$ zeros, and hence every block s_{a_n} is preceded by exactly k_{a_n} numbers. This implies that $z \notin Z = \bigcup_n Z_n$ by the definition of s_n.

To arrive at a contradiction, it remains to prove that $z \in \Phi(x)$, in other words, $z \leqslant^* x'$. By construction, every $z(i)$ is either zero or a number of the form $s_{a_j}(l)$, where $k_{a_j} \leqslant i$. In the latter case we have $z(i) \leqslant h(a_j) \leqslant x(a_j) \leqslant x'(i)$, as was to be proved.

Conversely, suppose that $\mathrm{BP}(\Delta^1_2)$ holds and every set of the form $\mathbf{L}[a] \cap \mathbb{N}^\omega$, $a \in \mathbb{N}^\omega$, is $\leqslant^*$-bounded in $\mathbb{N}^\omega$. Let us fix an $a \in \mathbb{N}^\omega$ and prove that the set $\mathbf{Coh}\,\mathbf{L}[a]$ is co-meagre; this is sufficient for the validity of $\mathrm{LM}(\Sigma^1_2)$ by Corollary 3.4.

To begin with, we note that $\mathbf{Coh}\,\mathbf{L}[a] \neq \varnothing$, again by Corollary 3.4. Let us fix an arbitrary element $b \in \mathbf{Coh}\,\mathbf{L}[a]$. Let $\mathcal{T}$ be the set of all trees $T \in \mathfrak{M}$, $T \subseteq \mathbb{N}^{<\omega}$, such that $[T]$ is a nowhere dense (closed) subset of $\mathbb{N}^\omega$ and T has no $\subseteq$-maximal elements. Then $b \notin \bigcup_{T\in\mathcal{T}}[T]$, because the union coincides with the complement of $\mathbf{Coh}\,\mathbf{L}[a]$. (Indeed, any meagre set can be covered by a union of countably many nowhere dense closed sets, and the notions used are absolute by Theorem 2.8 and Proposition 1.11(vi).)

Further, the set X of all $x \in \mathbb{N}^\omega$ such that $x(n) = b(n)$ for almost all n is countable. Let $X = \{x_n : n \in \mathbb{N}\}$ be an indexing of the elements of X in $\mathbf{L}[a,b]$. It follows from what was said above that $X \cap [T] = \varnothing$ for each $T \in \mathcal{T}$. We denote by $h_T(n)$ the smallest k such that the Baire interval $\mathcal{N}_{x_n\restriction k}$ is disjoint from $[T]$. Thus, $h_T \in \mathbf{L}[a,b] \cap \mathbb{N}^\omega$. However, under our assumptions the set $\mathbf{L}[a,b] \cap \mathbb{N}^\omega$ is $\leqslant^*$-bounded, that is, there is a point $z \in \mathbb{N}^\omega$ such that $h_T \leqslant^* z$ for each $T \in \mathcal{T}$. Hence, the dense $\mathbf{G}_\delta$ set $G = \bigcap_m \bigcup_{n\geqslant m} \mathcal{N}_{x_n\restriction z(n)}$ is disjoint from $[T]$ for $T \in \mathcal{T}$. In other words, G is included in $\mathbf{Coh}\,\mathbf{L}[a]$. $\square$

5C. In the class $\mathbf{\Sigma^1_2}$ measurability implies the Baire property. Here we prove the following theorem.

Theorem 5.4 (Raisonnier and Stern [80]). $\mathrm{LM}(\Sigma^1_2)$ *implies* $\mathrm{BP}(\Sigma^1_2)$.

We show below that the converse implication fails, a clear violation of the duality between the properties of measure and category which is observed in many other cases. On the other hand, the theorem cannot be regarded as a connection between measure in general and category in general. Indeed, the assertion of the theorem fails, for instance, for the classes Δ^1_2 and Δ^1_3 (see below) and thus is in fact a specific feature of the class Σ^1_2.

Proof. Assume $\mathrm{LM}(\Sigma^1_2)$ and fix an arbitrary $a \in \mathbb{N}^\omega$. Then $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$ by Theorem 3.3. We must prove that $\mathbf{Coh}\,\mathbf{L}[a]$ is a co-meagre set, in other words, that the union of all nowhere dense closed subsets of $\mathbb{N}^\omega$ with a code in $\mathbf{L}[a]$ is a co-meagre set. Since 2^ω contains a dense $\mathbf{G}_\delta$ set (with a recursive code) homeomorphic to $\mathbb{N}^\omega$, it suffices to prove the claim for the space 2^ω instead of $\mathbb{N}^\omega$. Let M be the family of all closed nowhere dense sets $C \subseteq 2^\omega$ with code in $\mathbf{L}[a]$. (By a *code* of a closed set $X \subseteq 2^\omega$ we mean the tree $T_X = \{u \restriction n : u \in X \wedge n \in \mathbb{N}\} \subseteq 2^{<\omega}$.) Thus, let us prove that the union of all the sets $C \in M$ is a meagre set in 2^ω.

By Theorem 5.1, there is a function $\varphi \in \mathcal{S}$ such that $x \in^* \varphi$ (that is, $x(n) \in \varphi(n)$ for almost all n) for each $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega$. We claim that $\sum_n \frac{1}{\#\varphi(n)} = \infty$. Indeed, let us write $U = \{n : \#\varphi(n) \geqslant n+1\}$. Then $\sum_{n\in U} \frac{1}{n+1} \leqslant \sum_{n\in U} \frac{\#\varphi(n)}{(n+1)2} < \infty$, and therefore $\sum_{n\notin U} \frac{1}{n+1} = \infty$ and $\sum_n \frac{1}{\#\varphi(n)} = \infty$, as was to be proved.

Our final objective is to get from φ a dense $\mathbf{G}_\delta$ set disjoint from every $C \in M$. This requires some additional work. First of all, if $C \in M$ and $n \in \mathbb{N}$, then there exist a number $k > n$ and a function $\sigma \in {}^{[n,k)}2$ such that the set $\mathcal{C}_{s^\wedge\sigma} = \{u \in 2^\omega : s^\wedge\sigma \subset u\}$ is disjoint from C for any $s \in {}^n2$. (This argument would not work for the space $\mathbb{N}^\omega$!) We denote by $k_C(n)$ and $\sigma_C(n)$ the smallest number $k > n$ and the lexicographically 'leftmost' function $\sigma \in {}^{[n,k)}2$ of the above form. The set $\mathcal{N}_{\sigma_C(n)} = \{u \in 2^\omega : \sigma_C(n) \subset u\}$ is still disjoint from C.

Clearly, if C has a code in $\mathbf{L}[a]$, then $k_C \in \mathbf{L}[a]$. Therefore, by Theorem 5.1, there is a function $\psi \in \mathcal{S}$ such that $k_C \in^* \psi$ for each $C \in M$. Then the function $g(i) = 1 + \max_{i' \leqslant i} \max \psi(i')\ \forall i$, satisfies $k_C \leqslant^* g$ for all $C \in M$. We note that g is an increasing positive function. We set $e_0 = 0$ and go on with $e_{l+1} = g(e_l)$ by induction on l. Let $C \in M$. For any l, if $e_{l+1} \geqslant k_C(e_l)$ (this is the case for almost all l, because g eventually dominates k_C), then there is a $\tau \in {}^{[e_l,e_{l+1})}2$ such that $\mathcal{C}_\tau \cap C = \varnothing$. We denote by $\tau_C(l)$ the 'leftmost' function $\tau \in {}^{[e_l,e_{l+1})}2$ with this property. If $e_{l+1} < k_C(e_l)$, then, to be definite, let $\tau_C(l)$ consist of zeros only. Thus, τ_C maps $\mathbb{N}$ into the (countable) set $\Sigma = \bigcup_{n<k\in\mathbb{N}} {}^{[n,k)}2$.

Lemma 5.5. *There is a function $h\colon \omega \to \Sigma$ such that the set $\{l : \tau_C(l) = h(l)\}$ is infinite for any $C \in M$.*

Proof of the lemma. Again by Theorem 5.1, there is a function $\eta \in \mathcal{S}$ such that $\tau_C \in^* \eta$ holds for any $C \in M$. One can assume that $\eta(l) \subseteq {}^{[e_l,e_{l+1})}2\ \forall l$. Let us equip the space $\Omega = \prod_{l\in\omega} \eta(l)$ with a measure ν which is the product of the uniform probability measures on the (finite) sets $\eta(l)$. We set $A^\sigma_l = \{w \in \Omega : w(l) = \sigma\}$ for any $l \in \mathbb{N}$ and $\sigma \in \varphi(l)$. These sets are obviously independent with respect to the measure ν; moreover, if $v \in \Omega$, then $\sum_l \nu(A^{v(l)}_l) = \sum_l \eta(l)^{-1} = \infty$, because $\eta \in \mathcal{S}$. By the Borel–Cantelli lemma, the set $X_v = \limsup_{l\in\mathbb{N}} A^{v(l)}_l$ has full ν-measure.

Let us now take an $a' \in \mathbb{N}^\omega$ such that both a (see the beginning of the proof of the theorem) and η belong to $\mathbf{L}[a]$. Then $\tau_C \in \mathbf{L}[a']$ for any $C \in M$. In our assumptions, $\mathbf{Rand}\,\mathbf{L}[a'] \neq \varnothing$, and therefore $\mathbf{Rand}_\nu\,\mathbf{L}[a'] \neq \varnothing$ by Lemma 3.2(3).

Thus, there is an $h \in \Omega$ belonging to each of the sets X_v, $v \in \Omega \cap \mathbf{L}$. This means that the set $\{l : h(l) = v(l)\}$ is infinite for any v of this kind, that is, h is the desired function by the choice of η.

$\square$ (*Lemma* 5.5)

To complete the proof of Theorem 5.4, we consider a function h given by the lemma. According to the definition of τ_C, we can assume that $h(l) \subseteq {}^{[e_l,e_{l+1})}2\ \forall\, l$. Let us take a point $z \in 2^\omega$ defined by the condition $z \restriction [e_l, e_{l+1}) = h(l)\ \forall\, l$. Let $Z = \{z_n : n \in \mathbb{N}\}$ be the (countable) set of all $z' \in 2^\omega$ such that $z'(k) = z(k)$ for almost all k. Then it follows from the definition of τ_C and z and from the choice of h that $Z \cap C = \varnothing$ for any $C \in M$. Thus, for any pair formed by some $C \in M$ and some n there is an index j such that $\mathcal{C}_{z_n \restriction j} \cap C = \varnothing$; let $j_C(n)$ be the smallest index j with this property. Then every function j_C, $C \in M$, belongs to $\mathbf{L}[a, z]$, and hence, as above, there is a point $r \in \mathbb{N}^\omega$ eventually dominating every function j_C, $C \in M$, that is, $j_C \leqslant^* r$. We write $Y_n = \mathcal{C}_{z_n \restriction r(n)}$. Then the $\mathbf{G}_\delta$ set $U = \limsup_n Y_n$ is dense in 2^ω (here the denseness of Z is important) and is disjoint from any set $C \in M$.

$\square$ (*Theorem* 5.4)

Historical and bibliographical remarks. The relation $\leqslant^*$ of eventual domination goes back to old papers of Du Bois Reymond in the 1870s (see, for instance, [18]), where it was used to study the comparative rate of growth of infinite sequences. Hausdorff [22] proved for this relation his famous (ω_1, ω_1)-gap theorem (which was re-proved in [24] and in fact became known from this new proof, because the paper [22] was obviously ahead of its time and moreover was published in a little-known provincial journal).

Starting from the 1960s, the relation $\leqslant^*$ (together with some similar relations including almost disjointness of sets of natural numbers) became the topic of numerous investigations (see, for instance, the surveys [16], [71]).

§ 6. 'Elementary' proof of one of the theorems

Let us return to the results indicated by the non-trivial implications in the diagram in the Introduction. Obviously, these results express propositions of a purely descriptive set-theoretic nature and are related to properties of point sets (in contrast to, say, the equivalences in Theorems 3.3 and 3.9 in which some notions related to Gödel constructibility are involved). On the other hand, the arguments forming the proofs of these assertions (that is, the proofs above of Theorems 3.3 and 3.9) invoke objects (like arbitrary ordinals and constructible sets) and methods (like forcing) which are beyond the framework of objects naturally considered in descriptive set theory, where the latter objects include:

($\star$) natural numbers and real numbers, countable ordinals, points of $\mathbb{N}^\omega$, Polish spaces in general and their points, projective and especially Borel subsets of Polish spaces, countable and transfinite (of length $\leqslant \omega_1$) 'effective' sequences of points and of projective (or Borel) sets (for example, those arising in the definition of constructible sets up to the step ω_1), as well as simple combinations of these concepts.

For example, let us consider the following theorem, which is one of the two implications in the diagram in the Introduction that come out of the block $\mathrm{PK}(\Pi^1_1)$.[28]

Theorem 6.1 (Lyubetskii's theorem). $\mathrm{PK}(\Pi^1_1)$ *implies* $\mathrm{LM}(\Sigma^1_2)$.

The question of whether or not this theorem can be proved within the framework of the notions listed in $(\star)$ is of interest and importance from the point of view of the philosophy and methodology of mathematics. In this sense it can be compared, for instance, with the question of whether Fermat's Last Theorem can be re-proved by methods of elementary number theory.

If we seek a proof of Theorem 6.1 in the even more restricted framework of classical descriptive set theory as it was in Luzin's times, then the question remains open. The problem of eliminating the constructibility and forcing from the proof of Theorem 6.1 remains unsolved. However, one can achieve the more modest goal of modifying the arguments in § 3 and § 4, including the use of forcing, so as to essentially keep within the framework of $(\star)$. This section presents such a modification based of some ideas outlined in [69].

We call the proof below 'elementary', because here this term refers to restrictions concerning the tools used in the proof rather than to the complexity of the arguments. In fact, this proof is somewhat more complicated, because we are to verify basic properties of forcing in a new situation, properties established long ago in the standard presentation.

6A. Analysis of the proof. The proof of Theorem 6.1 can be obtained from the following parts, which are fragments of the proofs of Theorems 3.3 and 3.9 in §§ 3, 4:

Part 1: the implication $\mathrm{PK}(\Pi^1_1) \implies \forall\, a \in \mathbb{N}^\omega\ (\mathbf{L}[a] \cap \mathbb{N}^\omega$ *countable*), which was proved (in a more effective form) in subsection 3C;

Part 2: the implication $\forall\, a \in \mathbb{N}^\omega\ (\mathbf{L}[a] \cap \mathbb{N}^\omega$ *countable*) $\implies \boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$;

Part 3: the implication that *if* $a \in \mathbb{N}^\omega$ *and* $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$, *then all* $\Sigma^1_2(a)$ *sets are* $\boldsymbol{\lambda}$*-measurable*. This is the special case $\mathcal{I} = \mathcal{I}_{\boldsymbol{\lambda}}$ of Corollary 4.16.

In principle, Part 1 satisfies the requirement of staying within the framework indicated in $(\star)$. (The Gödel construction is used only up to the step ω_1.) Part 2 is entirely trivial. However, Part 3 employs forcing, which essentially goes outside the framework of $(\star)$ in the standard presentation, for example, because it involves true classes of sets, in particular, classes of the form $\mathbf{L}[a]$ and their generic extensions. Thus, our objective is to prove Corollary 4.16 in the framework of the objects in the list $(\star)$.

6B. Descriptive continua. To begin with, we need a type of structures in $(\star)$ which can adequately enough replace transitive models of **ZFC** (for example, the classes $\mathbf{L}[a]$) as ground models for generic extensions. By a *descriptive continuum* we mean any set $\mathsf{M} \subseteq \mathbb{N}^\omega$ such that the structure $\langle \mathbb{N}; \mathsf{M}\rangle$ is an $\mathbf{L}[a]$-model of second-order arithmetic. This includes the following three requirements.

[28] The text below can be applied also to the other implication $\mathrm{PK}(\Pi^1_1) \Longrightarrow \mathrm{BP}(\Sigma^1_2)$ with minor changes, which we omit for lack of space.

Comprehension: if $\varphi(k,l,x_1,\dots,x_n)$ is an analytic formula (as in subsection 1C), $a_1,\dots,a_n \in \mathsf{M}$, and $\{\langle k,l\rangle : \varphi^{\mathsf{M}}(k,l,a_1,\dots,a_n)\}$ is the graph of an (everywhere defined) function $b\colon \mathbb{N}\to\mathbb{N}$, then $b\in\mathsf{M}$.

Countable dependent choice: if $\varphi(y,z,x_1,\dots,x_n)$ is an analytic formula, a_1, $\dots$, $a_n \in \mathsf{M}$, and $\forall\, y\in\mathsf{M}\ \exists\, z\in\mathsf{M}\ \varphi^{\mathsf{M}}(y,z,a_1,\dots,a_n)$, then there is a $b\in\mathsf{M}$ such that $\forall\, k\ \varphi^{\mathsf{M}}((b)_k,(b)_{k+1},a_1,\dots,a_n)$.

Absoluteness of well ordering: if $w\in\mathsf{M}\smallsetminus\mathbf{WO}$, that is, $Q_w=\{q_n : w(n)=0\}$ is not a well-ordered subset of $\mathbb{Q}$ (in the notation of Example 1.10.1), then there is an $a\in\mathsf{M}$ encoding a decreasing sequence in Q_w in the sense that $a(n)\in Q_w$ and $q_{a(n+1)}<q_{a(n)}$ for all n.

In these definitions φ^{M} stands for the *relativization* of the formula φ to M, that is, every quantifier $\exists\, x$ and $\forall\, x$ (which means $\exists\, x\in\mathbb{N}^\omega$ and $\forall\, x\in\mathbb{N}^\omega$) is changed to $\exists\, x\in\mathsf{M}$ and $\forall\, x\in\mathsf{M}$, respectively.

For example, $\mathbb{N}^\omega$ itself is a descriptive continuum, as well as any set of the form $\mathfrak{M}\cap\mathbb{N}^\omega$, where $\mathfrak{M}$ is a transitive model of **ZFC** theory or at least of $\mathbf{ZFC}^-$ theory (without the power set axiom). Conversely, if $\mathsf{M}\subseteq\mathbb{N}^\omega$ is a descriptive continuum, then there is a transitive model $\mathfrak{M}$ of $\mathbf{ZFC}^-$ theory (which need not be a model of **ZFC**) such that $\mathsf{M}=\mathfrak{M}\cap\mathbb{N}^\omega$. (We do not use this result below.)

We write $\operatorname{hgt}\mathsf{M}=\sup_{w\in\mathsf{M}\cap\mathbf{WO}}|w|$ (the *height* of M) for any descriptive continuum M and recall that $|w|<\omega_1$ is the ordinal encoded by w.

The following theorem expresses in essence the same property as the absoluteness theorem (Theorem 2.8), and the proof is the same. As usual, a closed analytic formula Φ is said to be *absolute for* M if $\Phi\iff\Phi^{\mathsf{M}}$.

Theorem 6.2. *Suppose that* M *is a descriptive continuum. In this case*

(i) *every* Σ^1_1 *formula* Φ *with parameters in* M *is absolute for* M;

(ii) *if* $\operatorname{hgt}\mathsf{M}=\omega_1$ (this is a typical case below), *then every* Σ^1_2 *formula* Φ *with parameters in* M *is absolute for* M.

6C. Generic extensions of descriptive continua. In what follows we fix a point $a\in\mathbb{N}^\omega$ and an $\mathbf{L}[a]$-absolute σ-CAC ideal $\mathfrak{I}$ (see subsection 3E). The descriptive continuum $\mathsf{M}=\mathbf{L}[a]\cap\mathbb{N}^\omega$ will be the ground model, and for the forcing we take the set $\mathbb{P}^{\mathsf{M}}_{\mathfrak{I}}=\mathbb{P}^{\mathbf{L}[a]}_{\mathfrak{I}}=\{p\in\mathsf{M}\cap\mathbf{BC} : \mathbf{B}_p\notin\mathfrak{I}\}$. Since we intend to argue as close to M as possible and, correspondingly, to refer to $\mathbf{L}[a]$ as little as possible, we begin with the *translation to the* M-*terminology* of the $\mathbf{L}[a]$-meaning of the notion of $\mathbf{L}[a]$-absolute σ-CAC ideal.

Proposition 6.3. *The set* $\operatorname{cod}\mathfrak{I}\cap\mathsf{M}=\{p\in\mathbf{BC}\cap\mathsf{M} : \mathbf{B}_p\in\mathfrak{I}\}$ *is definable in* M *by a* Π^1_1 *formula with the parameter* a, *and the set* $\mathbb{P}^{\mathsf{M}}_{\mathfrak{I}}$ (*complementary to* $\mathbf{BC}\cap\mathsf{M}$) *is definable in* M *by a* Σ^1_1 *formula with the parameter* a.

If the set $A\subseteq\mathbf{BC}\cap\mathsf{M}$ *is definable in* M *by an analytic formula with parameters in* M *and* $\mathbf{B}_c\cap\mathbf{B}_{c'}\in\mathfrak{I}$ *for* $c\neq c'\in A$, *then* A *is countable, and there is a* $z\in\mathsf{M}$ *such that* $A=\{(z)_k : k\in\mathbb{N}\}$.

Finally (= Lemma 4.10), *if a set* $D\subseteq\mathbb{P}^{\mathsf{M}}_{\mathfrak{I}}$ *is definable in* M *by an analytic formula with parameters in* M, *then there is a* $c\in\mathsf{M}$ *such that* $A=\{(c)_n : n\in\mathbb{N}\}\subseteq D$ *and* $\bigcup_{p\in A}\mathbf{B}_p$ *is an* $\mathfrak{I}$-*full set.*

The extension. It was proved in subsection 4C that the forcing $\mathbb{P}^{\mathsf{M}}_{\mathcal{J}}$ naturally produces $\mathcal{J}$-random points $x \in \mathbb{N}^\omega$, that is, elements of the set $\mathbf{Rand}_{\mathcal{J}}\,\mathsf{M}$. However, the problem is to define an extension of M by x in the framework of the structure of $\mathbb{N}^\omega$. Our idea is to define the extension as the family of all points of the form $F(x)$, where $F\colon \mathbb{N}^\omega \to \mathbb{N}^\omega$ is a Borel function with a code in M.

Let us fix a recursive homeomorphism $H\colon (\mathbb{N}^\omega)^2 \xrightarrow{\text{onto}} \mathbb{N}^\omega$ together with a pair of mutually inverse functions $H_1, H_2\colon \mathbb{N}^\omega \xrightarrow{\text{onto}} \mathbb{N}^\omega$. We set $\mathsf{F}_c = \{\langle H_1(z), H_2(z)\rangle : z \in \mathbf{B}_c\}$ for any $c \in \mathbf{BC}$; this is a Borel subset of $\mathbb{N}^\omega \times \mathbb{N}^\omega$. Let $\mathbf{BF}$ be the set of all $\tau \in \mathbf{BC}$ such that F_τ is the graph of a (Borel) function from $\mathbb{N}^\omega$ to $\mathbb{N}^\omega$. If $\tau \in \mathbf{BF}$, then we set $\tau\text{'}x = \mathsf{F}_\tau(x)$ for any $x \in \mathbb{N}^\omega$.

We write $\mathsf{M}\lceil x\rceil = \{\tau\text{'}x : \tau \in \mathsf{M} \cap \mathbf{BF}\}$.[29]

It can hardly be expected that $\mathsf{M}\lceil x\rceil$ is again a descriptive continuum in the most general case; however, this holds in some important cases. (See, for instance, Theorem 6.6 below.)

Forcing. By a $\underline{\pi}$-*formula* we mean any analytic formula Φ (see subsection 1C) without parameters in $\mathbb{N}^\omega$ and in which some or all free variables of type 1 are replaced by expressions of the form $\tau\text{'}\underline{\pi}$, where $\tau \in \mathbf{BF}$ and $\underline{\pi}$ is a special symbol. In this case if $x \in \mathbb{N}^\omega$, then we denote by $\Phi\lceil x\rceil$ the result of replacing every occurrence of the form $\tau\text{'}\underline{\pi}$ in Φ by $\tau\text{'}x = \mathsf{F}_\tau(x)$; this result is obviously an analytic formula with parameters in $\mathbb{N}^\omega$.

We set $\operatorname{Par}\Phi = \{\tau \in \mathbf{BF} : \tau\text{'}\underline{\pi} \text{ occurs in } \Phi\}$.

Let us define a binary relation $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi$, where it is assumed that $p \in \mathbb{P}^{\mathsf{M}}_{\mathcal{J}}$ and Φ is a closed $\underline{\pi}$-formula with $\operatorname{Par}\Phi \subseteq \mathsf{M} = \mathbf{L}[a] \cap \mathbb{N}^\omega$.

The definition is carried out by induction on the complexity of Φ:

(1) if Φ is a bounded formula, then $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi \iff \{x \in \mathbf{B}_p : \neg\Phi\lceil x\rceil\} \in \mathcal{J}$;
(2) $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \forall\, k\ \Phi(k) \iff \forall\, k \in \mathbb{N}\ (p \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi(k))$;
(3) $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \exists\, k\ \Phi(k) \iff \forall\, q \leqslant p\ \exists\, r \leqslant q\ \exists\, k \in \mathbb{N}\ (r \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi(k))$;
(4) $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \forall\, x\ \Phi(x) \iff \forall\, \tau \in \mathbf{BF} \cap \mathsf{M}\ (p \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi(\tau\text{'}\underline{\pi}))$;
(5) $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \exists\, x\ \Phi(x) \iff \forall\, q \leqslant p\ \exists\, r \leqslant q\ \exists\, \tau \in \mathbf{BF} \cap \mathsf{M}\ (r \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi(\tau\text{'}\underline{\pi}))$;
(6) $p \Vdash^{\mathsf{M}}_{\mathcal{J}} \neg\Phi(x) \iff \forall\, q \leqslant p\ \neg(q \Vdash^{\mathsf{M}}_{\mathcal{J}} \Phi)$.

The variables q and r range over $\mathbb{P}^{\mathsf{M}}_{\mathcal{J}}$ in (3), (5), and (6).

It is clear that the definitions (3) and (5) can be reduced to the definitions (2), (4), (6) by replacing the quantifier $\exists$ by $\neg\forall\neg$.

If Φ is a bounded formula, then the forcing $\Vdash^{\mathsf{M}}_{\mathcal{J}}$ of the formula $\neg\Phi$ can be defined by using (1), and it can be reduced to the forcing of Φ by using (6). However, one can readily see that both ways lead to the same result.

[29]We write $\lceil x\rceil$ instead of the more common $[x]$ to avoid confusion with the usual construction of generic extensions.

Theorem 6.4. *Suppose that $\varphi(k_1,\dots,k_j,x_1,\dots,x_n)$ is a parameter-free analytic formula with the variables $k_1,\dots,k_j,x_1,\dots,x_n$. Then the set*

$$F_\varphi = \bigl\{\langle p,k_1,\dots,k_j,\tau_1,\dots,\tau_n\rangle : k_1,\dots,k_j\in\mathbb{N} \wedge \tau_1,\dots,\tau_n\in\mathbf{BF}\cap\mathsf{M} \\ \wedge\ p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}} \wedge p \Vdash^{\mathsf{M}}_{\mathcal{I}} \varphi(k_1,\dots,k_j,\tau_1\text{'}\underline{\pi},\dots,\tau_n\text{'}\underline{\pi})\bigr\}$$

is definable by an analytic formula with the parameter a, relativized to M.

Proof. We use induction on the complexity of φ. In the case of bounded formulae we assume for simplicity that φ is $\varphi(k,x)$. Then

$$F_\varphi = \bigl\{\langle p,k,\tau\rangle : k\in\mathbb{N}\wedge p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}} \wedge \tau\in\mathbf{BF}\cap\mathsf{M}\wedge\forall\,x\in\mathbf{B}_p\ \varphi(k,\tau\text{'}x)\bigr\}$$

by the definition in (1). However, the formula defining this set is a Π^1_2 formula. (This is determined by the term $\tau\in\mathbf{BF}$; the other terms are even simpler.) Hence, the formula is absolute for $\mathsf{M}=\mathbf{L}[a]\cap\mathbb{N}^\omega$ by Theorem 2.8.

The inductive step (which can be broken up into cases corresponding to (2)–(6) above) is a rather routine exercise based on the definability of the sets $\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}$ and $\mathbf{BF}\cap\mathsf{M}$ in M. □

We recall that $\mathsf{M}=\mathbf{L}[a]\cap\mathbb{N}^\omega$ and we let $\mathbf{Rand}_{\mathcal{I}}\,\mathsf{M}=\mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}[a]$.

Theorem 6.5. *Suppose that $x\in\mathbf{Rand}_{\mathcal{I}}\,\mathsf{M}$ and Φ is a closed $\underline{\pi}$-formula with* $\operatorname{Par}\Phi\subseteq\mathsf{M}$. *Then*

$$\Phi\lceil x\rceil \text{ is true in } \mathsf{M}\lceil x\rceil \iff \exists\,p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}\ (x\in\mathbf{B}_p\wedge p\Vdash^{\mathsf{M}}_{\mathcal{I}}\Phi).$$

Proof. We argue by induction on the complexity of Φ. Assume that Φ is a bounded formula. If $p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}$, $x\in\mathbf{B}_p$, and $p\Vdash^{\mathsf{M}}_{\mathcal{I}}\Phi$, then the set $X=\{x'\in\mathbf{B}_p : \neg\,\Phi\lceil x'\rceil\}$ belongs to $\mathcal{I}$ and is a Borel set with a code in M, and hence $x\notin X$ and $\Phi\lceil x\rceil$. But the formula $\Phi\lceil x\rceil$ is absolute. Conversely, if the right-hand side of the equivalence in the theorem fails, then standard arguments give the condition $q\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}$, where $x\in\mathbf{B}_q$ and $q\Vdash^{\mathsf{M}}_{\mathcal{I}}\neg\,\Phi$. However, $\neg\,\Phi$ is again a bounded formula, and it follows from what was proved above that $\Phi\lceil x\rceil$ fails.

Consider the inductive step for (4). Suppose that Φ is $\forall\,y\ \Psi(y)$. It follows from the definition of $\mathsf{M}\lceil x\rceil$ that $\Phi\lceil x\rceil$ is true in $\mathsf{M}\lceil x\rceil$ if and only if $\Psi(\tau\text{'}\underline{\pi})\lceil x\rceil$ is true in $\mathsf{M}\lceil x\rceil$ for all $\tau\in\mathbf{BF}\cap\mathsf{M}$. The right-hand side of (4) has a similar structure. This enables us to reduce the equivalence of the theorem for the formula Φ to the induction assumption that the theorem holds for all formulae of the form $\Psi(\tau\text{'}\underline{\pi})$.

The inductive step in (2) is similar to that in (4): one considers numbers $k\in\mathbb{N}$ instead of codes τ. For the step (6) the result follows from the equivalence

$$\neg\exists\,p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}\ (x\in\mathbf{B}_p\wedge p\Vdash^{\mathsf{M}}_{\mathcal{I}}\Phi) \iff \exists\,p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{I}}\ (x\in\mathbf{B}_p\wedge p\Vdash^{\mathsf{M}}_{\mathcal{I}}\neg\Phi),$$

which can be proved by standard arguments. □

Theorem 6.6. *If $x\in\mathbf{Rand}_{\mathcal{I}}\,\mathsf{M}$, then the set $\mathsf{M}\lceil x\rceil$ is a descriptive continuum.*

Proof. In principle, the result follows from Lemma 4.12, because the equality $\omega_1^{\mathbf{L}[a]}=\omega_1^{\mathbf{L}[a,x]}$ enables one to easily prove that $\mathsf{M}\lceil x\rceil=\mathbf{L}[a,x]\cap\mathbb{N}^\omega$. Thus, here it is of importance **how** to prove rather than **what** to prove. Namely, in accordance with our main goal, we wish to give an 'M-proof' which does not appeal to forcing properties for models of **ZFC**.

We restrict ourselves to the verification of comprehension for $\mathsf{M}\lceil x\rceil$, because the two other conditions can be verified by similar arguments. For simplicity, let $n=1$, that is, a formula of the form $\varphi(k,l,x)$ is considered. Suppose that $a_1\in\mathbb{N}^\omega\cap\mathsf{M}\lceil x\rceil$ and $b=\{\langle k,l\rangle : \varphi^{\mathsf{M}}(k,l,a_1)\}\in\mathbb{N}^\omega$. We claim that $b\in\mathsf{M}\lceil x\rceil$. By definition, $a_1=\tau_1\text{'}x$, where $\tau_1\in\mathsf{M}\cap\mathbf{BF}$. By Theorem 6.5, there is a 'condition' $r\in\mathbb{P}^{\mathsf{M}}_{\mathcal{J}}$ such that $x\in\mathbf{B}_r$ and $r\Vdash^{\mathsf{M}}_{\mathcal{J}}\forall\,k\ \exists!\,l\ \varphi(k,l,\tau_1)$. We set

$$D_{kl}=\bigl\{p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{J}} : \mathbf{B}_p\cap\mathbf{B}_r=\varnothing\ \vee\ (\mathbf{B}_p\subseteq\mathbf{B}_r\wedge p\Vdash^{\mathsf{M}}_{\mathcal{J}}\forall\,k\ \exists\,!\,l\ \varphi(k,l,\tau_1))\bigr\}.$$

By the choice of r, the set $D_k=\bigcup_l D_{kl}$ is dense in $\mathbb{P}^{\mathsf{M}}_{\mathcal{J}}$ for any k. In addition, all the sets D_{kl} and D_k are definable in M by Theorem 6.4. Applying Proposition 6.3 (the last part), we find a $u\in\mathbb{N}^\omega\cap\mathsf{M}$ such that the set $U_k=\{((u)_k)_l : l\in\mathbb{N}\}$ is included in D_k for any k and the union $\bigcup_{c\in U_k}\mathbf{B}_c$ is an $\mathcal{J}$-full set. Dividing this structure in M in accordance with the relation to the condition r, we can find a $v\in\mathbb{N}^\omega\cap\mathsf{M}$ such that every set $V_k=\{((v)_k)_l : l\in\mathbb{N}\}$ satisfies the equality $V_k=\{((v)_k)_l : l\in\mathbb{N}\wedge\mathbf{B}_{((v)_k)_l}\subseteq\mathbf{B}_r\}$. In this case the set $\bigcup_{c\in V_k}\mathbf{B}_c$ is an $\mathcal{J}$-full subset of $\mathbf{B}_r$ (in the sense that the corresponding difference belongs to $\mathcal{J}$).

We write $V_{kl}=V_k\cap D_{kl}$. One can assume that $\mathbf{B}_c\cap\mathbf{B}_{c'}=\varnothing$ for $c\in V_{kl}$ and $c'\in V_{kl'}$, $l\neq l'$. (Generally speaking, we have only $\mathbf{B}_c\cap\mathbf{B}_{c'}\in\mathcal{J}$ in this case. However, the undesirable intersections can be removed. Indeed, there are only countably many of these intersections, and hence the entire part to be removed is still a set in $\mathcal{J}$, after which the remainder must be encoded in M.) Under this assumption, there is a Borel function $\mathsf{F}_\tau\colon\mathbb{N}^\omega\to\mathbb{N}^\omega$ with $\tau\in\mathbf{BF}\cap\mathsf{M}$ such that $\forall\,x'\in\mathbf{B}_c\ (\tau\text{'}x'(k)=l)$ (where $\tau\text{'}y=\mathsf{F}_\tau(y)$) holds for all k and l and $c\in V_{kl}$. In particular, this holds for the given point $x\in\mathbf{Rand}_{\mathcal{J}}\,\mathsf{M}$. Using standard forcing arguments, one can now show that the point $y=\tau\text{'}x\in\mathsf{M}\lceil x\rceil$ satisfies the condition $\forall\,k\ \varphi^{\mathsf{M}}(k,y(k),a_1)$, as was to be proved. $\square$

6D. Completion of the 'elementary' proof of Theorem 6.1. We again assume that $a\in\mathbb{N}^\omega$, $\mathsf{M}=\mathbf{L}[a]\cap\mathbb{N}^\omega$, $\mathcal{J}$ is an $\mathbf{L}[a]$-absolute σ-CAC ideal, $\mathbb{P}^{\mathsf{M}}_{\mathcal{J}}=\mathbb{P}^{\mathbf{L}[a]}_{\mathcal{J}}$, and $\mathbf{Rand}_{\mathcal{J}}\,\mathsf{M}=\mathbf{Rand}_{\mathcal{J}}\,\mathbf{L}[a]$. The following theorem is analogous to Theorem 4.17.

Theorem 6.7. *Suppose that $\varphi(x)$ is an analytic formula with parameters in M. Then there is a Borel code $c\in\mathbf{BC}\cap\mathsf{M}$ such that*

$$\{x\in\mathbf{Rand}_{\mathcal{J}}\,\mathsf{M} : \varphi(x)\ \textit{is true in}\ \mathsf{M}\lceil x\rceil\}=\mathbf{Rand}_{\mathcal{J}}\,\mathsf{M}\cap\mathbf{B}_c.$$

Proof. We can assume that φ is $\varphi(z,x)$, with the single parameter $z\in\mathsf{M}$. Let $c_z\in\mathbf{BF}\cap\mathsf{M}$ be some code of the constant function $F(x)=z\ \forall\,x$. Thus, $c_z\text{'}\underline{\pi}$ is always interpreted as z. (This is an analogue of $\check{z}$ in the exposition of forcing in subsection 4A.) Let $\varepsilon\in\mathbf{BF}\cap\mathsf{M}$ be a code of the function $F(x)=x\ \forall\,x$. Then $\varphi(z,x)$ is identical to $\varphi(c_z\text{'}\underline{\pi},\varepsilon\text{'}\underline{\pi})\lceil x\rceil$ for any $x\in\mathbb{N}^\omega$. By Theorem 6.4, the set

$$D=\bigl\{p\in\mathbb{P}^{\mathsf{M}}_{\mathcal{J}} : p\Vdash^{\mathsf{M}}_{\mathcal{J}}\varphi(c_z\text{'}\underline{\pi},\varepsilon\text{'}\underline{\pi})\ \text{or}\ p\Vdash^{\mathsf{M}}_{\mathcal{J}}\neg\varphi(c_z\text{'}\underline{\pi},\varepsilon\text{'}\underline{\pi})\bigr\}$$

is definable in M, moreover, this set is dense in $\mathbb{P}_{\mathcal{J}}^{\mathsf{M}}$ by (6) in the definition of $\Vdash_{\mathcal{J}}^{\mathsf{M}}$. As in the proof of Lemma 4.12 (which is applicable here because we recall that $\mathsf{M} = \mathbf{L}[a] \cap \mathbb{N}^\omega$), this implies the existence of a $p \in \mathsf{M}$ such that $(p)_n \in D$ for any n and the set $\{(p)_n : n \in \mathbb{N}\}$ is dense in $\mathbb{P}_{\mathcal{J}}^{\mathsf{M}}$. The last fact means that $\bigcup_n \mathbf{B}_{(p)_n}$ is an $\mathcal{J}$-full set. We set

$$u = \{n : (p)_n \Vdash_{\mathcal{J}}^{\mathsf{M}} \varphi(c_z\text{'}\underline{\pi}, \varepsilon\text{'}\underline{\pi})\} \quad \text{and} \quad v = \mathbb{N} \smallsetminus u = \{n : p_n \Vdash_{\mathcal{J}}^{\mathsf{M}} \neg\varphi(c_z\text{'}\underline{\pi}, \varepsilon\text{'}\underline{\pi})\}.$$

There are Borel codes c, $c' \in \mathbf{BC} \cap \mathsf{M}$ such that $\mathbf{B}_c = \bigcup_{n \in u} \mathbf{B}_{(p)_n}$ and $\mathbf{B}_{c'} = \mathbb{N}^\omega \smallsetminus \mathbf{B}_c = \bigcup_{n \in v} \mathbf{B}_{(p)_n}$. We claim that c is the desired code. Let $x \in \mathbf{Rand}_{\mathcal{J}} \mathsf{M}$. Then $x \in \mathbf{B}_c \cup \mathbf{B}_{c'}$. If $x \in \mathbf{B}_c$, then $x \in \mathbf{B}_{(p)_n}$ for some $n \in u$. Hence, it follows from Theorem 6.5 that the formula $\varphi(c_z\text{'}\underline{\pi}, \varepsilon\text{'}\underline{\pi})[x]$ (that is, the formula $\varphi(z, x)$) is true in $\mathsf{M}[x]$. If $x \notin \mathbf{B}_c$, then $x \in \mathbf{B}_{c'}$, and the formula $\varphi(z, x)$ fails in $\mathsf{M}[x]$ for the same reasons. $\square$

To complete the proof of Theorem 6.1, it remains to repeat the simple proof of Corollary 4.16 in which the reference to Theorem 2.8 is replaced by a reference to Theorems 6.6 and 6.2.

$\square$ (*Theorem* 6.1)

§ 7. Undecidability of the problems

In this section we prove the following two theorems.

Theorem 7.1. *None of the following implications is derivable in* **ZFC**:

(I) $\neg\,\mathrm{PK}(\Pi_1^1) \Longrightarrow \neg\,\mathrm{PK}(\mathrm{OD})$;
(II) $\neg\,\mathrm{BP}(\Delta_2^1) \Longrightarrow \neg\,\mathrm{BP}(\mathrm{OD})$;
(III) $\neg\,\mathrm{LM}(\Delta_2^1) \Longrightarrow \neg\,\mathrm{LM}(\mathrm{OD})$.

Theorem 7.2. *If* **ZFCI** *theory is consistent*, *then the proposition* PK(**ROD**) $\wedge$ BP(**ROD**) $\wedge$ LM(**ROD**) *does not contradict* **ZFC** *theory.*[30]

We recall that OD stands for the class of all ordinal definable sets and the symbol **ROD** for the class of all real-ordinal definable sets. (For the exact definitions of these notions, see below.) Thus, Theorem 7.1 claims that the existence of counterexamples to the regularity properties (in the strongest possible form, because PK holds for Σ_1^1 and the properties LM and BP hold for the classes Σ_1^1 and Π_1^1) does not imply the existence of *effective* counterexamples, not even under the obviously weakest possible understanding of effectiveness as $\in$-definability with ordinals as parameters.

After necessary definitions and comments in subsection 7A, we concentrate on the proof of Theorem 7.1, and then, using this, we continue with the proof of Theorem 7.2.

7A. Definitions and comments on the theorems. One can say that a set x is *definable* if there is an $\in$-formula $\varphi(v)$ that is parameter-free and has only one free

[30]In all metamathematical results, including consistency theorems, we tacitly assume the consistency of **ZFC** itself. However, Theorem 7.2 uses the stronger assumption that the theory **ZFCI** is consistent (the latter theory consists of the axioms of **ZFC** together with the axiom "there is a strongly inaccessible cardinal").

variable v and for which x is the only set satisfying $\varphi(x)$, that is, $\forall v\ (\varphi(v) \iff x = v)$. However, this notion admits no formal rigorous definition in **ZFC**; in other words, there is no $\in$-formula $\delta(x)$ distinguishing at once all definable sets. (We do not go into the reason for this.) On the other hand, a wider notion of *ordinal definable* set (that is, a set definable by an $\in$-formula with ordinals as parameters) admits such a definition. Let us consider a formula $\mathrm{od}\, x$ saying the following (see [75]):

> there is an ordinal α and an $\in$-formula $\varphi(v)$ with ordinals smaller than α as parameters and with a single free variable v such that $x \in \mathbf{V}_\alpha$, and x is the only set satisfying $\mathbf{V}_\alpha \models \varphi(x)$.

We recall that $\mathbf{V}_\alpha$ is the αth von Neumann level (see subsection 2A) and the symbol $\models$ stands for the model-theoretic truth relation for a formula in a structure.

We write $\mathrm{OD} = \{x : \mathrm{od}\, x\}$ for the class of all *ordinal definable* sets.

Thus, every $x \in \mathrm{OD}$ is definable by an $\in$-formula of the form $\mathbf{V}_\alpha \models \varphi(x)$ in which ordinals $\leqslant \alpha$ can occur as parameters. Conversely, if x is ordinal definable informally, then, applying the reflection principle (see, for instance, Theorem 16 in [29]), we find an ordinal α such that the definition can be relativized to $\mathbf{V}_\alpha$, and hence $x \in \mathrm{OD}$.

The class $\mathbf{ROD} = \{x : \mathrm{rod}\, x\}$ of all *real-ordinal definable* sets,[31] that is, the sets definable by formulae containing ordinals and elements of $\mathbb{N}^\omega$ as parameters, is defined similarly. Here $\mathrm{rod}\, x$ stands for a modification of the formula $\mathrm{od}\, x$ which means that φ can contain not only ordinals but also elements of $\mathbb{N}^\omega \cap \mathbf{V}_\alpha$ as parameters. (For the details relating to ordinal definability, see [75] or [29], § 14.)

It is clear that $\mathrm{Ord} \subseteq \mathrm{OD}$; moreover, $\mathbf{L} \subseteq \mathrm{OD}$ because for any $x \in \mathbf{L}$ there is an ordinal ξ such that $x = \mathbf{F}_\xi$. Further, $\mathit{\Sigma}^1_\infty \subseteq \mathrm{OD}$, and the class Σ^1_∞ of all projective subsets of $\mathbb{N}^\omega$ is included in **ROD**. Thus, Theorem 7.2 implies the following corollary.

Corollary 7.3. *Suppose that* **ZFCI** *theory is consistent.*

(i) *The assertion that all projective sets have the perfect kernel property and the Baire property and are $\boldsymbol{\lambda}$-measurable does not contradict* **ZFC**, *and hence this assertion is undecidable in* **ZFC** *by Theorem* 3.11.

(ii) *Therefore, the assertions* $\mathrm{PK}(\Pi^1_1)$, $\mathrm{LM}(\Delta^1_2)$, $\mathrm{BP}(\Delta^1_2)$, $\mathrm{LM}(\Sigma^1_2)$, *and* $\mathrm{BP}(\Sigma^1_2)$ (*see the diagram in the Introduction*) *are also consistent and undecidable in* **ZFC** *by Theorem* 3.11.

Theorem 7.1 is connected with the following form of the regularity problems for point sets: is it possible to effectively give counterexamples[32] to the regularity properties if such counterexamples do exist? The theorem answers this question in the negative.

[31] The first word reflects the possibility of taking points of $\mathbb{N}^\omega$ as parameters of a definition, along with ordinals. The word *real* as a noun, as applied to points of $\mathbb{N}^\omega$, is rather typical for the English-language literature concerning descriptive set theory and is based on the identification, discussed in footnote 12, of the points of $\mathbb{N}^\omega$ with the irrational real numbers. Unfortunately, there is no adequate translation into Russian of the word real in this sense that is concordant with the traditions of Russian mathematical language.

[32] See the references in footnote 9 on the discussions in early descriptive set theory concerning 'effective' examples in contrast to 'pure' existence proofs.

The result itself can be presented in a form in which there are no ordinals at all, that is, in a form of 'pure' definability. However, for the above reasons, the corollary looks more 'metamathematical' than the theorem itself.

Corollary 7.4. *There is no $\in$-formula with a single free variable and satisfying at least one of the following three requirements*:

(i) *probably in* $\mathbf{ZFC} + \neg\,\mathrm{PK}(\Pi^1_1)$, *it defines a subset of* $\mathbb{N}^\omega$ *without the perfect kernel property*;
(ii) *provably in* $\mathbf{ZFC} + \neg\,\mathrm{BP}(\Delta^1_2)$, *it defines a subset of* $\mathbb{N}^\omega$ *that does not have the Baire property*;
(iii) *provably in* $\mathbf{ZFC} + \neg\,\mathrm{LM}(\Delta^1_2)$, *it defines a* $\boldsymbol{\lambda}$*-non-measurable subset of* $\mathbb{N}^\omega$.

The proofs of Theorems 7.1 and 7.2, given below in this section, include an analysis of three different models of **ZFC** obtained as extensions of the class **L** by

1) a single generic collapse function $\omega \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$, which will prove the parts (II) and (III) of Theorem 7.1;
2) a family of $\aleph_2^{\mathbf{L}}$ generic collapse functions $\omega \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$, which will prove the part (I) of Theorem 7.1;
3) a collapse up to a strongly inaccessible cardinal, which will prove Theorem 7.2.

However, the manipulations in the three models have many common points.

7B. First model; one collapse function. To prove the parts (II) and (III) of Theorem 7.1, we take the class **L** of all constructible sets as the ground model and the forcing $\mathbb{C}(\omega_1^{\mathbf{L}}) = \mathrm{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$ designed for the 'collapse' of $\omega_1^{\mathbf{L}}$. Thus, $\mathbb{C}(\omega_1^{\mathbf{L}})$ consists of all functions p such that $\mathrm{dom}\, p \in \mathbb{N}$ and $\mathrm{ran}\, p \subseteq \omega_1^{\mathbf{L}}$. We equip $\mathbb{C}(\omega_1^{\mathbf{L}})$ with the order opposite to inclusion. Then any $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set $G \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ produces a generic collapse function $f[G] = \bigcup G \colon \mathbb{N} \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$, and at the same time we have $G = \{f[G] \restriction n : n \in \mathbb{N}\}$. By Theorem 4.4, the next theorem implies Theorem 7.1 (II), (III).

Theorem 7.5. *Let* $G \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ *be a* $\mathbb{C}(\omega_1^{\mathbf{L}})$*-generic set over* **L**. *Then* $\mathrm{BP}(\mathrm{OD})$ *and* $\mathrm{LM}(\mathrm{OD})$ *hold in* $\mathbf{L}[G]$, *but* $\mathrm{LM}(\Delta^1_2)$ *and* $\mathrm{BP}(\Delta^1_2)$ *fail in* $\mathbf{L}[G]$.

In the course of the proof of this theorem (that is, until the end of subsection 7C) we fix some $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set G over **L**. The 'negative' part of the theorem is not difficult. We have $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]}$, because the function $f[G] \in \mathbf{L}[G]$ maps ω onto $\omega_1^{\mathbf{L}}$, and hence there is a $w \in \mathbf{L}[G] \cap \mathbf{WO}_{\omega_1^{\mathbf{L}}}$. This enables one to obtain a point $a \in \mathbb{N}^\omega$ such that $\mathbf{L}[a] = \mathbf{L}[f[G]] = \mathbf{L}[G]$. The violation of $\mathrm{BP}(\Delta^1_2)$ and $\mathrm{LM}(\Delta^1_2)$ in $\mathbf{L}[G]$ follows now from Theorem 3.11.

The 'positive' part requires much more work. By Lemma 3.8, it suffices to show that in $\mathbf{L}[G]$ every OD set is $\mathcal{I}$-measurable for any **L**-absolute σ-CAC ideal $\mathcal{I}$. The proof is based on the following lemma in which Λ (the empty function) is the weakest element of $\mathbb{C}(\omega_1^{\mathbf{L}})$ and the $\breve{z}_i$ are the canonical $\mathbb{C}(\omega_1^{\mathbf{L}})$-names of sets z_i (it is clear that $\breve{z}_i[G] = z_i$ for $i = 1, \dots, n$).

Lemma 7.6. *Suppose that* $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$, $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$, $\varphi(v_1, \dots, v_n)$ *is an* $\in$*-formula, and* $z_1, \dots, z_n \in \mathbf{L}[x]$ *are arbitrary sets. Then*

$$\varphi(z_1, \dots, z_n) \text{ is true in } \mathbf{L}[G] \iff \Lambda \Vdash^{\mathbf{L}[x]}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \varphi(\breve{z}_1, \dots, \breve{z}_n).$$

Therefore, if $X \in \mathbf{L}[G]$, $X \subseteq \mathbf{L}$, *and* $X \in \mathrm{OD}$ *in* $\mathbf{L}[G]$, *then* $X \in \mathbf{L}$.

Assuming the validity of this lemma, let us consider, *arguing in* $\mathbf{L}[G]$, an arbitrary $\mathbf{L}$-absolute σ-CAC ideal $\mathcal{I}$. We note that the set $\mathbf{L} \cap \mathbb{N}^\omega$ is countable in $\mathbf{L}[G]$ (since $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]}$), and therefore the set $\mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}$ is $\mathcal{I}$-full. We now consider any OD set $X \subseteq \mathbb{N}^\omega$. Thus, $X = \{x \in \mathbb{N}^\omega : \varphi(x, \alpha_1, \dots, \alpha_n)\}$, where $\alpha_1, \dots, \alpha_n$ are ordinals. Let $\psi(x)$ be the formula

$$\text{“}\Lambda \Vdash^{\mathbf{L}[x]}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \varphi(\breve{x}, \breve{\alpha}_1, \dots, \breve{\alpha}_n) \text{ is true in } \mathbf{L}[x]\text{”}.$$

The equivalence $x \in X \iff \psi(x)$ holds in $\mathbf{L}[G]$ for any $x \in \mathbf{Rand}_{\mathcal{I}}\,\mathbf{L}$ by Lemma 7.6. (We note that the equality $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$ follows from Lemma 4.12.) The $\mathcal{I}$-measurability of X follows now from Theorem 4.17.

Remark 7.7. The above proof of BP(OD) and LM(OD) in $\mathbf{L}[G]$ can readily be transformed into a proof of BP(OD(a)) and LM(OD(a)) for any $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ such that $\omega_1^{\mathbf{L}[a]} = \omega_1^{\mathbf{L}}$, where OD($a$) means that a can be used along with ordinals as a parameter of definability.

7C. The proof of the key lemma. The proof of Lemma 7.6 uses the following lemma.

Lemma 7.8. *If* $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ *satisfies* $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$, *then the class* $\mathbf{L}[G]$ *is a* $\mathbb{C}(\omega_1^{\mathbf{L}})$*-generic extension of* $\mathbf{L}[x]$ (recall that $\mathbb{C}(\omega_1^{\mathbf{L}}) = \mathrm{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$).

Proof. Let us fix a $\mathbb{C}(\omega_1^{\mathbf{L}})$-name $t \in \mathbf{L}$ such that $x = t[G]$ and define sets $A_\xi \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ by induction on $\xi < \mathrm{Ord}$. We set

$$A_0 = \bigl\{p \in \mathbb{C}(\omega_1^{\mathbf{L}}) : \forall\, k\ \forall\, \gamma < \omega_1^{\mathbf{L}}\ \bigl((p \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\breve{k}) = \breve{\gamma}) \implies x(k) = \gamma\bigr)\bigr\},$$

$$A_{\xi+1} = \bigl\{p \in A_\xi : \text{for any set } D \in \mathbf{L} \text{ dense in } \mathbb{C}(\omega_1^{\mathbf{L}}) \text{ and such that } D \subseteq \mathbb{C}(\omega_1^{\mathbf{L}}) \text{ there is a } q \in A_\xi \cap D \text{ such that } q \leqslant p\bigr\},$$

$$A_\vartheta = \bigcap_{\xi < \vartheta} A_\xi \quad \text{for the limit ordinals } \vartheta.$$

(As above, we write $q \leqslant p$ meaning that $p \subseteq q$ in $\mathbb{C}(\omega_1^{\mathbf{L}}) = \mathrm{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$.) Since the sets $\mathbb{C}(\omega_1^{\mathbf{L}})$ decrease, there is an ordinal ζ such that $A_\zeta = A_{\zeta+1} = A_\eta$ for any $\eta > \zeta$. It can be shown that the ‘limit’ set $\Sigma = A_\zeta$ consists of all conditions p which for t force nothing incompatible with the existing properties of $x = t[G]$, but we do not dwell on this topic.

The following assertions hold for the set Σ.

(1) $\Sigma \in \mathbf{L}[x]$. (Indeed, $\mathbb{C}(\omega_1^{\mathbf{L}}) \in \mathbf{L}$, and the relation $\Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})}$ is definable in $\mathbf{L}$, and hence $A_0 \in \mathbf{L}[x]$ and the sequence of sets A_ξ belongs to $\mathbf{L}[x]$.)

(2) If a set $D \in \mathbf{L}$, $D \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$, is dense in $\mathbb{C}(\omega_1^{\mathbf{L}})$, then $D \cap \Sigma$ is dense in Σ. (This follows from the equality $\Sigma = A_\zeta = A_{\zeta+1}$.) We immediately note the following corollary: the set Σ has no $\subseteq$-maximal elements.

(3) $G \subseteq \Sigma$. (The relation $G \subseteq A_\xi$ follows by induction on ξ from the genericity of G.)

(4) If $G' \subseteq \Sigma$ is $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic over $\mathbf{L}$, then $t[G'] = t[G] = x$. (Generally, one can easily see that this holds even for $G' \subseteq A_0$.)

(5) G is a Σ-generic set over $\mathbf{L}[x]$.

To prove (5), suppose the contrary. By Theorem 4.3, there is a 'condition' $p \in G$ such that *any* $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set $G' \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ over $\mathbf{L}$ containing p is not Σ-generic over $\mathbf{L}[x]$. (A remark: it follows from (4) that $t[G'] = x$, and hence the set Σ constructed in $\mathbf{L}[G']$ coincides with the set Σ constructed in $\mathbf{L}[G]$.) We now consider an arbitrary set $G' \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ which is Σ-generic over $\mathbf{L}[x]$ and contains p (Theorem 4.4). Then G' is also $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic over $\mathbf{L}$ by (2), a contradiction.

Thus, $\mathbf{L}[G] = \mathbf{L}[x][G]$ is a Σ-generic extension of $\mathbf{L}[x]$. Therefore, by Theorem 4.2(ii), to prove Lemma 7.8, it suffices to prove that the set Σ contains a dense subset $\Sigma' \in \mathbf{L}[x]$ order isomorphic to $\mathbb{C}(\omega_1^{\mathbf{L}})$ in $\mathbf{L}[x]$. We recall that $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$ by assumption, and therefore $\omega_1^{\mathbf{L}}$ is still the first uncountable ordinal in $\mathbf{L}[x]$. Hence, $\mathbb{C}(\omega_1^{\mathbf{L}})$ is a tree of height ω with ω_1-branchings in $\mathbf{L}[x]$. Thus, to construct a desired set Σ', it suffices to show that for any 'condition' $p \in \Sigma$ the set $\Sigma_p = \{q \in \Sigma : q \leqslant p\}$ is uncountable in $\mathbf{L}[x]$.

Suppose the contrary. Let Σ_p be countable in $\mathbf{L}[x]$. We consider any Σ-generic set $G' \subseteq \Sigma$ over $\mathbf{L}[x]$ containing p. Then $\omega_1^{\mathbf{L}[x,G']} = \omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$. As above, it follows that G' is also $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic over $\mathbf{L}$, which implies that $\omega_1^{\mathbf{L}}$ is countable in $\mathbf{L}[G']$, a contradiction. □

Proof of Lemma 7.6. Suppose the contrary. Then, by Theorem 4.3 and Lemma 7.8, there are 'conditions' $p, q \in \mathbb{C}(\omega_1^{\mathbf{L}})$ such that $p \Vdash^{\mathbf{L}[x]}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \varphi(\breve{z}_1, \dots, \breve{z}_n)$ and at the same time $q \Vdash^{\mathbf{L}[x]}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \neg\varphi(\breve{z}_1, \dots, \breve{z}_n)$. We can assume that $\operatorname{dom} p = \operatorname{dom} q = m \in \mathbb{N}$. Denote by $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant p}$ the set of all $p' \in \mathbb{C}(\omega_1^{\mathbf{L}})$, $p' \leqslant p$. We define $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant q}$ similarly. If $p' \in \mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant p}$, then we define $h(p') \in \mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant q}$ in such a way that $\operatorname{dom} h(p') = \operatorname{dom} p'$ and $h(p')(k) = q(k)$ for any $k \in \operatorname{dom} p$ and $h(p')(k) = p'(k)$ for any $k \in \operatorname{dom} p' \smallsetminus \operatorname{dom} p$. Clearly, $h \in \mathbf{L}$ is an order isomorphism of $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant p}$ onto $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant q}$.

Let us now consider an arbitrary $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant p}$-generic set G_p over $\mathbf{L}[x]$. Then $p \in G_p$ and $G_q = \{h(p') : p' \in G_p\} \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant q}$ is a $\mathbb{C}(\omega_1^{\mathbf{L}})_{\leqslant q}$-generic set over $\mathbf{L}[x]$ containing q. One can readily see that the set $G'_p = G_p \cup \{p' \in \mathbb{C}(\omega_1^{\mathbf{L}}) : p' \geqslant p\}$ is $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic over $\mathbf{L}[x]$. It follows from the choice of p that the formula $\varphi(z_1, \dots, z_n)$ is true in $\mathbf{L}[x][G_p]$. For the same reason, this formula is false in $\mathbf{L}[x][G_q]$. However, $\mathbf{L}[x][G_p] = \mathbf{L}[x][G_q]$ (because $h \in \mathbf{L}$), a contradiction.

To prove the last assertion, we suppose that $X = \{x \in \mathbf{L} : \varphi(x, \alpha_1, \dots, \alpha_n)\}$ in $\mathbf{L}[G]$, where $\alpha_1, \dots, \alpha_n$ are ordinals. Then

$$x \in X \iff \Lambda \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \varphi(\breve{x}, \breve{\alpha}_1, \dots, \breve{\alpha}_n),$$

as already proved above. However, the relation $\Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})}$ is expressible in $\mathbf{L}$ by Theorem 4.3. $\square$

$\square$ (*Theorem* 7.5 *and Theorem* 7.1, *parts* (II) *and* (III))

7D. Second model; $\aleph_2^{\mathbf{L}}$ collapse functions. To prove the part (I) of Theorem 7.1, we again take $\mathbf{L}$ as the ground model, and for the forcing we take the product $\mathbf{P}$ of $\aleph_2$ copies of the forcing $\mathbb{C}(\omega_1^{\mathbf{L}}) = \mathrm{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$ with finite base. In other words, $\mathbf{P}$ consists of all functions p such that the set $\operatorname{dom} p \subseteq \omega_2^{\mathbf{L}}$ is finite and $p(\xi) \in \mathbb{C}(\omega_1^{\mathbf{L}})$ for any $\xi \in \operatorname{dom} p$. The order on $\mathbf{P}$ is defined in the natural way: $p \leqslant q$ (that is, p is stronger than q) if $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p(\xi) \leqslant q(\xi)$ in $\mathbb{C}(\omega_1^{\mathbf{L}})$ for any $\xi \in \operatorname{dom} q$. Thus, the empty function Λ is the largest (and the weakest) element of $\mathbf{P}$. We set

$$\mathbf{P}_{<\vartheta} = \{p \in \mathbf{P} : \operatorname{dom} p \subseteq \vartheta\}, \qquad \mathbf{P}_{\geqslant\vartheta} = \{p \in \mathbf{P} : \operatorname{dom} p \subseteq \omega_2^{\mathbf{L}} \smallsetminus \vartheta\},$$

and also $\mathbf{P}_{\leqslant\vartheta} = \mathbf{P}_{<\vartheta+1}$ and $\mathbf{P}_{>\vartheta} = \mathbf{P}_{\geqslant\vartheta+1}$ for any $\vartheta < \omega_2^{\mathbf{L}}$.

The next theorem suffices for the proof of Theorem 7.1(I).

Theorem 7.9. *Suppose that $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic over $\mathbf{L}$. Then* $\mathrm{PK}(\mathrm{OD})$ *holds and* $\mathrm{PK}(\Pi^1_1)$ *fails in* $\mathbf{L}[G]$.

In the course of the proof of this theorem, that is, up to the end of subsection 7E, we fix a $\mathbf{P}$-generic set $G \subseteq \mathbf{P}$ over $\mathbf{L}$. It produces a collection of $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic sets $G_\xi = \{p(\xi) : p \in G\}$, $\xi < \omega_2^{\mathbf{L}}$, and a collection of collapse functions $f_\xi[G] = \bigcup G_\xi \colon \mathbb{N} \xrightarrow{\text{onto}} \omega_1^{\mathbf{L}}$. We set $G_{<\vartheta} = G \cap \mathbf{P}_{<\vartheta}$ and define $G_{\leqslant\vartheta}$, $G_{>\vartheta}$, and $G_{\geqslant\vartheta}$ similarly.

The 'negative' part of the theorem presents no great difficulties and uses the following simple result.

Lemma 7.10. *Every antichain $A \subseteq \mathbf{P}$, $A \in \mathbf{L}$, is of cardinality $\leqslant \omega_1^{\mathbf{L}}$ in* $\mathbf{L}$.

Proof. We argue in $\mathbf{L}$. One can assume that A is a *maximal* antichain. We set $\xi_0 = 0$. If $\xi_n < \omega_2^{\mathbf{L}}$ has already been defined, then the set $\mathbf{P}_{<\xi_n}$ is obviously of cardinality $\leqslant \omega_1^{\mathbf{L}}$, and hence (by the maximality of A) there is an ordinal ξ, $\xi_n < \xi < \omega_2^{\mathbf{L}}$, such that for any $p \in \mathbf{P}_{<\xi_n}$ there exists a 'condition' $q \in A \cap \mathbf{P}_{<\xi}$ compatible with p in $\mathbf{P}$. We denote by ξ_{n+1} the smallest such ordinal ξ. Then $\vartheta = \sup_n \xi_n < \omega_2^{\mathbf{L}}$. It remains to prove that $A \subseteq \mathbf{P}_{<\vartheta}$. Suppose not; let $r \in A \smallsetminus \mathbf{P}_{<\vartheta}$. Then the condition $p = r \restriction (\vartheta \cap \operatorname{dom} p)$ belongs to $\mathbf{P}_{<\vartheta}$, and hence $p \in \mathbf{P}_{<\xi_n}$ for some n. In this case there is by definition a 'condition' $q \in A \cap \mathbf{P}_{<\vartheta}$ compatible with p. Then q is compatible with r as well (since $\operatorname{dom} q \cap \operatorname{dom} r = \operatorname{dom} p \cap \operatorname{dom} r$ by construction). However, q and r belong to A, and clearly $q \neq r$ because $r \notin \mathbf{P}_{<\vartheta}$, a contradiction.

It follows immediately from the lemma that $\omega_1^{\mathbf{L}[G]} \leqslant \omega_2^{\mathbf{L}}$ by Theorem 4.2(i). On the other hand, the function $f_0[G] \in \mathbf{L}[G]$ (generally, each of the functions $f_\xi[G]$) maps ω onto $\omega_1^{\mathbf{L}}$, and hence $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]}$. It follows that $\omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$, and moreover there is an $a \in \mathbf{L}[G] \cap \mathbf{WO}_{\omega_1^{\mathbf{L}}}$. This object a satisfies the condition $\omega_1^{\mathbf{L}[a]} = \omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$, and therefore $\mathbf{L}[a] \cap \mathbb{N}^\omega$ is countable in $\mathbf{L}[G]$. To derive the formula $\neg\,\mathrm{PK}(\Pi^1_1)$ in $\mathbf{L}[G]$, it remains to apply Theorem 3.11.

The 'positive' part of Theorem 7.9, that is, the proof of the formula $\mathrm{PK}(\mathrm{OD})$ in $\mathbf{L}[G]$ (see subsection 7E) is more cumbersome and uses several lemmas.

Lemma 7.11. *If $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$, then $x \in \mathbf{L}[G_{<\vartheta}]$ for some $\vartheta < \omega_2^{\mathbf{L}}$.*

Proof. By Theorem 4.1, there is a $\mathbf{P}$-name $t \in \mathbf{L}$ such that $x = t[G]$. *Arguing in* $\mathbf{L}$, we set $D_{nk} = \{p \in \mathbf{P} : p \Vdash_{\mathbf{L}}^{\mathbf{P}} t(\check{n}) = \check{k}\}$ and $D_n = \bigcup_{k \in \mathbb{N}} D_{nk}$, choose a maximal antichain $A_n \subseteq D_n$ in each D_n, and let $A = \bigcup_n A_n$. It follows from Lemma 7.11 that $\operatorname{card} A < \Omega$. Thus, there is an ordinal $\vartheta < \omega_2^{\mathbf{L}}$ such that $A \subseteq \mathbf{P}_{<\vartheta}$. Then one can readily see that

$$x(n) = k \iff G \cap D_{nk} \neq \varnothing \iff G \cap D_{nk} \cap A \neq \varnothing \iff G_{<\vartheta} \cap D_{nk} \neq \varnothing,$$

which implies that $x \in \mathbf{L}[G_{<\vartheta}]$. $\square$

Lemma 7.12 (compare with Lemma 7.8). *If $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$, then the class $\mathbf{L}[G]$ is a $\mathbf{P}$-generic extension of $\mathbf{L}[x]$.*

Proof. Using Lemma 7.11, we find an ordinal $\vartheta < \omega_2^{\mathbf{L}}$ such that $x \in \mathbf{L}[G_{<\vartheta}]$. One can assume that $\vartheta \geqslant \omega_1^{\mathbf{L}}$. For simplicity, suppose that $\vartheta = \omega_1^{\mathbf{L}}$; the general case can be reduced to this one by using any bijection $b\colon \omega_1^{\mathbf{L}} \xrightarrow{\text{onto}} \vartheta$ in $\mathbf{L}$. Under this assumption, we consider the set $\mathbf{P}'$ of all 'conditions' $p \in \mathbf{P}_{<\omega_1^{\mathbf{L}}}$ such that either $p = \Lambda$ or $\operatorname{dom} p = \{0\} \cup \{p(0)(k) : k \in \operatorname{dom} p(0)\}$ and $\operatorname{dom} p(\xi) = \operatorname{dom} p(0)$ for every $\xi \in \operatorname{dom} p$.

One can readily see that $\mathbf{P}' \in \mathbf{L}$, $\mathbf{P}'$ is dense in $\mathbf{P}_{<\omega_1^{\mathbf{L}}}$, and $\mathbf{P}'$ is order isomorphic to the set $\mathbb{C}(\omega_1^{\mathbf{L}}) = \operatorname{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$ in $\mathbf{L}$. (Each of these two sets is a tree of height ω with $\omega_1^{\mathbf{L}}$-branchings with respect to the ordering $\subseteq$, that is, the ordering inverse to $\leqslant$). Hence, there is a $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set $F \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ over $\mathbf{L}$ such that $\mathbf{L}[G_{<\vartheta}] = \mathbf{L}[F]$. As in the proof of Lemma 7.8, there is a tree $\Sigma \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$, $\Sigma \in \mathbf{L}[x]$, without $\subseteq$-maximal elements and such that $F \subseteq \Sigma$ and F is Σ-generic over $\mathbf{L}[x]$.

We now consider the class $\mathbf{L}[G_{\leqslant\omega_1^{\mathbf{L}}}]$. Obviously, the set $\mathbf{P}_{\leqslant\omega_1^{\mathbf{L}}}$ is order isomorphic to the product $\mathbf{P}_{<\omega_1^{\mathbf{L}}} \times \mathbb{C}(\omega_1^{\mathbf{L}})$. Thus, the class $\mathbf{L}[G_{\leqslant\omega_1^{\mathbf{L}}}]$ is a $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic extension of the class $\mathbf{L}[G_{<\omega_1^{\mathbf{L}}}] = \mathbf{L}[F]$ by Theorem 4.2(iii), and thus also a $\Sigma \times \mathbb{C}(\omega_1^{\mathbf{L}})$-generic extension of $\mathbf{L}[x]$ by Theorem 4.2(iv).

However, due to the above properties of Σ, the set $\Sigma \times \mathbb{C}(\omega_1^{\mathbf{L}})$ is order isomorphic to $\mathbb{C}(\omega_1^{\mathbf{L}})$ itself. It follows that $\mathbf{L}[G_{\leqslant\omega_1^{\mathbf{L}}}]$ is a $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic extension (and therefore also a $\mathbf{P}_{\leqslant\omega_1^{\mathbf{L}}}$-generic extension) of the class $\mathbf{L}[x]$. This enables us to make the final step. Since $\mathbf{P}$ is isomorphic to the product $\mathbf{P}_{\leqslant\omega_1^{\mathbf{L}}} \times \mathbf{P}_{>\omega_1^{\mathbf{L}}}$, the class $\mathbf{L}[G]$ is a $\mathbf{P}_{>\omega_1^{\mathbf{L}}}$-generic extension of $\mathbf{L}[G_{\leqslant\omega_1^{\mathbf{L}}}]$, and hence also a $\mathbf{P}_{\leqslant\omega_1^{\mathbf{L}}} \times \mathbf{P}_{>\omega_1^{\mathbf{L}}}$-generic extension of $\mathbf{L}[x]$ by the above. It remains to go from $\mathbf{P}_{\leqslant\omega_1^{\mathbf{L}}} \times \mathbf{P}_{>\omega_1^{\mathbf{L}}}$ back to $\mathbf{P}$. $\square$

Lemma 7.13. *Suppose that $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$, $\varphi(v_1, \dots, v_n)$ is an $\in$-formula, and $z_1, \dots, z_n \in \mathbf{L}[x]$ are arbitrary sets. Then*

$$\varphi(z_1, \dots, z_n) \text{ is true in } \mathbf{L}[G] \iff \Lambda \Vdash_{\mathbf{P}}^{\mathbf{L}[x]} \varphi(\check{z}_1, \dots, \check{z}_n).$$

Proof. Arguing as in the proof of Lemma 7.6 (with reference to Lemma 7.12 instead of Lemma 7.8), we need only show that for any 'conditions' $p, q \in \mathbf{P}$ (under the simplifying assumptions $\operatorname{dom} p = \operatorname{dom} q = u \subseteq \omega_2^{\mathbf{L}}$ and $\operatorname{dom} p(\xi) = \operatorname{dom} q(\xi)$ for any $\xi \in u$) the sets $\mathbf{P}_{\leqslant p} = \{p' \in \mathbf{P} : p' \leqslant p\}$ and $\mathbf{P}_{\leqslant q}$ are order isomorphic in $\mathbf{L}$. The construction of the desired isomorphism is not difficult. $\square$

7E. Perfect kernel property in the second model. Continuing the proof of Theorem 7.9, we claim that the assertion PK(OD) holds in any $\mathbf{P}$-generic model $\mathbf{L}[G]$. To this end, we consider an arbitrary set $X \in \mathbf{L}[G]$, $X \subseteq \mathbb{N}^\omega$, which is an OD set in $\mathbf{L}[G]$. Let $X = \{x : \varphi(x)\}$ in $\mathbf{L}[G]$, where φ is assumed for simplicity to be a parameter-free formula. Let $\psi(x)$ be the formula "$\Lambda \Vdash^{\mathbf{L}[x]}_{\mathbf{P}} \varphi(\check{x})$ is true in $\mathbf{L}[x]$". Then $\forall\, x\ (\varphi(x) \Longleftrightarrow \psi(x))$ in $\mathbf{L}[G]$ by Lemma 7.13, and therefore $X = \{x : \psi(x)\}$.

If $X \subseteq \mathbf{L}$, then X is at most countable in $\mathbf{L}[G]$, because $\omega_1^{\mathbf{L}}$ is countable in $\mathbf{L}[G]$. It remains to prove that X has a perfect subset in $\mathbf{L}[G]$ if $X \not\subseteq \mathbf{L}$. Let $x \in X \smallsetminus \mathbf{L}$. Then by Lemma 7.11, there is an ordinal $\vartheta < \omega_2^{\mathbf{L}}$ such that $x \in \mathbf{L}[G_{<\vartheta}]$. It was shown in the proof of Lemma 7.11 that the $\mathbf{P}_{<\vartheta}$-generic extensions coincide with the $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic extensions (where $\mathbb{C}(\omega_1^{\mathbf{L}}) = \mathrm{Coll}(\mathbb{N}, \omega_1^{\mathbf{L}})$). Thus, there is a $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set $F_0 \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ over $\mathbf{L}$, $F_0 \in \mathbf{L}[G]$, such that $x \in \mathbf{L}[F_0]$. Let us fix a $\mathbb{C}(\omega_1^{\mathbf{L}})$-name $t \in \mathbf{L}$ with $x = t[F_0]$. Since the formula ψ is obviously absolute, there is a 'condition' $p_0 \in F_0$ forcing $\psi(t) \wedge t \notin \mathbf{L}$. We assume for brevity that $p_0 = \Lambda$, that is, $\Lambda \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} \psi(t) \wedge t \notin \mathbf{L}$; the general case requires only simple and obvious corrections. Under this assumption,

($*$) $t[F] \in X$ for any $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set $F \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$ over $\mathbf{L}$, $F \in \mathbf{L}[G]$.

The idea of the construction below is to define in $\mathbf{L}[G]$ a 'perfect set' of generic subsets of $\mathbb{C}(\omega_1^{\mathbf{L}})$ on which the map $F \mapsto t[F]$ is continuous and bijective.

For $p, q \in \mathbb{C}(\omega_1^{\mathbf{L}})$ we set $p \perp q$ if there exist numbers n and $k \neq \ell$ such that $p \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\check{n}) = \check{k}$ and $q \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\check{n}) = \check{\ell}$. We claim that

(†) if $p, q \in \mathbb{C}(\omega_1^{\mathbf{L}})$, then there are 'conditions' $p' \leqslant p$ and $q' \leqslant q$ (that is, stronger 'conditions') such that $p' \perp q'$.

Indeed, otherwise there are no 'conditions' $p', p'' \leqslant p$ forcing elementarily incompatible properties of t, for example, such that $p' \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\check{n}) = \check{k}$ and $p'' \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\check{n}) = \check{\ell}$ for some n and $k \neq \ell$. It follows that all values of t are already 'decided' by p; in other words, for any n there is a k_n such that $p \Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})} t(\check{n}) = \check{k}_n$. Then the 'condition' p forces $t = \check{a}$, where $a(n) = k_n\ \forall\, n$, and hence $a \in \mathbf{L}$, because $\Vdash^{\mathbf{L}}_{\mathbb{C}(\omega_1^{\mathbf{L}})}$ is expressible in the ground model $\mathbf{L}$. This contradicts the assumption $\Lambda \Vdash t \notin \mathbf{L}$ and proves (†).

Let us consider an auxiliary forcing Π consisting of all functions π such that $\operatorname{dom}\pi$ is a subset of $2^{<\omega}$ of the form $2^{<m} = \{s \in 2^{<\omega} : m > \operatorname{lh} s\}$ (we denote $m \in \mathbb{N}$ by $m = \operatorname{hgt}\pi$ (the *height* of π)) and $\operatorname{ran}\pi \subseteq \omega_1^{\mathbf{L}}$. We equip Π with the order opposite to inclusion: $\pi \leqslant \rho$ (π is stronger than ρ) if $\operatorname{dom}\rho \subseteq \operatorname{dom}\pi$ (then obviously $\operatorname{hgt}\rho \leqslant \operatorname{hgt}\pi$) and $\rho = \pi \restriction \operatorname{dom}\rho$. Clearly, $\varnothing$ (that is, the empty function, or, equivalently, Λ) is the largest element of Π.

The forcing Π can be called a *ramified power set* of $\mathbb{C}(\omega_1^{\mathbf{L}})$. Indeed, if $\pi \in \Pi$, $m = \operatorname{hgt}\pi$, and $s \in 2^{<m}$, then $\pi/s \in \mathbb{C}(\omega_1^{\mathbf{L}})$ can be defined in such a way that $\operatorname{dom}(\pi/s) = 1 + \operatorname{lh} s$ and $(\pi/s)(k) = \pi(s \restriction k)\ \forall\, k \leqslant \operatorname{lh} s$. For instance, if $s = \langle i, j\rangle$, then $\operatorname{dom}(\pi/s) = 3$, $(\pi/s)(0) = \pi(\Lambda)$, $(\pi/s)(1) = \pi(\langle i\rangle)$, and $(\pi/s)(2) = \pi(s)$. At the same time, $\Pi \in \mathbf{L}$, and Π is a tree of height ω with $\omega_1^{\mathbf{L}}$-branchings (in the sense of the order $\subseteq$, that is, the order inverse to $\leqslant$); therefore, Π and $\mathbb{C}(\omega_1^{\mathbf{L}})$ are order isomorphic in $\mathbf{L}$.

We claim that, if $D \in \mathbf{L}$ is a dense subset of $\mathbb{C}(\omega_1^{\mathbf{L}})$ and $k \in \mathbb{N}$, then the following sets belong to $\mathbf{L}$ and are dense in Π:

$$\Delta_D = \bigl\{\pi \in \Pi : \forall\, s \in {}^n 2\ (\pi/s \in D),\ \text{where}\ n = \operatorname{hgt}\pi - 1\bigr\};$$
$$\Delta_k = \bigl\{\pi \in \Pi : k \leqslant n = \operatorname{hgt}\pi - 1 \wedge \forall\, s, s' \in {}^n 2\ (s \restriction k \neq s' \restriction k \Longrightarrow \pi/s \perp \pi/s')\bigr\}.$$

(Here ${}^n 2 = \{s \in 2^{<\omega} : n = \operatorname{lh} s\}$.) To prove the denseness of Δ_D, suppose that $\rho \in \Pi$ and $\operatorname{hgt}\rho = m+1 \geqslant 1$. It follows from the denseness of D that for any $\sigma \in {}^m 2$ there is a 'condition' $p_\sigma \in D$ such that $p_\sigma \leqslant \rho/\sigma$ in $\mathbb{C}(\omega_1^{\mathbf{L}})$, that is, $\rho/\sigma \subseteq p_\sigma$. Since the set ${}^m 2$ is finite, we can assume without loss of generality that all the p_σ have the same value of $\operatorname{dom} p_\sigma = n + 1 > m$. It is now easy to define a 'condition' $\pi \in \Pi$ such that $\operatorname{hgt}\pi = n + 1$ and $\pi/s = p_{s \restriction m} \in D\ \forall\, s \in {}^n 2$, which implies that $\pi \in \Delta_D$.

To verify the denseness of Δ_k, suppose again that $\rho \in \Pi$ and $\operatorname{hgt}\rho = m + 1 \geqslant 1$. We can assume that $m \geqslant k$. Using the result (†), one can choose a 'condition' $p_\sigma \in \mathbb{C}(\omega_1^{\mathbf{L}})$ for any $\sigma \in {}^m 2$ such that $p_\sigma \perp p_{\sigma'}$ is true if $\sigma \neq \sigma'$. We can again assume that for any $\sigma \in {}^m 2$ there is an $n > m$ for which $\operatorname{dom} p_\sigma = n + 1$. There is a 'condition' $\pi \in \Pi$ such that $\operatorname{hgt}\pi = n + 1$ and $\pi/s = p_{s \restriction m}$ for any $s \in {}^n 2$. If $s, s' \in {}^n 2$ and $s \restriction k \neq s' \restriction k$, then all the more so $s \restriction m \neq s' \restriction m$, and hence $\pi/s = p_{s \restriction m} \perp p_{s' \restriction m} = \pi/s'$. Thus, $\pi \in \Delta_k$.

We have $\pi \leqslant \rho$ in both cases, which proves the denseness of Δ_D and Δ_k.

As indicated above, the sets $\mathbb{C}(\omega_1^{\mathbf{L}})$ and Π are order isomorphic in $\mathbf{L}$. Since, as we have seen, $\mathbf{L}[G]$ contains a $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic set over $\mathbf{L}$, it follows that $\mathbf{L}[G]$ contains a Π-generic set $\Gamma \subseteq \Pi$ over $\mathbf{L}$. In this case $\gamma = \bigcup \Gamma$ is a map $2^{<\omega} \to \omega_1^{\mathbf{L}}$. We set $\Gamma/u = \{\pi/(u \restriction n) : \pi \in \Gamma \wedge n < \operatorname{hgt}\pi\}$ for each $u \in 2^\omega$; then $\Gamma/u \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$. We claim that the following assertions are true in $\mathbf{L}[G]$:

(a) for any $u \in 2^\omega$ the set Γ/u is $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic over $\mathbf{L}$;

(b) if $u \neq v \in 2^\omega$, then $t[\Gamma/u] \neq t[\Gamma/v]$;

(c) the map $u \mapsto t[\Gamma/u]$ is a continuous function $2^\omega \to \mathbb{N}^\omega$ with a code in $\mathbf{L}$;

(d) $t[\Gamma/u] \in X$ for any $u \in 2^\omega$.

To prove (a), suppose that a set $D \in \mathbf{L}$, $D \subseteq \mathbb{C}(\omega_1^{\mathbf{L}})$, is dense in $\mathbb{C}(\omega_1^{\mathbf{L}})$. Then Δ_D is dense in Π, and hence there is an index $n \in \mathbb{N}$ such that $\pi = \gamma \restriction 2^{<n+1} \in \Delta_D$. We set $s = u \restriction n$. Then $\pi/s \in D$ by the definition of Δ_D. On the other hand, one can readily see that $\pi/s \in \Gamma/u$.

The assertion (b) can be proved in a similar way, but one must use the sets Δ_k.

(c) Suppose that $u \in 2^\omega$ and $t[\Gamma/u](m) = k$. By (a), there is a 'condition' $p \in \Gamma/u$ forcing $t(\check m) = \check k$. By definition, t is of the form $\pi/(u \restriction n)$, where $\pi \in \Gamma$ and $n < \operatorname{hgt}\pi$. Thus, $p \in \Gamma/v$, and hence $t[\Gamma/v](m) = k$ whenever $v \in \mathbb{N}^\omega$ and $v \restriction n = u \restriction n$.

The assertion (d) follows from (a) and (∗).

Combining (b), (c), and (d), we immediately obtain in $\mathbf{L}[G]$ a perfect subset $\{t[\Gamma/u] : u \in 2^\omega\}$ of the set X, as was to be proved.

$\square$ (*Theorem* 7.9 *and Theorem* 7.1, *part* I *and thus the entire theorem*)

Remark 7.14. Arguing as above in the proof of Theorem 7.5, one can prove that LM(OD), BP(OD) hold in any $\mathbf{P}$-generic model $\mathbf{L}[G]$. On the other hand, the methods discussed in § 8 give $\neg\,\mathrm{LM}(\Delta^1_2)$ and $\neg\,\mathrm{BP}(\Sigma^1_2)$; however, they also give

$\mathrm{BP}(\Delta^1_2)$, because for any $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ with $\omega_1^{\mathbf{L}[a]} = \omega_2^{\mathbf{L}}$ the set $\mathbf{Coh}\,\mathbf{L}[a]$ is non-empty but not co-meagre, and $\mathbf{Rand}\,\mathbf{L}[a] = \varnothing$.

It follows that the conjunction of PK(OD), LM(OD), and BP(OD) is consistent with $\neg\,\mathrm{PK}(\Pi^1_1) \wedge \neg\,\mathrm{LM}(\Delta^1_2) \wedge \neg\,\mathrm{BP}(\Sigma^1_2)$. $\square$

We conjecture that $\neg\,\mathrm{BP}(\Sigma^1_2)$ can be improved to $\neg\,\mathrm{BP}(\Delta^1_2)$. However, the most interesting question we cannot answer is as follows: is it true that PK(OD) holds in the $\mathbb{C}(\omega_1^{\mathbf{L}})$-generic model treated above in subsection 7B? Generally, is PK(OD) consistent with the assumption that $\mathbb{N}^\omega \subseteq \mathbf{L}[a]$ for some $a \in \mathbb{N}^\omega$?

7F. Solovay model. Here we begin the proof of Theorem 7.2. The forcing used in the proof is still the product of 'collapse' forcings of the form $\mathrm{Coll}(\mathbb{N},\xi)$; however, unlike the forcing $\mathbf{P}$ of subsection 7D, where all factors had the same value $\xi = \omega_2^{\mathbf{L}}$, ξ will vary in this new version from 0 up to a fixed strongly inaccessible cardinal in the ground model.

Let us pass to the details. We fix a strongly inaccessible cardinal Ω in the class $\mathbf{L}$ of all constructible sets and consider the forcing $\mathcal{P}$ consisting of all functions p such that $\operatorname{dom} p \subseteq \Omega$ is finite and $p(\xi) \in \mathrm{Coll}(\mathbb{N},\xi)$ for any $\xi \in \operatorname{dom} p$, that is, $p(\xi)$ is a function from some $m = \{0,\dots,m-1\}$ into $\xi = \{\eta : \eta < \xi\}$ for any $\xi \in \operatorname{dom} p$. Let the set $\mathcal{P}$ be equipped with an order similar to that used above for $\mathbf{P}$, namely, $p \leqslant q$ (p is a stronger condition) whenever $\operatorname{dom} q \subseteq \operatorname{dom} p$ and $p(\xi) \leqslant q(\xi)$ (that is, $q(\xi) = p(\xi) \restriction \operatorname{dom} q(\xi)$) for any $\xi \in \operatorname{dom} p$. Both the forcing $\mathcal{P}$ itself and the order thus defined belong to $\mathbf{L}$. We set

$$\mathcal{P}_{<\vartheta} = \{p \in \mathcal{P} : \operatorname{dom} p \subseteq \vartheta\}, \qquad \mathcal{P}_{\geqslant\vartheta} = \{p \in \mathcal{P} : \operatorname{dom} p \subseteq \Omega \smallsetminus \vartheta\},$$

and also $\mathcal{P}_{\leqslant\vartheta} = \mathcal{P}_{<\vartheta+1}$ and $\mathcal{P}_{>\vartheta} = \mathcal{P}_{\geqslant\vartheta+1}$.

The proof of Theorem 7.2 from the following assertion is similar to the proof of Theorem 7.1.

Theorem 7.15. *If a set* $\mathcal{G} \subseteq \mathcal{P}$ *is* $\mathcal{P}$*-generic over* $\mathbf{L}$, *then* $\mathrm{PK}(\mathbf{ROD})$, $\mathrm{LM}(\mathbf{ROD})$, *and* $\mathrm{BP}(\mathbf{ROD})$ *hold in* $\mathbf{L}[\mathcal{G}]$.

When proving the theorem, we fix a $\mathcal{P}$-generic set $\mathcal{G} \subseteq \mathcal{P}$ over $\mathbf{L}$. We write $\mathcal{G}_{<\vartheta} = \mathcal{G} \cap \mathcal{P}_{<\vartheta}$ and define $\mathcal{G}_{\geqslant\vartheta}$, $\mathcal{G}_{\leqslant\vartheta}$, and $\mathcal{G}_{>\vartheta}$ similarly.

We begin with some crucial technical results relating to the *Solovay model* $\mathbf{L}[\mathcal{G}]$.

Lemma 7.16. *Every antichain* $A \subseteq \mathcal{P}$, $A \in \mathbf{L}$, *is of cardinality* $< \Omega$ *in* $\mathbf{L}$.

Proof. We argue in $\mathbf{L}$. One can assume that A is a maximal antichain. Let $\xi_0 = \omega$. If $\xi_n < \Omega$ has been defined, then the cardinality of the set $\mathcal{P}_{<\xi_n}$ is obviously less than Ω, and hence, since Ω is inaccessible and A is maximal, there is an ordinal ξ, $\xi_n < \xi < \Omega$, such that for any $p \in \mathcal{P}_{<\xi_n}$ there is a 'condition' $q \in A \cap \mathcal{P}_{<\xi}$ compatible with p in $\mathcal{P}$. Let ξ_{n+1} be the smallest such ordinal ξ. Let $\vartheta = \sup_n \xi_n$. We have $\vartheta < \Omega$, again because Ω is inaccessible, and $A \subseteq \mathcal{P}_{<\vartheta}$ (see the proof of Lemma 7.10), which implies the desired result, because the set $\mathcal{P}_{<\vartheta}$ has cardinality $\operatorname{card}\vartheta < \Omega$. $\square$

Corollary 7.17. $\omega_1^{\mathbf{L}[\mathcal{G}]} = \Omega$.

Proof. The inequality $\omega_1^{\mathbf{L}[\mathcal{G}]} \leqslant \Omega$ follows from Lemma 7.16 and Theorem 4.2(i). Conversely, every set $\mathcal{G}_\xi = \{p(\xi) : p \in \mathcal{G}\}$ is $\mathrm{Coll}(\mathbb{N}, \xi)$-generic over $\mathbf{L}$ by Theorem 4.12(iii). Thus, $f_\xi[\mathcal{G}] = \bigcup \mathcal{G}_\xi$ is a map from $\mathbb{N}$ onto ξ. It follows that $\xi < \omega_1^{\mathbf{L}[\mathcal{G}]}$. Since $\xi < \Omega$ is arbitrary, we have $\Omega \leqslant \omega_1^{\mathbf{L}[\mathcal{G}]}$. $\square$

Lemma 7.18. *If $x \in \mathbf{L}[\mathcal{G}] \cap \mathbb{N}^\omega$, then there is an ordinal $\vartheta < \Omega$ such that $x \in \mathbf{L}[\mathcal{G}_{<\vartheta}]$, and hence $\omega_1^{\mathbf{L}[x]} < \Omega = \omega_1^{\mathbf{L}[\mathcal{G}]}$ and $\mathbf{L}[a] \cap \mathbb{N}^\omega$ is countable in $\mathbf{L}[G]$.*

Proof. The existence of ϑ is proved as in Lemma 7.11. Further, the map $\langle p, q\rangle \mapsto p \cup q$ is an order isomorphism between $\mathcal{P}_{<\vartheta} \times \mathcal{P}_{\geqslant\vartheta}$ and $\mathcal{P}$. By Theorem 4.2(iii), this readily implies that the set $\mathcal{G}_{<\vartheta}$ is $\mathcal{P}_{<\vartheta}$-generic over $\mathbf{L}$. Since $\operatorname{card} \mathcal{P}_{<\vartheta} < \Omega$ in $\mathbf{L}$, there is an ordinal $\varkappa < \Omega$ for which it is true in $\mathbf{L}$ that $\varkappa$ is a regular cardinal and $\operatorname{card} \mathcal{P}_{<\vartheta} < \varkappa$. Then $\omega_1^{\mathbf{L}[x]} \leqslant \omega_1^{\mathbf{L}[\mathcal{G}_{<\vartheta}]} \leqslant \varkappa < \Omega$ by Theorem 4.2(i), and hence $\omega_1^{\mathbf{L}[x]} < \Omega = \omega_1^{\mathbf{L}[\mathcal{G}]}$. $\square$

Lemma 7.19 (compare with Lemmas 7.8 and 7.12). *If $x \in \mathbf{L}[\mathcal{G}] \cap \mathbb{N}^\omega$, then the class $\mathbf{L}[\mathcal{G}]$ is a $\mathcal{P}$-generic extension of $\mathbf{L}[x]$.*

Proof (a sketch). Lemma 7.18 gives an ordinal $\vartheta < \Omega$ such that $x \in \mathbf{L}[\mathcal{G}_{<\vartheta}]$. We can assume that $\vartheta = \varkappa+1$, where $\varkappa < \Omega$ is a cardinal in $\mathbf{L}$. The set $\mathcal{P}'$ of all 'conditions' $p \in \mathcal{P}_{\leqslant\varkappa}$ such that $\xi \in \operatorname{dom} p$, $\operatorname{dom} p \smallsetminus \{\xi\} = \operatorname{ran} p(\xi)$, and $\operatorname{dom} p(\eta) = \operatorname{dom} p(\xi)$ for any $\eta \in \operatorname{dom} p$ is dense in $\mathcal{P}_{\leqslant\varkappa}$ and is a tree with $\varkappa$-branchings (since $\varkappa$ is a cardinal). Therefore, $\mathcal{P}'$ is order isomorphic to the set $\mathbb{C}(\varkappa) = \mathrm{Coll}(\mathbb{N}, \varkappa)$ in $\mathbf{L}$. We conclude that $\mathbf{L}[\mathcal{G}_{\leqslant\varkappa}] = \mathbf{L}[F]$, where the set $F \subseteq \mathbb{C}(\varkappa)$ is $\mathbb{C}(\varkappa)$-generic over $\mathbf{L}$. As in the proof of Lemma 7.8, there is a tree $\Sigma \subseteq \mathbb{C}(\varkappa)$, $\Sigma \in \mathbf{L}[x]$, having no $\subseteq$-maximal elements and such that $F \subseteq \Sigma$ and F is Σ-generic over $\mathbf{L}[x]$. The rest follows the proof of Lemma 7.12 and, in particular, uses the obvious fact that the set $\mathbb{C}(\vartheta) = \mathrm{Coll}(\mathbb{N}, \vartheta)$ is order isomorphic to $\Sigma \times \mathbb{C}(\vartheta)$. This implies that $\mathbf{L}[\mathcal{G}_{\leqslant\vartheta}]$ is a $\mathbb{C}(\vartheta)$-generic extension of the class $\mathbf{L}[a]$. $\square$

Lemma 7.20. *Suppose that $x \in \mathbf{L}[\mathcal{G}] \cap \mathbb{N}^\omega$, $\varphi(v_1, \dots, v_n)$ is an $\in$-formula, and $z_1, \dots, z_n \in \mathbf{L}[x]$ are arbitrary sets. Then*

$$\varphi(z_1, \dots, z_n) \text{ is true in } \mathbf{L}[\mathcal{G}] \iff \Lambda \Vdash_{\mathcal{P}}^{\mathbf{L}[x]} \varphi(\check{z}_1, \dots, \check{z}_n).$$

Proof. See the proof of Lemma 7.13. $\square$

Proof of Theorem 7.15. We begin with measurability and the Baire property in the Solovay model $\mathbf{L}[\mathcal{G}]$. Let $a \in \mathbf{L}[\mathcal{G}] \cap \mathbb{N}^\omega$. Arguing in $\mathbf{L}[\mathcal{G}]$, we consider an arbitrary $\mathbf{L}$-absolute σ-CAC ideal $\mathcal{I}$ of Borel subsets of $\mathbb{N}^\omega$ and an $\mathbf{ROD}$ set $X \subseteq \mathbb{N}^\omega$. We claim that X is $\mathcal{I}$-measurable.

One can assume that X belongs to $\mathrm{OD}(a)$ (otherwise the parameters in $\mathbb{N}^\omega$ in the definition of X can simply be adjoined to a.) Then $X = \{x : \varphi(a, x)\}$, where φ contains only a and ordinals as parameters. Suppose for brevity that a is the only parameter of φ (the general case is completely analogous). Then $X = \{x : \psi(a, x)\}$ by Lemma 7.20, where $\psi(a, x)$ is the formula "$\Lambda \Vdash_{\mathcal{P}}^{\mathbf{L}[a,x]} \varphi(\check{a}, \check{x})$ is true in $\mathbf{L}[a, x]$". The set $\mathbf{Rand}_{\mathcal{I}}\, \mathbf{L}[a]$ is $\mathcal{I}$-full by Lemma 7.18. In this case the $\mathcal{I}$-measurability of X in $\mathbf{L}[\mathcal{G}]$ follows from Theorem 4.17.

Let us pass to the perfect kernel property. We consider in $\mathbf{L}[\mathcal{G}]$ an **ROD** set $X = \{x : \varphi(a, x)\} = \{x : \psi(a, x)\} \subseteq \mathbb{N}^\omega$, where a, φ, and ψ are as above. Assuming that X is uncountable in $\mathbf{L}[\mathcal{G}]$, we have $X \not\subseteq \mathbf{L}[a]$. Let us fix some $x \in X \smallsetminus \mathbf{L}[a]$. Then $a, x \in \mathbf{L}[\mathcal{G}_{<\vartheta}]$ for a suitable $\vartheta < \Omega$ by Lemma 7.18. Moreover, the class $\mathbf{L}[\mathcal{G}_{\leqslant\vartheta}]$ is a $\mathbb{C}(\vartheta)$-generic extension of $\mathbf{L}[a]$, where $\mathbb{C}(\vartheta) = \mathrm{Coll}(\mathbb{N}, \vartheta)$ (see the proof of Lemma 7.19), that is, one can find a $\mathbb{C}(\vartheta)$-generic set $F_0 \subseteq \mathbb{C}(\vartheta)$ over $\mathbf{L}[a]$ such that $\mathbf{L}[\mathcal{G}_{\leqslant\vartheta}] = \mathbf{L}[a][F]$. Moreover, $x \in \mathbf{L}[a][F] \smallsetminus \mathbf{L}[a]$. The rest of the proof repeats the manipulations in subsection 7E with the only difference that $\mathbf{L}$ must be replaced everywhere by $\mathbf{L}[a]$.

$\square$ (*Theorems* 7.15 *and* 7.2)

Historical and bibliographical remarks. For the notions concerning ordinal definability, see [75] or [29], § 14.

Theorem 7.2 and its Corollary 7.3 are due to Solovay [89]. (For proofs in Russian which differ somewhat in details from those in [89], see [29], [35].) The model $\mathbf{L}[\mathcal{G}]$ in Theorem 7.15 is usually called the Solovay model, though the first publication about it is due to Levy [49].

Part (ii) of Corollary 7.3 and Corollary 7.4 in its parts (ii) and (iii), and in its part (i) in a somewhat weakened form (see the remarks concerning Theorem 7.1 below), that is, assertions on the undecidability of the classical problems in the diagram in the Introduction, were obtained independently by Lyubetskii [17], [68]. Theorem 7.1 in its part (II) is presented in [65], [68], [69]. Theorem 7.1 in its part (III) is completely similar. The proof for part (I) was obtained in the course of writing the present paper, and a weaker result claiming that, in **ZFC**, $\neg\,\mathrm{PK}(\Pi^1_1)$ does not imply the existence of an OD sieve defining an uncountable Π^1_1 set without perfect subsets was announced by Lyubetskii in [17], [67] (the proof was published in [68]). The crucial lemma, Lemma 7.19, was proved by Solovay [89]; Lemmas 7.8 and 7.12 are variations on the same theme.

§ 8. Irreversibility of the implications

Let us return to the diagram in the Introduction. Is the graph of connections among the five propositions there complete? The next theorem answers this question in the affirmative.

Theorem 8.1. *None of the following implications is derivable in* **ZFC**:

(1) $\mathrm{LM}(\Sigma^1_2) \implies \mathrm{PK}(\Pi^1_1)$; (2) $\mathrm{BP}(\Sigma^1_2) \wedge \mathrm{LM}(\Delta^1_2) \implies \mathrm{LM}(\Sigma^1_2)$;
(3) $\mathrm{BP}(\Sigma^1_2) \implies \mathrm{LM}(\Delta^1_2)$; (4) $\mathrm{BP}(\Delta^1_2) \wedge \mathrm{LM}(\Delta^1_2) \implies \mathrm{BP}(\Sigma^1_2)$;
(5) $\mathrm{LM}(\Delta^1_2) \implies \mathrm{BP}(\Delta^1_2)$.

The proofs of the five parts of the theorem can be carried out according to a common scheme based on iterations of forcing. The length of the iterations is ω_1, and the iterated forcings are taken from a list including four elementary forcings considered in subsection 8A. The longest and technically most complicated arguments are used in the proof of non-deducibility of the implications (2) and (4), in particular, because the iterations in this cases are of mixed character. In these arguments, as well as in the proof of non-deducibility of (3), we refer to the results of § 5 concerning the relationships between the regularity properties and the properties of the relation $\leqslant^*$ on $\mathbb{N}^\omega$ and $\boldsymbol{\ell}_1$. The proof of non-deducibility of (5) differs from the other proofs in that it requires an iteration with countable rather than finite support.

The exposition in this section assumes that the reader is somewhat acquainted with the method of iterated forcing (for related references, see the historical and bibliographical remarks below.)

8A. Four key forcings. The proof of Theorem 8.1 uses generic models connected with the following four forcings.

Cohen forcing: $\mathbf{C} = \mathbb{N}^{<\omega}$ with the order such that $s \leqslant t$ (s is stronger) if $t \subseteq s$;

dominating forcing: $\mathbf{D}$ consists of all pairs of the form $\langle s, f\rangle$ such that $s \in \mathbb{N}^{<\omega}$, $f \in \mathbb{N}^{\omega}$, $s \subset f$, with the order such that $\langle s, f\rangle \leqslant \langle t, g\rangle$ ($\langle s, f\rangle$ is stronger) if $t \subseteq s$ and $\forall\, n\ (g(n) \leqslant f(n))$;

random forcing: $\mathbf{B}$ consists of all trees $T \subseteq 2^{<\omega}$ (see subsection 1A) having no $\subseteq$-maximal elements and such that $\boldsymbol{\lambda}([T]) > 0$ with the order for which $T \leqslant S$ (T is stronger) if $T \subseteq S$;

$\boldsymbol{\varepsilon}$-random forcing: if $0 < \varepsilon < 1$, then $\mathbf{A}_\varepsilon$ consists of all trees $T \in \mathbf{B}$ such that $\boldsymbol{\lambda}([T]) > \varepsilon$, with the same order as $\mathbf{B}$.

Clearly, $\mathbf{C}$ is a countable set, which is the same for all models. On the other hand, if $\mathfrak{M}$ is a transitive model of **ZFC**, then $(\mathbf{D})^{\mathfrak{M}}$ (that is, $\mathbf{D}$ defined in $\mathfrak{M}$) coincides with $\mathbf{D} \cap \mathfrak{M}$ by Theorem 2.8 (the absoluteness theorem), and the same holds for $\mathbf{B}$ and $\mathbf{A}_\varepsilon$. It is customary to say "a $\mathbf{D}$-generic set over $\mathfrak{M}$", or "a $\mathbf{D}$-generic extension of $\mathfrak{M}$", meaning $(\mathbf{D})^{\mathfrak{M}}$-generic sets and extensions, of course (generally, $\mathbf{D} \in \mathfrak{M}$ means that $\mathbb{N}^{\omega} \subseteq \mathfrak{M}$), and we shall follow this practice. The same applies to $\mathbf{B}$ and $\mathbf{A}_\varepsilon$.

Every $\mathbf{C}$-generic set $G \subseteq \mathbf{C}$ generates a point $a_G = \bigcup G \in \mathbb{N}^{\omega}$ which is a Cohen point over the ground model (see subsection 4D.)

The $\mathbf{D}$-generic sets G also generate points in $\mathbb{N}^{\omega}$, namely, we can set $a_G = \bigcup\{s : \exists\, f\ (\langle s, f\rangle \in G)\}$. The second components f of the 'conditions' $\langle s, f\rangle$ play the role of lower bounds for a_G: it is clear that $a_G(n) \geqslant f(n)$ whenever $\langle s, f\rangle \in G$ and $n \geqslant \operatorname{lh} s$. This readily implies that $f \leqslant^* a_G$ for any function $f \in \mathbb{N}^{\omega}$ in the ground model, where $\leqslant^*$ is the relation of eventual domination, that is, $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for almost all (all but finitely many) $n \in \mathbb{N}$.

If a set $G \subseteq \mathbf{B}$ is generic, then $T_G = \bigcap G$ is a branch in $2^{<\omega}$. In other words, the intersection $[G] = \bigcap_{T \in G}[T]$ contains a unique point a_G of 2^{ω}. The point a_G is $\boldsymbol{\lambda}$-random over the ground model by Corollary 4.14.

Finally, any $\mathbf{A}_\varepsilon$-generic set G generates a tree $T_G = \bigcap G \subseteq 2^{<\omega}$; the closed set $[T_G] \subseteq 2^{\omega}$ satisfies the condition $\boldsymbol{\lambda}([T_G]) = \varepsilon$, and any point of the tree is random over the ground model (see subsection 8B).

Lemma 8.2. *Each of the four forcings satisfies the* CAC.

Proof. It is clear that the CAC holds for $\mathbf{C}$ because the set itself is countable. For $\mathbf{D}$ note that any two 'conditions' $\langle s, f\rangle$ and $\langle t, g\rangle$ with $s = t$ are compatible in $\mathbf{D}$. For $\mathbf{B}$ note that if $T, S \in \mathbf{B}$ are incompatible in $\mathbf{B}$, then $\boldsymbol{\lambda}([T] \cap [S]) = 0$. Finally, let us consider $\mathbf{A}_\varepsilon$. Suppose the contrary: let $A \subseteq \mathbf{A}_\varepsilon$ be an uncountable antichain. There is a $\delta > 0$ such that the set $A' = \{T \in A : \boldsymbol{\lambda}([T]) > \varepsilon + 3\delta\}$ is also uncountable. Every set $T \in A'$ can be covered by an appropriate set U_T with

$\boldsymbol{\lambda}(U_T) < \boldsymbol{\lambda}(T) + \delta$ which is the complement of the union of *finitely many* Baire intervals. Then $\boldsymbol{\lambda}(U_T \cap U_{T'}) \leqslant \boldsymbol{\lambda}([T] \cap [T']) + 2\delta \leqslant \varepsilon + 2\delta$, and hence $U_T \neq U_{T'}$ for any $T \neq T' \in A'$. It follows that the set $\{U_T : T \in A'\}$ is uncountable together with A'. We arrive at a contradiction, because there are only countably many finite unions of intervals. $\square$

We note that each of the four forcings admits an equivalent representation (in the sense of Theorem 4.2(ii)) in the form $\mathbb{P}_{\mathcal{J}}$ for a suitable σ-CAC ideal $\mathcal{J}$ that is $\mathfrak{M}$-absolute for any transitive model $\mathfrak{M} \models \mathbf{ZFC}$. The ideals $\mathcal{J}_{\text{cat}}$ and $\mathcal{J}_{\boldsymbol{\lambda}}$ work for $\mathbf{C}$ and $\mathbf{B}$ (see subsection 4D), and the ideal of meagre sets in the sense of a certain non-Polish topology works for $\mathbf{D}$ (see subsection 9C). Concerning $\mathbf{A}_\varepsilon$, see [10], 3.4.B.

8B. $\mathbf{LM}(\boldsymbol{\Sigma}^1_2) \not\Longrightarrow \mathbf{PK}(\boldsymbol{\Pi}^1_1)$. In principle, this is the easiest part of Theorem 8.1. One can use the standard model (we denote it by $\mathbf{L}[X]$) in which Martin's axiom $\mathbf{MA}_{\aleph_1}$ holds for families of cardinality $\aleph_1$, and in addition $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$. (For the construction of such a model, see [11], Chap. 5, § 6, [73], or [29], § 22.) The equality $\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}}$ implies $\neg\,\mathrm{PK}(\Pi^1_1)$ in $\mathbf{L}[X]$ by Corollary 3.4. On the other hand, it is known that $\mathbf{MA}_{\aleph_1}$ implies the additivity of the ideal of measure zero sets with respect to unions of $\aleph_1$ sets (see [73] or [29], Theorem 50). Thus, $\mathrm{LM}(\Sigma^1_2)$ by Corollary 1.7.

We now want to present a more direct (but somewhat longer) argument which yields a model in which the continuum hypothesis $2^{\aleph_0} = \aleph_1$ holds (unlike the construction based on $\mathbf{MA}_{\aleph_1}$). By Corollary 3.4, in order to prove that $\mathrm{LM}(\Sigma^1_2) \not\Longrightarrow \mathrm{PK}(\Pi^1_1)$, it is sufficient to construct a model in which one has $\omega_1^{\mathbf{L}[a]} = \omega_1$ and $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathbf{L}[a]) = 1$ for any $a \in \mathbb{N}^\omega$.

We first consider a more elementary question: for a transitive model $\mathfrak{M}$ of $\mathbf{ZFC}$, how can one construct a generic extension $\mathfrak{M}[G]$ of it such that $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathfrak{M}) = 1$ and $\omega_1^{\mathfrak{M}} = \omega_1^{\mathfrak{M}[G]}$? This can be done by using the ε-random forcing $\mathbf{A}_\varepsilon$, where we set $\varepsilon = \frac{1}{2}$ for definiteness.

Lemma 8.3. *Suppose that a set G is $(\mathbf{A}_\varepsilon)^{\mathfrak{M}}$-generic over $\mathfrak{M}$. Then $\omega_1^{\mathfrak{M}} = \omega_1^{\mathfrak{M}[G]}$ and $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathfrak{M}) = 1$ in $\mathfrak{M}[G]$.*

Proof. It is clear that $[G] = \{[T] : T \in G\}$ is a system of closed subsets of 2^ω of measure $> \varepsilon$ with the finite intersection property, and hence $X = \bigcap[G]$ is also a closed subset of 2^ω with $\boldsymbol{\lambda}(X) \geqslant \varepsilon$; in fact, $\boldsymbol{\lambda}(X) = \varepsilon$. *We claim that* $\bigcap[G] \subseteq \mathbf{Rand}\,\mathfrak{M}$. Indeed, let $c \in \mathbf{BC} \cap \mathfrak{M}$ and $\boldsymbol{\lambda}(\mathbf{B}_c) = 0$. Then the set $D_c = \{T \in (\mathbf{A}_\varepsilon)^{\mathfrak{M}} : [T] \cap \mathbf{B}_c = \varnothing\}$ is dense in $(\mathbf{A}_\varepsilon)^{\mathfrak{M}}$ and belongs to $\mathfrak{M}$. It follows that $D_c \cap G \neq \varnothing$, and therefore $\mathbf{B}_c \cap X = \varnothing$, as was to be proved.

Thus, it is true in $\mathfrak{M}[G]$ that there is a closed set $X = \bigcap[G] \subseteq \mathbf{Rand}\,\mathfrak{M}$ of $\boldsymbol{\lambda}$-measure $\varepsilon > 0$. Moreover, for any $\varepsilon' > \varepsilon$ there is a $T \in G$ (in which case $X \subseteq [T]$) with $\varepsilon < \boldsymbol{\lambda}([T]) < \varepsilon'$. This readily implies, by using Lemma 1.3, that in fact $\boldsymbol{\lambda}(\mathbf{Rand}\,\mathfrak{M}) = 1$ in $\mathfrak{M}[G]$.

$\square$ (*Lemma* 8.3)

The proof of Theorem 8.1(1) itself consists in iterating the forcing $\mathbf{A}_\varepsilon$. According to general theorems on iterated forcing with finite support, there is a forcing $\mathbb{P} \in \mathbf{L}$ satisfying the CAC in $\mathbf{L}$ and producing generic objects of the form $G = \{G_\xi\}_{\xi<\omega_1^{\mathbf{L}}}$, and hence every object G_ξ is $(A_\varepsilon)^{\mathbf{L}[G_{<\xi}]}$-generic over $\mathbf{L}[G_{<\xi}]$, where $G_{<\xi} = \{G_\eta\}_{\eta<\xi}$. It follows from the CAC that $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$; moreover, for any point $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ there is an ordinal $\xi < \omega_1^{\mathbf{L}}$ such that $a \in \mathbf{L}[G_{<\xi}] \cap \mathbb{N}^\omega$. Then by Lemma 8.3, $\mathbf{Rand}\,\mathbf{L}[a]$ is a set of full $\boldsymbol{\lambda}$-measure in $\mathbf{L}[G_{\leqslant\xi}]$, and therefore in $\mathbf{L}[G]$.

$\square$ (*Theorem* 8.1(1))

8C. $\mathbf{BP}(\boldsymbol{\Sigma}^1_2) \not\Longrightarrow \mathbf{LM}(\boldsymbol{\Delta}^1_2)$. We use here an iteration of the dominating forcing $\mathbf{D}$. We are going to use Theorem 5.3 and need the following lemma.

Lemma 8.4. *If $a \in \mathbb{N}^\omega$ and if $G \subseteq (\mathbf{D})^{\mathbf{L}[a]}$ is $(\mathbf{D})^{\mathbf{L}[a]}$-generic over $\mathbf{L}[a]$, then $\mathbf{L}[a,G] \cap \mathbf{Coh}\,\mathbf{L}[a] \neq \varnothing$.*

Proof. Let $y = \bigcup\{s : \exists f\ (\langle s,f\rangle \in G)\}$. We recall that $(m)_1 = i$ if $m = 2^i(2j+1)-1$ for some j. Let $\overline{y}(n) = (y(n))_1\ \forall n$, so that $\overline{y} \in \mathbb{N}^\omega$. We claim that $\overline{y} \in \mathbf{Coh}\,\mathbf{L}[a]$. It suffices to show that $\overline{y} \notin [T]$ for any tree $T \in \mathbf{L}[a]$, $T \subseteq \mathbb{N}^{<\omega}$, such that $[T]$ is a nowhere dense subset of $\mathbb{N}^\omega$. (See the proof of Theorem 5.3.) If T is as above, then the set D_T of all 'conditions' $\langle s,f\rangle \in (\mathbf{D})^{\mathbf{L}[a]}$ satisfying $\mathcal{N}_{\overline{s}} \cap [T] = \varnothing$ (where $\overline{s} \in \mathbb{N}^{<\omega}$ is obtained from s as $\overline{y}$ is from y; $\operatorname{lh}\overline{s} = \operatorname{lh} s$) is dense in $(\mathbf{D})^{\mathbf{L}[a]}$ and belongs to $\mathbf{L}[a]$ together with T. Therefore, $D_T \cap G \neq \varnothing$, and the desired result follows. $\square$

The proof of Theorem 8.1(3) uses an iteration of the forcing $\mathbf{D}$ of length $\omega_1^{\mathbf{L}}$. Unfortunately, here we have to discuss technical details of iterated forcing with finite support.

Arguing in $\mathbf{L}$, we use induction on ξ, $0 < \xi \leqslant \omega_1^{\mathbf{L}}$, to define a forcing $\mathbb{P}_\xi$ consisting of functions p defined on ξ, along with the following objects $\overline{\mathbb{Q}}^\xi$, Q^ξ, $\mathbf{1}_\xi$.

- (1°) $\mathbb{P}_0 = \{\Lambda\}$ and $\mathbf{1}_0 = \Lambda$ (the empty function).
- (2°) If a forcing $\mathbb{P}_\xi$ has been defined (and consists of functions p with $\operatorname{dom} p = \xi$), and if $\mathbf{1}_\xi$ is the largest (that is, the weakest) element of $\mathbb{P}_\xi$, then we fix a $\mathbb{P}_\xi$-term $\overline{\mathbb{Q}}^\xi$ such that $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\overline{\mathbb{Q}}^\xi$ is the forcing $\mathbf{D}$". We also fix a $\mathbb{P}_\xi$-term $\mathbf{1}^\xi$ such that $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\mathbf{1}^\xi$ is the pair $\langle\breve{\Lambda},\breve{\mathbf{0}}\rangle$". (We note that the pair $\langle\Lambda,\mathbf{0}\rangle$ is the weakest element of the forcing $\mathbf{D}$; here $\mathbf{0}(k) = 0\ \forall\, k$.)
- (3°) We set $Q^\xi = \{q : \exists p\ (\langle p,q\rangle \in \overline{\mathbb{Q}}^\xi\}$. It follows from the definitions in subsection 4A that $\overline{\mathbb{Q}}^\xi[G] \subseteq \{t[G] : t \in Q^\xi\}$ for all G, and hence the requirement $p(\xi) \in Q^\xi$ below in 4° (introduced to ensure that $\mathbb{P}_{\xi+1}$ is a set rather than a true class in $\mathfrak{M}$) leads to no loss of generality.
- (4°) $\mathbb{P}_{\xi+1}$ consists of all functions p, $\xi+1 = \operatorname{dom} p$, such that $p \restriction \xi \in \mathbb{P}_\xi$, $p(\xi) \in Q^\xi$, and $p\restriction\xi \Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$p(\xi) \in \overline{\mathbb{Q}}^\xi$".
- (5°) The order on $\mathbb{P}_{\xi+1}$ is defined as follows: $p \leqslant q$ if $p\restriction\xi \leqslant q\restriction\xi$ and $p\restriction\xi \Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$p(\xi) \leqslant q(\xi)$ in the sense of $\overline{\mathbb{Q}}^\xi$". The element $\mathbf{1}_{\xi+1} \in \mathbb{P}_{\xi+1}$ is defined as follows: $\mathbf{1}_{\xi+1}\restriction\xi = \mathbf{1}_\xi$ and $\mathbf{1}_{\xi+1}(\xi) = \mathbf{1}^\xi$.
- (6°) (The *finite support* assumption.) If $\vartheta \leqslant \omega_2^{\mathbf{L}}$ is a limit ordinal, then $\mathbb{P}_\vartheta$ consists of all the functions p, $\vartheta = \operatorname{dom} p$, such that $p\restriction\xi \in \mathbb{P}_\xi$ and $\|p\| = \{\xi < \vartheta : \neg p\restriction\xi \Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$p(\xi) = \mathbf{1}^\xi$"$\}$ is a finite set.

(7°) The order on $\mathbb{P}_\vartheta$ is defined as follows: $p \leqslant q$ if $p \restriction \xi \leqslant q \restriction \xi$ for all $\xi < \vartheta$. The element $\mathbf{1}_\vartheta \in \mathbb{P}_\vartheta$ is defined as follows: $\mathbf{1}_\vartheta(\xi) = \mathbf{1}^\xi$ for all $\xi < \vartheta$.

The forcing $\mathbb{P} = \mathbb{P}_{\omega_1^{\mathbf{L}}} \in \mathbf{L}$ satisfies the CAC (in $\mathbf{L}$). Indeed, the iterations of finitely supported CAC forcings satisfy the CAC, and the fact that the forcing $\mathbf{D}$ satisfies the CAC follows from Lemma 8.2.

Let us fix a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathbf{L}$. Then any set of the form $G_\xi = \{p \restriction \xi : p \in G\}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$.

We claim that $\mathrm{BP}(\Sigma^1_2)$ *is true in* $\mathbf{L}[G]$. By Theorem 5.3, it suffices to show that for any $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$

1) $\mathbf{Coh}\,\mathbf{L}[a] \cap \mathbf{L}[G] \neq \varnothing$,
2) the set $\mathbf{L}[a] \cap \mathbb{N}^\omega$ is $\leqslant^*$-bounded in $\mathbf{L}[G]$.

It follows from the CAC by the usual arguments that $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$, and hence $a \in \mathbf{L}[G_\xi] \cap \mathbb{N}^\omega$ for some $0 < \xi < \omega_1^{\mathbf{L}}$. However, $\mathbf{L}[G_{\xi+1}]$ is a $(\mathbf{D})^{\mathbf{L}[G_\xi]}$-generic extension of $\mathbf{L}[G_\xi]$ by the general properties of iterated forcing. We conclude that the property 1) follows from Lemma 8.4 and 2) is a simple property of the $\mathbf{D}$-generic extensions (see subsection 8A).

We claim that $\mathrm{LM}(\Delta^1_2)$ *is false in* $\mathbf{L}[G]$. According to Corollary 3.4(IV), it suffices to show that the set $\mathbf{Rand}\,\mathbf{L}[a]$ is empty in $\mathbf{L}[G]$ for at least one $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$. Let us take some $a \in \mathbf{L} \cap \mathbb{N}^\omega$, say $a = \mathbf{0}$, and consider an arbitrary $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$. We must prove that $x \notin \mathbf{Rand}\,\mathbf{L}[a]$. As above, $x \in \mathbf{L}[G_\xi]$ for some $0 < \xi < \omega_1^{\mathbf{L}}$. We have $x = \mathbf{x}[G]$ for some $\mathbb{P}_\xi$-name $\mathbf{x} \in \mathbf{L}$.

We argue in $\mathbf{L}$. A set $P \subseteq \mathbb{P}_\xi$ is said to have the *finite intersection property* if for any finite $P' \subseteq P$ there is a 'condition' $r \in \mathbb{P}_\xi$ satisfying $r \leqslant p$ for each $p \in P'$. We claim that $\mathbb{P}_\xi$ itself has the σ-*finite intersection property*, that is, it admits a representation in the form $\mathbb{P}_\xi = \bigcup_{n \in \mathbb{N}} P_n$, where all the sets P_n have the finite intersection property.

We argue by induction on ξ. If $\xi = 0$, then there is nothing to prove. Suppose that $\vartheta < \omega_2^{\mathbf{L}}$ is a limit ordinal and that $\mathbb{P}_\xi = \bigcup_n P_n^\xi$ for any $\xi < \vartheta$, where all the sets P_n^ξ have the finite intersection property. Then $\mathbb{P}_\vartheta = \bigcup_{\xi<\vartheta} \bigcup_n P_{\xi n}$, where $P_{\xi n} = \{p \in \mathbb{P}_\vartheta : \|p\| \subseteq \xi \wedge p \restriction \xi \in P_n^\xi\}$. It remains to consider the step $\xi \to \xi + 1$. Suppose that $\mathbb{P}_\xi = \bigcup_n P_n$, where all the sets P_n have the finite intersection property. For any $n \in \mathbb{N}$ and $s \in \mathbb{N}^{<\omega}$ we define

$$P_{ns} = \bigl\{p \in \mathbb{P}_{\xi+1} : \exists\, q \in P_n \ (q \leqslant p \restriction \xi \wedge q \Vdash^{\mathbf{L}}_{\mathbb{P}_\xi} \text{ ``} s_{p(\xi)} = \check{s}\text{''})\bigr\},$$

where s_p stands for s for any 'condition' $p = \langle s, f\rangle \in \mathbf{D}$. One can readily see that every set P_{ns} has the finite intersection property and $\mathbb{P}_{\xi+1} = \bigcup_{s \in \mathbb{N}^{<\omega}} \bigcup_n P_{sn}$, which completes the proof of the σ-finite intersection property.

Thus, we have a representation $\mathbb{P}_\xi = \bigcup_n P_n$ (still in $\mathbf{L}$), where all the sets P_n have the finite intersection property. Let $\mathcal{T}_n$ be the set of all trees $T \subseteq 2^{<\omega}$ having no $\subseteq$-maximal elements and such that there is a 'condition' $p \in P_n$ which $\mathbb{P}_\xi$-forces the formula $\mathbf{x} \in [T]$. Since P_n has the finite intersection property, it follows that the sets $[T]$, $T \in \mathcal{T}_n$, form a system (of closed subsets of 2^ω) which has the finite intersection property. Hence, there is a point $x_n \in \bigcap_{T \in \mathcal{T}_n} [T]$. We can choose a sequence of trees $T_k \subseteq 2^{<\omega}$ in such a way that the sets $[T_k]$ become pairwise disjoint and disjoint from $X = \{x_n : n \in \mathbb{N}\}$, and satisfy $\sum_k \boldsymbol{\lambda}([T_k]) = 1$.

We now claim, this time in $\mathbf{L}[G]$, that x does not belong to the $\mathbf{F}_\sigma$ set $\bigcup_k [T_k]$, and hence does not belong to $\mathbf{Rand}\,\mathbf{L}[a]$, because the complement of the set $\bigcup_k [T_k]$ has $\boldsymbol{\lambda}$-measure zero. Suppose not: let x belong to one of the sets $[T_k]$. This is forced by some condition $p \in G$. Then $p \in P_n$ for some n, and hence $T_k \in \mathcal{T}_n$. By construction, this implies that $x_n \in [T_k]$, contradicting the choice of the trees T_k.

$\square$ (*Theorem* 8.1(3))

8D. $\mathbf{BP}(\boldsymbol{\Delta}^1_2) \wedge \mathbf{LM}(\boldsymbol{\Delta}^1_2) \not\Longrightarrow \mathbf{BP}(\boldsymbol{\Sigma}^1_2)$. A desired model can be obtained by simultaneous iteration of the Cohen forcing $\mathbf{C}$ and the random forcing $\mathbf{B}$ of length $\omega_1^{\mathbf{L}}$ and with finite support. Namely, $\mathbf{C}$ acts at the even steps (an even ordinal is any ordinal which is the sum of a limit ordinal and an even natural number, including 0) and $\mathbf{B}$ acts at the odd steps. (One can also reverse the definition.)

More precisely, the iterated forcing $\mathbb{P}$ is constructed by the scheme 1°–7° in subsection 8C with the following correction.

(2°) If a forcing $\mathbb{P}_\xi$ is already defined (and consists of functions p with $\xi = \operatorname{dom} p$) and if $\mathbf{1}_\xi$ is the largest (and hence the weakest) element of $\mathbb{P}_\xi$, then we fix a $\mathbb{P}_\xi$-term $\overline{\mathbb{Q}}^\xi$ such that $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\overline{\mathbb{Q}}^\xi = \mathbf{C}$" if ξ is even and $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\overline{\mathbb{Q}}^\xi = \mathbf{B}$" if ξ is odd. We also fix a $\mathbb{P}_\xi$-term $\mathbf{1}^\xi$ such that $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\mathbf{1}^\xi = \check{\Lambda}$" if ξ is even and $\Vdash^{\mathbf{L}}_{\mathbb{P}_\xi}$ "$\mathbf{1}^\xi = {}^{\smile}2^{<\omega}$" if ξ is odd. (We note that Λ is the weakest element of $\mathbf{C}$ and $2^{<\omega}$ is the weakest element of $\mathbf{B}$.)

Let us fix a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathbf{L}$. Every set of the form $G_\xi = \{p \restriction \xi : p \in G\}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$. Moreover, $\mathbf{L}[G] = \mathbf{L}[\{a_\xi\}_{\xi<\omega_1^{\mathbf{L}}}]$, where $a_\xi \in \mathbf{Coh}\,\mathbf{L}[\{a_\eta\}_{\eta<\xi}]$ if ξ is even and $a_\xi \in \mathbf{Rand}\,\mathbf{L}[\{a_\eta\}_{\eta<\xi}]$ if ξ is odd.

Our objective is to prove that $\mathrm{BP}(\Delta^1_2)$ and $\mathrm{LM}(\Delta^1_2)$ hold in such a model $\mathbf{L}[G]$ but $\mathrm{BP}(\Sigma^1_2)$ fails. The 'affirmative' part of this claim easily follows from the results obtained above. Indeed, since $\mathbf{C}$ and $\mathbf{B}$ are CAC forcings, the iterated forcing $\mathbb{P}$ also satisfies the CAC, and therefore $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$. It follows that for any $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ there is an ordinal $\xi < \omega_1^{\mathbf{L}}$ for which $a \in \mathbf{L}[G_\xi]$. This implies that $\mathbf{Coh}\,\mathbf{L}[a] \neq \varnothing$ and $\mathbf{Rand}\,\mathbf{L}[a] \neq \varnothing$, and hence $\mathrm{LM}(\Delta^1_2)$ and $\mathrm{BP}(\Delta^1_2)$ hold in $\mathbf{L}[G]$ by Corollary 3.4.

As far as the 'negative' part is concerned, in order to derive $\neg\,\mathrm{BP}(\Sigma^1_2)$, it suffices (by Theorem 5.3) to show that at least one set of the form $\mathbf{L}[a] \cap \mathbb{N}^\omega$, $a \in \mathbb{N}^\omega$, is not bounded in $\mathbf{L}[G]$. We claim that $\mathbf{L} \cap \mathbb{N}^\omega$ is not bounded in $\mathbf{L}[G]$. The proof of this claim is based on the fact that a certain property of forcings is preserved by the iteration.

Definition 8.5. A forcing $\mathcal{P}$ is said to be $\leqslant^*$-*good* if for any $\mathcal{P}$-generic set $G \subseteq \mathcal{P}$ over the universe $\mathbf{V}$ and for any $f \in \mathbf{V}[G] \cap \mathbb{N}^\omega$ there is a $g \in \mathbf{V} \cap \mathbb{N}^\omega$ such that $\forall\, x \in \mathbf{V} \cap \mathbb{N}^\omega$ $(x \leqslant^* f \implies x \leqslant^* g)$.

Theorem 8.6. *It is true in* $\mathbf{L}$ *that every forcing* $\mathbb{P}_\xi$, $\xi < \omega_2^{\mathbf{L}}$, *is* $\leqslant^*$-*good.*

It follows from the theorem that in every $\mathbb{P}_\xi$-generic extension of $\mathbf{L}$ the set $\mathbf{L} \cap \mathbb{N}^\omega$ is not $\leqslant^*$-bounded. In particular, this is true in any class of the form $\mathbf{L}[G_\xi]$. However, as mentioned above, any $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ belongs to $\mathbf{L}[G_\xi] \cap \mathbb{N}^\omega$ for some $\xi < \omega_2^{\mathbf{L}}$, and therefore the set $\mathbf{L} \cap \mathbb{N}^\omega$ is not $\leqslant^*$-bounded even in $\mathbf{L}[G]$, as was to be proved. Thus, the problem is reduced to the proof of Theorem 8.6.

We precede the proof by two lemmas.

Lemma 8.7. *The forcings* $\mathbf{B}$ *and* $\mathbf{C}$ *are* $\leqslant^*$*-good.*

Proof. We first consider the forcing $\mathbf{B}$. Suppose that a set $G \subseteq \mathbf{B}$ is generic over $\mathbf{V}$ and that $f \in \mathbb{N}^\omega \cap \mathbf{V}[G]$. Then $f = \underline{f}[G]$, where $\underline{f} \in \mathbf{V}$. One can assume that $\Vdash_{\mathbf{B}} \underline{f} \in \mathbb{N}^\omega$. We claim that the set

$$D' = \{T' \in \mathbf{B} : \exists\, g \in \mathbb{N}^\omega\ (T' \Vdash_{\mathbf{B}} \underline{f} \leqslant^* \breve{g})\}$$

is dense in $\mathbf{B}$. (Since G is generic, this implies that $f \leqslant^* g$ for some $g \in \mathbf{V} \cap \mathbb{N}^\omega$, and the 'goodness' is obvious.) Let $T_0 \in \mathbf{B}$. For any n the set

$$D_n = \{T \in \mathbf{B} : T \subseteq T_0 \wedge \exists\, k\ (T \Vdash_{\mathbf{B}} \underline{f}(\breve{n}) = \breve{k})\}$$

is dense in $\mathbf{B}$ below T_0 for any n. Let us choose a maximal antichain $A_n \subseteq D_n$; it is countable by the CAC. For any n we can find a finite subset $A'_n \subseteq A_n$ such that the set $X_n = \bigcup_{T \in A'_n} [T]$ (a closed subset of $[T_0]$) satisfies $\boldsymbol{\lambda}(X_n) \geqslant \alpha(1 - 2^{-n-2})$, where $\alpha = \boldsymbol{\lambda}([T_0])$. Then $T_n = \bigcup A'_n$ is a tree in $2^{<\omega}$ (with no maximal elements) and $[T_n] = X_n$, and therefore $T_n \in \mathbf{B}$. Since A'_n is finite, there is a k_n such that $T_n \Vdash_{\mathbf{B}} \underline{f}(\breve{n}) \leqslant \breve{k}_n$. We finally set $T' = \bigcap_n T_n$. It follows from the construction that $\boldsymbol{\lambda}([T']) \geqslant \frac{\alpha}{2}$, and hence $T' \in \mathbf{B}$. Moreover, $T' \Vdash_{\mathbf{B}} \underline{f}(\breve{n}) \leqslant \breve{k}_n$ for any n, that is, $T' \Vdash_{\mathbf{B}} \underline{f} \leqslant^* \breve{g}$, where $g(n) = k_n\ \forall\, n$, and hence $T' \in D'$.

We use other arguments for the forcing $\mathbf{C} = \mathbb{N}^{<\omega}$. Let

$$F_{ns} = \{x \in \mathbb{N}^\omega : s \Vdash_{\mathbf{C}} \breve{x} \leqslant_n \underline{f}\} \quad \text{for} \quad s \in \mathbf{C} = \mathbb{N}^{<\omega} \quad \text{and} \quad n \in \mathbb{N},$$

where $x \leqslant_n y$ means that $x(i) \leqslant y(i)$ for all $i \geqslant n$. It is clear that $M_{nsi} = \max_{h \in F_{ns}} h(i)$ is finite if $i \geqslant n$. We set $f_{ns}(i) = M_{nsi}$ for $i \geqslant n$ and $f_{ns}(i) = 0$ for $i < n$. Thus, $x \leqslant_n f_{ns}$ for any $x \in F_{ns}$. It remains to choose $g \in \mathbb{N}^\omega$ in such a way that $f_{ns} \leqslant^* g$ for any n and s. $\square$

The next lemma characterizes the $\leqslant^*$-good forcings in terms of the forcing relation. We recall that $\Vdash_{\mathcal{P}}$ means $\Vdash_{\mathcal{P}}^{\mathbf{V}}$.

Lemma 8.8. *A* CAC *forcing* $\mathcal{P}$ *is* $\leqslant^*$*-good if and only if for any name* t *such that* $\Vdash_{\mathcal{P}} t \in \mathbb{N}^\omega$ *there is a function* $g \in \mathbb{N}^\omega$ *for which*

$$\forall\, x \in \mathbb{N}^\omega\ \big((\exists\, p \in \mathcal{P}\ (p \Vdash_{\mathcal{P}} \breve{x} \leqslant^* t)) \implies x \leqslant^* g\big).$$

Proof. In the non-trivial direction, suppose that $\mathcal{P}$ is a good forcing and $\Vdash_{\mathcal{P}} t \in \mathbb{N}^\omega$. Then the set

$$D = \{p \in \mathcal{P} : \exists\, g = g_p \in \mathbb{N}^\omega\ \forall\, x \in \mathbb{N}^\omega\ (p \Vdash_{\mathcal{P}} (\breve{x} \leqslant^* t \Longrightarrow \breve{x} \leqslant^* \breve{g}))\}$$

is dense in $\mathcal{P}$. It follows from the CAC that there is a countable maximal antichain $A \subseteq D$. Since A is countable, there is a $g \in \mathbb{N}^\omega$ such that $g_p \leqslant^* g$ for all $p \in A$. Clearly, g ensures the 'goodness' of $\mathcal{P}$. $\square$

Proof of Theorem 8.6. The 'goodness' of $\mathbb{P}_\xi$ in $\mathbf{L}$ is proved by induction on $\xi < \omega_2^{\mathbf{L}}$. Lemma 8.7 ensures the inductive step $\xi \to \xi + 1$, because, in the iteration under consideration, the extension $\mathbf{L}[G_{\xi+1}]$ of $\mathbf{L}[G_\xi]$ is either $\mathbf{B}$-generic or $\mathbf{C}$-generic. To carry out the limit step, we consider a limit ordinal $\xi < \omega_2^{\mathbf{L}}$. There is a strictly increasing sequence $\{\xi_n\}_{n\in\mathbb{N}} \in \mathbf{L}$ of smaller ordinals with $\xi = \sup_n \xi_n$. Let us choose an arbitrary $f \in \mathbf{L}[G] \cap \mathbb{N}^\omega$. There is a name $\underline{f} \in \mathbf{L}$ such that $f = \underline{f}[G]$. One can assume that $\Vdash_{\mathbb{P}}^{\mathbf{L}}$ "$\underline{f} \in \mathbb{N}^\omega$".

Let us consider an arbitrary n. Arguing in $\mathbf{L}[G_{\xi_n}]$, we can construct a sequence $\{p_n^k\}_{k\in\mathbb{N}} \in \mathbf{L}[G_{\xi_n}]$ of 'conditions' $p_n^k \in \mathbb{P}$ such that for any k one has $p_n^{k+1} \leqslant p_n^k$ (that is, p_n^{k+1} is stronger in $\mathbb{P}$), $p_n^k \upharpoonright \xi_n \in G_{\xi_n}$, and $p_n^k \Vdash_{\mathbb{P}}^{\mathbf{L}}$ "$\underline{f} \upharpoonright \breve{k} = {}^{\smile}s_k$" for some $s_k \in \mathbb{N}^{<\omega}$. Then $s_k \subset s_{k+1}$, and therefore $f_n = \bigcup_k s_k \in \mathbf{L}[G_{\xi_n}] \cap \mathbb{N}^\omega$. There is a name $\underline{f_n} \in \mathbf{L}$ such that $f_n = \underline{f_n}[G_{\xi_n}]$ and $\Vdash_{\mathbb{P}_{\xi_n}}^{\mathbf{L}}$ "$\underline{f_n} \in \mathbb{N}^\omega$". This construction can be carried out in such a way that the sequence $\{\underline{f_n}\}_{n\in\mathbb{N}}$ of names belongs to $\mathbf{L}$. Since $\mathbb{P}_{\xi_n}$ is 'good', it follows from Lemma 8.8 that for any n there is a function $g_n \in \mathbf{L} \cap \mathbb{N}^\omega$ for which

$$\forall\, x \in \mathbf{L} \cap \mathbb{N}^\omega \ \bigl((\exists\, p \in \mathbb{P}_{\xi_n} \ (p \Vdash_{\mathbb{P}_{\xi_n}}^{\mathbf{L}} \breve{x} \leqslant^* \underline{f_n})) \implies x \leqslant^* g_n\bigr).$$

One can again assume that $\{g_n\}_{n\in\mathbb{N}} \in \mathbf{L}$. Under this assumption, there is a $g \in \mathbf{L} \cap \mathbb{N}^\omega$ such that $g_n \leqslant^* g\ \forall\, n$. We claim that g proves that $\mathbb{P}_\xi$ is 'good' with respect to f, that is, $x \leqslant^* f \implies x \leqslant^* g$ for any $x \in \mathbf{L} \cap \mathbb{N}^\omega$.

Suppose the contrary. Let $x \in \mathbf{L}[a] \cap \mathbb{N}^\omega$, $x \leqslant^* f$, and $x \not\leqslant^* g$. Then one can find a number m and a 'condition' $p \in G$ such that $p \Vdash_{\mathbb{P}}^{\mathbf{L}} \breve{x} \leqslant^*{}_m \underline{f}$, where $x \leqslant^*{}_m y$ means that $x(k) \leqslant y(k)$ for any $k \geqslant m$. Since the iteration under consideration is finitely supported, it follows that $\|p\| \subseteq \xi_n$ for some n. Under our assumptions we have $x \not\leqslant^* g_n$; in particular, $x \not\leqslant^*_m g_n$. In other words, there is a number $k \geqslant m$ such that $j = g_n(k) < x(k)$. By the definition of g_n, this means that $p_n^k \Vdash_{\mathbb{P}}^{\mathbf{L}}$ "$\underline{f}(\breve{k}) = \breve{j}$".

We recall that $p_n^k \upharpoonright \xi_n \in G_{\xi_n}$ by definition. This readily implies the existence of a $\mathbb{P}$-generic set $G' \subseteq \mathbb{P}$ over $\mathbf{L}$ such that $G'_{\xi_n} = G_{\xi_n}$ and $p_n^k \in G'$. Then $f' = \underline{f}[G']$ satisfies $f'(k) = j$. On the other hand, the 'condition' p belongs to G', because $\|p\| \subseteq \xi_n$ and $G'_{\xi_n} = G_{\xi_n}$. This gives $x \leqslant^*{}_m f'$ by the choice of p; in particular, $x(k) \leqslant f'(k)$ because $k \geqslant m$. However, $x(k) > j$, a contradiction.

$\square$ (*Theorems* 8.6 *and* 8.1(4))

8E. $\mathbf{BP}(\boldsymbol{\Sigma}_2^1) \wedge \mathbf{LM}(\boldsymbol{\Delta}_2^1) \not\Longrightarrow \mathbf{LM}(\boldsymbol{\Sigma}_2^1)$. A model to prove the underivability of this implication is obtained by iterating the dominating forcing $\mathbf{D}$ and the random forcing $\mathbf{B}$ of length $\omega_1^{\mathbf{L}}$ and with finite support. Thus, we consider an iterated forcing $\mathbb{P}$ constructed according to the scheme 1°–7° in subsection 8C with the following correction.

2° If a forcing $\mathbb{P}_\xi$ is defined (and consists of functions p with $\xi = \operatorname{dom} p$) and $\mathbf{1}_\xi$ is the largest (hence the weakest) element of $\mathbb{P}_\xi$, then we fix a $\mathbb{P}_\xi$-term $\overline{\mathbb{Q}}^\xi$ such that $\Vdash_{\mathbb{P}_\xi}^{\mathbf{L}}$ "$\overline{\mathbb{Q}}^\xi = \mathbf{D}$" if ξ is even and $\Vdash_{\mathbb{P}_\xi}^{\mathbf{L}}$ "$\overline{\mathbb{Q}}^\xi = \mathbf{B}$" if ξ is odd. We also fix a $\mathbb{P}_\xi$-term $\mathbf{1}^\xi$ such that $\Vdash_{\mathbb{P}_\xi}^{\mathbf{L}}$ "$\mathbf{1}^\xi = {}^{\smile}\langle\Lambda, \mathbf{0}\rangle$" if ξ is even and $\Vdash_{\mathbb{P}_\xi}^{\mathbf{L}}$ "$\mathbf{1}^\xi = {}^{\smile}2^{<\omega}$" if ξ is odd.

Let us fix a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathbf{L}$. Every set of the form $G_\xi = \{p \restriction \xi : p \in G\}$ is $\mathbb{P}_\xi$-generic over $\mathbf{L}$, and $\mathbf{L}[G_{\xi+1}]$ is a $(\mathbf{D})^{\mathbf{L}[G_\xi]}$-generic extension of $\mathbf{L}[G_\xi]$ if ξ is even and a $(\mathbf{B})^{\mathbf{L}[G_\xi]}$-generic extension if ξ is odd. Our objective is to prove that $\mathrm{BP}(\Sigma^1_2)$ and $\mathrm{LM}(\Delta^1_2)$ hold and $\mathrm{LM}(\Sigma^1_2)$ fails in $\mathbf{L}[G]$. The 'positive' part of this claim is analogous to some results obtained above. Namely, $\mathrm{BP}(\Sigma^1_2)$ holds for the same reasons as in the model defined in subsection 8C, and $\mathrm{LM}(\Delta^1_2)$ holds for the same reasons as in the model defined in subsection 8D.

Let us pass to the 'negative' part. We are going to use Theorem 5.3 to derive $\neg\,\mathrm{LM}(\Sigma^1_2)$ in $\mathbf{L}[G]$. It suffices to prove that the set $\boldsymbol{\ell}_1 \cap \mathbf{L}$ is not $\leqslant^*$-bounded in $\boldsymbol{\ell}_1 \cap \mathbf{L}[G]$, that is, no function $f \in \boldsymbol{\ell}_1 \cap \mathbf{L}[G]$ satisfies $g \leqslant^* f$ for any $g \in \boldsymbol{\ell}_1 \cap \mathbf{L}$. The proof of this assertion uses the fact that a certain property of forcing is preserved under the iteration used.

The following definition is a modification of Definition 8.5.

Definition 8.9. A forcing $\mathcal{P}$ is said to be $\boldsymbol{\ell}_1$-*good* if for any $\mathcal{P}$-generic set $G \subseteq \mathcal{P}$ over the universe $\mathbf{V}$ and any $f \in \mathbf{V}[G] \cap \boldsymbol{\ell}_1$ there is a $g \in \mathbf{V} \cap \boldsymbol{\ell}_1$ such that $\forall\, x \in \mathbf{V} \cap \boldsymbol{\ell}_1$ $(x \leqslant^* f \implies x \leqslant^* g)$.

Lemma 8.10. *The forcings* $\mathbf{B}$ *and* $\mathbf{D}$ *are* $\boldsymbol{\ell}_1$-*good.*

Proof. The proof for the forcing $\mathbf{B}$ differs from the corresponding part of the proof of Lemma 8.7. Suppose that a set $G \subseteq \mathbf{B}$ is generic over $\mathbf{V}$ and that $f \in \boldsymbol{\ell}_1 \cap \mathbf{V}[G]$. By definition, the sum $\sum_n f(n)$ is finite, and hence it can be made as small as desired by changing finitely many values of f. Since this operation does not affect the relation $\leqslant^*$, we can assume that $\sum_n f(n) < 1$.

We have $f = \underline{f}[G]$, where $\underline{f} \in \mathbf{V}$. By virtue of what was said above, one can assume that $\Vdash_{\mathbf{B}} \sum_n \underline{f}(n) < 1$. Using the CAC, one can readily see that for any pair $n, i \in \mathbb{N}$ there is a Borel set $B_{ni} \subseteq \mathbb{N}^\omega$ such that

$$\forall\, T \in \mathbf{B}\ \bigl((T \Vdash_{\mathbf{B}} \underline{f}(n) \geqslant i\, 2^{-n}) \iff \boldsymbol{\lambda}([T] \smallsetminus B_{ni}) = 0\bigr). \tag{1}$$

We set $g_m(n) = \sup\{i : \boldsymbol{\lambda}(B_{ni}) > \frac{1}{m+1}\}$ and claim that

$$\sum_n g_m(n)\, 2^{-n} \leqslant m+1 \quad \forall\, m. \tag{2}$$

Suppose the contrary. Then $\sum_{n<n_0} g_m(n)\, 2^{-n} > m+1$ for some m and n_0. We set $U = \{\langle n,i,j\rangle : n < n_0 \wedge i < g_m(n) \wedge j < 2^{n_0-n}\}$ and $B_{nij} = B_{ni}$ for $\langle n,i,j\rangle \in U$. By the definition of $g_m(n)$ we have $\boldsymbol{\lambda}(B_{nij}) > \frac{1}{m+1}$ for $\langle n,i,j\rangle \in U$. Hence, it follows from the choice of m_0 and n that

$$\sum_{\langle n,i,j\rangle \in U} \boldsymbol{\lambda}(B_{nij}) > \frac{1}{m+1} \sum_{n<n_0} g_m(n)\, 2^{n_0-n} > 2^{n_0}.$$

Therefore, there is a set $V \subseteq U$ with $2^{n_0}+1$ elements such that the intersection $B = \bigcap_{\langle n,i,j\rangle\in V} B_{nij}$ satisfies the condition $\boldsymbol{\lambda}(B) > 0$.[33] We set

$$V_n = \{\langle i,j\rangle : \langle n,i,j\rangle \in V\}, \qquad N = \{n < n_0 : V_n \neq \varnothing\},$$
$$I_n = \{i : \exists\, j\ (\langle i,j\rangle \in V_n)\}, \qquad i_n = \sup I_n \quad (\text{for } n \in N).$$

Since $\boldsymbol{\lambda}(B) > 0$, there is a 'condition' $T \in \mathbf{B}$ with $[T] \subseteq B$. It follows from (1) that $T \Vdash_{\mathbf{B}}$ "$\underline{f}(n) \geqslant i_n\, 2^{-n}$" for every $n \in N$. (To make the notation less cluttered, we omit the symbols $\breve{}$ over i_n, and so on.) However, by the definition of U we have $i_n\, 2^{n_0-n} \geqslant \#V_n$. This implies that $T \Vdash_{\mathbf{B}}$ "$\underline{f}(n) \geqslant 2^{-n_0}\#V_n$" (for any $n \in N$). It follows that $T \Vdash_{\mathbf{B}}$ "$\sum_{n<n_0} \underline{f}(n) \geqslant 2^{-n_0}\sum_{n\in N}\#V_n = 2^{-n_0}\#V > 1$", which contradicts our assumption that $\Vdash_{\mathbf{B}}$ "$\sum_n \underline{f}(n) < 1$". This completes the proof of (2).

We now set $h_m(n) = g_m(n)\,2^{-n}$. Thus, $\sum_n h_m(n) < \infty$, and hence there is a function $g \in \boldsymbol{\ell}_1$ such that $h_m \leqslant^* g\ \forall\, m$. We claim that g implies the 'goodness' of $\mathbf{B}$ with respect to f. Let $h \in \boldsymbol{\ell}_1 \cap \mathbf{V}$ and $h \leqslant^* f$. There is a 'condition' $T \in G$ such that $T \Vdash_{\mathbf{B}} \forall\, n \geqslant k\ (h(n) \leqslant \underline{f}(n))$ for some k. Let us take m large enough that $\boldsymbol{\lambda}([T]) > \frac{1}{m+1}$. Then $\boldsymbol{\lambda}(B_{n,h(n)2^n}) \geqslant \boldsymbol{\lambda}([T]) > \frac{1}{m+1}$ holds for $n \geqslant k$. Therefore, $h(n)\,2^n \leqslant g_m(n)$ or, which is the same, $h(n) \leqslant h_m(n)$. Thus, $h \leqslant^* h_n$, and hence $h \leqslant^* g$, as was to be proved.

The proof for the forcing $\mathbf{D}$ is as follows. First, one can readily see that the partially ordered set $\mathbf{D}$ is *separated*, which means that for any pair of 'conditions' $p \nleqslant q$ in $\mathbf{D}$ there is a 'condition' $r \leqslant p$, $r \nleqslant q$. In this case (see [29], Theorem 29B) there is a complete Boolean algebra $\mathcal{B}$ such that $\mathbf{D}$ is order isomorphic to some dense subset of the set $\mathcal{P} = \mathcal{B} \smallsetminus \{0_{\mathcal{B}}\}$ of all non-zero elements of $\mathcal{B}$ with the order $a \leqslant b$ if $a \cdot b = b$. Thus, it suffices to prove that $\mathcal{P}$ is $\boldsymbol{\ell}_1$-good. We note that the set $\mathbf{D}$ has the σ-finite intersection property; indeed, $\mathbf{D} = \bigcup_{s\in\mathbb{N}^{<\omega}} D_s$, where each $D_s = \{\langle s,f\rangle : f \in \mathbb{N}^\omega\}$ has the finite intersection property. It follows that $\mathcal{P}$ has the σ-finite intersection property as well. Let $\mathcal{P} = \bigcup_n P_n$, where every set P_n has the finite intersection property. We can assume that every P_n is a *maximal* set with the finite intersection property, is closed with respect to $\cdot$ (the product operation in the Boolean algebra $\mathcal{B}$), satisfies $a \geqslant b \in P_n \Longrightarrow a \in P_n$, and necessarily contains one of any two complementary elements a, $-a$. In this case there is a finitely additive strictly positive measure on $\mathcal{P}$ given by $\mu(a) = \sum_{a\in P_n} 2^{-n}$ on $\mathcal{B}$. Having this measure, we can repeat for $\mathcal{P}$ the above proof given for $\mathbf{B}$. $\square$

The proof of the following assertion is similar to that of Theorem 8.6.

[33] Here we refer to the following lemma from measure theory: *if $M \leqslant K$ and a family $\{X_k : k < K\}$ of measurable sets satisfies the condition $\boldsymbol{\lambda}(\bigcap_{k\in V} X_k) > 0$ for any $V \subseteq K$ with* $\operatorname{card} V = M$, *then* $\sum_{k<K} \boldsymbol{\lambda}(X_k) \leqslant M-1$. The proof by induction on M, simultaneously for all $K \geqslant M$, uses the following decomposition (where $\overline{X_{K-1}}$ is the complement of X_{K-1}):

$$\sum_{k<K} \boldsymbol{\lambda}(X_k) = \boldsymbol{\lambda}(X_{K-1}) + \sum_{k<K-1} \boldsymbol{\lambda}(X_k \cap X_{K-1}) + \sum_{k<K-1} \boldsymbol{\lambda}(X_k \cap \overline{X_{K-1}})$$
$$\leqslant \boldsymbol{\lambda}(X_{K-1}) + (M-2)\,\boldsymbol{\lambda}(X_{K-1}) + (M-1)\,\boldsymbol{\lambda}(\overline{X_{K-1}}) = M-1.$$

Theorem 8.11. *It is true in* $\mathbf{L}$ *that all forcings* $\mathbb{P}_\xi$, $\xi < \omega_2^{\mathbf{L}}$, *are* $\boldsymbol{\ell}_1$*-good.*

In turn, this theorem implies that the set $\mathbf{L} \cap \boldsymbol{\ell}_1$ is not $\leqslant^*$-bounded in $\boldsymbol{\ell}_1$ in any $\mathbb{P}_\xi$-generic extension of $\mathbf{L}$. In particular, this holds in any class $\mathbf{L}[G_\xi]$. However, any $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$ belongs to $\mathbf{L}[G_\xi] \cap \mathbb{N}^\omega$ for a suitable $\xi < \omega_2^{\mathbf{L}}$ by the CAC, and hence $\mathbf{L} \cap \mathbb{N}^\omega$ is not $\leqslant^*$-bounded in $\boldsymbol{\ell}_1$ even in $\mathbf{L}[G]$, as was to be proved.

$\square$(*Theorem* 8.1(2))

8F. $\mathbf{LM}(\boldsymbol{\Delta}^1_2) \not\Longrightarrow \mathbf{BP}(\boldsymbol{\Delta}^1_2)$. Here we show that the implication (5) of Theorem 8.1 is underivable. In fact, it will even be proved that $\mathrm{LM}(\Delta^1_2) \not\Longrightarrow \mathrm{BP}(\Delta^1_2) \vee \mathrm{LM}(\Sigma^1_2)$; hence, in particular, $\mathrm{LM}(\Delta^1_2) \not\Longrightarrow \mathrm{LM}(\Sigma^1_2)$. This is, weaker than the assertion (2) in Theorem 8.1, of course, but on the other hand, it is simpler to prove. The generic model we use here can be characterized as an iterated generic extension of $\mathbf{L}$ of length $\omega_1^{\mathbf{L}}$ by the random forcing $\mathbf{B}$ with countable support.[34] Fortunately, in this case the rather complicated construction admits a simple geometric form.

Below we consider spaces of the form $\mathcal{C}^\vartheta$ for $\vartheta < \omega_1^{\mathbf{L}}$, where $\mathcal{C} = 2^\omega$, and use boldface characters like $\mathbf{x}$ and $\mathbf{y}$ to denote points of these spaces. Let us equip each space $\mathcal{C}^\vartheta$ with a Borel measure $\boldsymbol{\lambda}^\vartheta$ which is the product of ϑ copies of the measure $\boldsymbol{\lambda}$. The Borel sets in these spaces admit an encoding defined as follows. We fix (once and for all) a sequence $\{b_\vartheta\}_{\vartheta<\omega_1^{\mathbf{L}}} \in \mathbf{L}$ of bijections $b_\vartheta \colon \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N} \times \xi$. Every bijection b_ϑ induces a homeomorphism $H_\vartheta \colon 2^\omega \xrightarrow{\text{onto}} \mathcal{C}^\vartheta$. If $c \in \mathbf{BC}$, then we set $\mathbf{B}_c[\mathcal{C}^\vartheta] = H_\vartheta\text{"}\mathbf{B}_c$. In the sense of this encoding, for any $\vartheta < \omega_1^{\mathbf{L}}$ the ideal $\mathcal{I}_{\boldsymbol{\lambda}}[\mathcal{C}^\vartheta]$ of all Borel sets $X \subseteq \mathcal{C}^\vartheta$ with $\boldsymbol{\lambda}^\vartheta(X) = 0$ is an $\mathbf{L}$-absolute σ-CAC ideal in the Borel algebra of the space $\mathcal{C}^\vartheta$.[35]

For any $\vartheta < \omega_1^{\mathbf{L}}$ we set $\mathbb{P}^\vartheta = \{p \in \mathbf{BC} \cap \mathbf{L} : \boldsymbol{\lambda}^\vartheta(\mathbf{B}_p[\mathcal{C}^\vartheta]) > 0\}$. Any $p \in \mathbb{P}^\vartheta$ should now be 'visualized' as the set $\mathbf{B}_p[\mathcal{C}^\vartheta] = H_\vartheta\text{"}\mathbf{B}_p \subseteq \mathcal{C}^\vartheta$ rather than as $\mathbf{B}_p \subseteq \mathcal{C}$. Accordingly, we write $p \leqslant q$ (that is, p is stronger) if $\mathbf{B}_p[\mathcal{C}^\vartheta] \subseteq \mathbf{B}_q[\mathcal{C}^\vartheta]$.

Finally, we set $\mathbb{P} = \bigcup_{\vartheta<\omega_1^{\mathbf{L}}} \{\langle \vartheta, p\rangle : p \in \mathbb{P}^\vartheta\}$. We order $\mathbb{P}$ as follows: $\langle \vartheta, p\rangle \leqslant \langle \xi, q\rangle$ ($\langle \vartheta, p\rangle$ is stronger) if $\xi \leqslant \vartheta$ and $\mathbf{B}_p[\mathcal{C}^\vartheta] \restriction \xi \subseteq \mathbf{B}_q[\mathcal{C}^\xi]$, where $\mathbf{B}_p[\mathcal{C}^\vartheta] \restriction \xi = \{\mathbf{x} \restriction \xi : \mathbf{x} \in \mathbf{B}_p[\mathcal{C}^\vartheta]\}$. Any 'condition' $\langle \vartheta, p\rangle \in \mathbb{P}$ is again 'visualized' as the set $\mathbf{B}_p[\mathcal{C}^\vartheta] \subseteq \mathcal{C}^\vartheta$.

To prove Theorem 8.1(5), we fix a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ over $\mathbf{L}$ and write $G^\vartheta = \{p : \langle \vartheta, p\rangle \in G\}$; $G^\vartheta \subseteq \mathbb{P}^\vartheta$. As in Lemma 4.11, there is a unique 'point' $\mathbf{a} \in \mathcal{N}^{\omega_1{}^{\mathbf{L}}}$ (that is, $\mathbf{a} \colon \omega_1^{\mathbf{L}} \to \mathbb{N}^\omega$) such that $\mathbf{a} \restriction \vartheta \in \mathbf{B}_p[\mathcal{C}^\vartheta]$ if $\langle \vartheta, p\rangle \in G$. In other words, if $\vartheta < \omega_1^{\mathbf{L}}$, then $\mathbf{a} \restriction \vartheta$ is the only point in the intersection $\bigcap_{p\in G^\vartheta} \mathbf{B}_p[\mathcal{C}^\vartheta]$.

We claim that $\mathrm{LM}(\Delta^1_2)$ holds in $\mathbf{L}[G]$ and $\mathrm{LM}(\Sigma^1_2)$ and $\mathrm{BP}(\Delta^1_2)$ fail in $\mathbf{L}[G]$. By Theorem 3.3, it suffices to show that *the following three statements are true in* $\mathbf{L}[G]$ *for any* $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$:

(a) $\mathbf{Rand}\,\mathbf{L}[a] \neq \varnothing$;
(b) $\mathbf{Rand}\,\mathbf{L}[a]$ *is not a set of* $\boldsymbol{\lambda}$*-measure* 1;
(c) $\mathbf{Coh}\,\mathbf{L}[a] = \varnothing$.

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[34]Iterations with finite support do not work here because they produce generic extensions containing Cohen points of $\mathbb{N}^\omega$, and hence lead to $\mathrm{BP}(\Delta^1_2)$ by Corollary 3.4.

[35]We pay no attention to a certain discrepancy with the general setup in subsection 3E and hope that the reader will make the rather obvious corrections.

Lemma 8.12. *The forcing $\mathbb{P}$ satisfies the* CAC *in* $\mathbf{L}$, *and hence all cardinals of* $\mathbf{L}$ *remain cardinals in* $\mathbf{L}[G]$; *in particular,* $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}[\mathbf{a}]} = \omega_1^{\mathbf{L}}$. *Moreover,*

(i) *if* $x \in \mathbf{L}[G] \cap \mathbb{N}^\omega$, *then there is an ordinal* $\xi < \omega_1^{\mathbf{L}}$ *such that* $x \in \mathbf{L}[\mathbf{a} \restriction \xi]$;

(ii) *if* $\vartheta < \omega_1^{\mathbf{L}}$, *then* G^ϑ *is* $\mathbb{P}^\vartheta$*-generic over* $\mathbf{L}$;

(iii) *if* $\vartheta < \omega_1^{\mathbf{L}}$, *then the point* $\mathbf{a} \restriction \vartheta$ *is* $\boldsymbol{\lambda}^\vartheta$*-random over* $\mathbf{L}$ *in the sense that* $\mathbf{a} \restriction \vartheta \notin \mathbf{B}_c[\mathcal{C}^\vartheta]$ *whenever* $c \in \mathbf{BC} \cap \mathbf{L}$ *and* $\boldsymbol{\lambda}^\vartheta(\mathbf{B}_c[\mathcal{C}^\vartheta]) = 0$;

(iv) *if* $\vartheta < \omega_1^{\mathbf{L}}$, *then the point* $\mathbf{a}(\vartheta)$ *is random over* $\mathbf{L}[\mathbf{a} \restriction \vartheta]$, *that is,* $\mathbf{a}(\vartheta) \notin \mathbf{B}_c$ *whenever* $c \in \mathbf{BC} \cap \mathbf{L}[\mathbf{a} \restriction \vartheta]$ *and* $\boldsymbol{\lambda}(\mathbf{B}_c) = 0$.

Proof. To prove the CAC for $\mathbb{P}$, we note that otherwise $\mathbf{L}$ would contain uncountably many Borel subsets of $\mathcal{C}^{\omega_1^{\mathbf{L}}}$ of positive $\boldsymbol{\lambda}^{\omega_1^{\mathbf{L}}}$-measure with pairwise intersections of $\boldsymbol{\lambda}^{\omega_1^{\mathbf{L}}}$-measure zero, which is impossible.

(i) As usual, this follows from the fact that the antichains are countable.

(ii) The proof is reduced to showing that if a set $D \subseteq \mathbb{P}^\vartheta$ is dense in $\mathbb{P}^\vartheta$, then the set $D' = \{\langle \xi, p\rangle \in \mathbb{P} : \xi \geqslant \vartheta \wedge \exists\, q \in D\ (\mathbf{B}_p[\mathcal{C}^\xi] \restriction \vartheta = \mathbf{B}_q[\mathcal{C}^\vartheta])\}$ is dense in $\mathbb{P}$ (and $D' \in \mathbf{L}$ if $D \in \mathbf{L}$). This implies (iii) (in fact, this is Lemma 4.12 in a somewhat different situation).

(iv) One can assume that $\mathbf{B}_c$ is a $\mathbf{G}_\delta$ set, that is, $\mathbf{B}_c = \mathbf{U}_z = \bigcap_n \bigcup_{z(2^n \cdot 3^m)=0} \mathcal{N}_{s_m}$ for a suitable $z \in \mathbf{L}[\mathbf{a} \restriction \vartheta] \cap \mathbb{N}^\omega$. We have $z = \mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{a} \restriction \vartheta)]$ for some $\vartheta < \omega_1^{\mathbf{L}}$ by Theorem 2.6(i). Suppose the contrary: let $\mathbf{a}(\vartheta) \in \mathbf{U}_z$. There is a 'condition' $\langle \vartheta + 1, p\rangle \in \mathbb{P}$ forcing $\boldsymbol{\lambda}(\mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{a} \restriction \vartheta)]}) = 0$ and $\mathbf{a}(\vartheta) \in \mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{a} \restriction \vartheta)]}$. (As is often customary, we identify sets like $\mathbf{a}(\vartheta)$ and $\mathbf{a} \restriction \vartheta$ in generic extensions with their names.) We set $P = \mathbf{B}_p[\mathcal{C}^{\vartheta+1}]$; obviously, $\boldsymbol{\lambda}^{\vartheta+1}(P) > 0$. Moreover, by the choice of P, the set $X = \{\mathbf{x} \in P \restriction \vartheta : \boldsymbol{\lambda}(\mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{x})]}) > 0\}$ has $\boldsymbol{\lambda}^\vartheta$-measure zero (an analogue of Lemma 4.13).

It follows from considerations connected with the Fubini theorem that there is a stronger 'condition' $\langle \vartheta + 1, q\rangle \in \mathbb{P}$ (that is, the set $Q = \mathbf{B}_q[\mathcal{C}^{\vartheta+1}]$ satisfies $Q \subseteq P$) such that the cross-section $Q_{\mathbf{x}} = \{y \in \mathbb{N}^\omega : \langle \mathbf{x}, y\rangle \in Q\}$ is disjoint from $\mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{x})]}$ for any $\mathbf{x} \in Y = Q \restriction \vartheta$. However, this set Q forces $a_\vartheta \notin \mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{a} \restriction \vartheta)]}$ (for example, by Lemma 4.13, or, more precisely, by its analogue in this case), which contradicts the choice of the 'condition' $\langle \vartheta + 1, p\rangle$. $\square$

We can now give the proofs of the assertions (a), (b), and (c).

(a) Let $a \in \mathbf{L}[G] \cap \mathbb{N}^\omega$. We have $a \in \mathbf{L}[\mathbf{a} \restriction \vartheta]$ for some $\vartheta < \omega_1^{\mathbf{L}}$ by Lemma 8.12(i). Then $a_\vartheta \in \mathbf{Rand}\,\mathbf{L}[a]$ by Lemma 8.12(iv).

(b) It suffices to show that $\mathbf{L} \cap \mathbb{N}^\omega$ is not a set of $\boldsymbol{\lambda}$-measure zero in $\mathbf{L}[G]$. To do this, one must show that $\mathbf{L} \cap \mathbb{N}^\omega$ is not a set of measure zero in $\mathbf{L}[\mathbf{a} \restriction \vartheta]$ for any $\vartheta < \omega_1^{\mathbf{L}}$. Suppose the contrary. Then there exists a $z \in \mathbf{L}[\mathbf{a} \restriction \vartheta] \cap \mathbb{N}^\omega$ such that $\mathbf{L} \cap \mathbb{N}^\omega \subseteq \mathbf{U}_z$ and $\boldsymbol{\lambda}(\mathbf{U}_z) = 0$. There is an ordinal $\xi < \omega_1^{\mathbf{L}}$ such that $z = \mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{a} \restriction \vartheta)]$. A contradiction will be obtained from a consideration of the cross-sections $P_{\mathbf{x}} = \{y : P(\mathbf{x}, y)\}$ and $P^y = \{\mathbf{x} : P(\mathbf{x}, y)\}$ of the $\Sigma_1^1(\mathbf{L})$ set

$$P = \{\langle \mathbf{x}, y\rangle \in \mathbb{N}^\vartheta \times \mathbb{N}^\omega : y \in \mathbf{U}_{\mathbf{f}_\xi[{H_\vartheta}^{-1}(\mathbf{x})]}\}.$$

By definition, $P_{\mathbf{a} \restriction \vartheta} = \mathbf{U}_z$ is a set of $\boldsymbol{\lambda}$-measure zero and $\mathbf{L} \cap \mathbb{N}^\omega \subseteq P_{\mathbf{a} \restriction \vartheta}$. However, this proposition is an absolute formula by Theorem 2.8 ("$\boldsymbol{\lambda}(\mathbf{U}_z) = 0$" is an arithmetic formula), and hence it can be relativized to $\mathbf{L}[\mathbf{a} \restriction \vartheta]$. Thus, this formula is forced in $\mathbf{L}$ by a condition $\langle\vartheta, p\rangle \in \mathbb{P}^\vartheta$ such that $\mathbf{a} \restriction \vartheta \in \mathbf{B}_p[\mathcal{C}^\vartheta]$. (This is Lemma 4.12 for the ideal of sets of $\boldsymbol{\lambda}^\vartheta$-measure zero.) Then $p \in \mathbf{BC} \cap \mathbf{L}$, and $X = \mathbf{B}_p[\mathcal{C}^\vartheta]$ is a set of positive $\boldsymbol{\lambda}^\vartheta$-measure. By the choice of p, Lemma 4.13 implies that the following is true in $\mathbf{L}$: 1) for any $y \in \mathbb{N}^\omega$ the set $\mathbf{B}_p[\mathcal{C}^\vartheta] \smallsetminus P^y$ has positive $\boldsymbol{\lambda}$-measure, and 2) $\boldsymbol{\lambda}(P_{\mathbf{x}}) = 0$ for almost all points $\mathbf{x} \in \mathbf{B}_p[\mathcal{C}^\vartheta]$ (modulo a set of $\boldsymbol{\lambda}^\vartheta$-measure zero). However, this contradicts the Fubini theorem.

(c) Arguments similar to those used above reduce the problem to the following form: 1) if $\vartheta < \omega_1^{\mathbf{L}}$ and $x \in \mathbf{L}[\mathbf{a} \restriction \vartheta]$, then there is a code $c \in \mathbf{BC} \cap \mathbf{L}[a]$ such that $\mathbf{B}_c$ is a meagre set and $x \in \mathbf{B}_c$. And then to another form: 2) if $X \subseteq \mathbb{N}^\omega$ is a Borel set with $\boldsymbol{\lambda}(X) > 0$ and $F\colon X \to \mathbb{N}^\omega$ is a Borel function, then there is a Borel set $X' \subseteq X$, again with $\boldsymbol{\lambda}(X') > 0$, such that $F``X'$ is a meagre set. Clearly, for any $\varepsilon > 0$ and $s \in \mathbb{N}^{<\omega}$ there is an $s' \in \mathbb{N}^{<\omega}$ such that $s \subseteq s'$ and $\boldsymbol{\lambda}(F^{-1}(\mathcal{N}_{s'})) < \varepsilon$. Therefore, there is an open dense set $Y \subseteq \mathbb{N}^\omega$ such that $\boldsymbol{\lambda}(F^{-1}(Y)) < \varepsilon$. It remains to take $X' = X \smallsetminus (F^{-1}(Y))$.

$\square$ (*Theorem* 8.1(5))

Historical and bibliographical remarks. The history of the results considered in this section extends from the late 1960s to the mid-1990s and is connected in general with the development of forcing and iterated forcing. Original references on iterated forcing are [12], [90] (see also monographs like [10], [29], [30], [44]). There is no suitable source in Russian except for [11], Chapter 4, where iterations of length two are considered and iterations of arbitrary length in the special case in which the objective is the consistency of Martin's axiom.

As mentioned above, the assertion $\mathrm{LM}(\Sigma^1_2) \not\Longleftarrow \mathrm{PK}(\Pi^1_1)$ follows from early work on Martin's axiom. The result was obtained independently by Lyubetskii (see [65], [66], [68]; a proof based on iteration of the forcing $\mathbf{A}_\varepsilon$ is given in [69]).

The assertion $\mathrm{BP}(\Sigma^1_2) \wedge \mathrm{LM}(\Delta^1_2) \not\Longrightarrow \mathrm{LM}(\Sigma^1_2)$ is Theorem 9.3.6 in [10], given there with a reference to the paper [26]; however there is no result of this kind in [26].

As far as the assertion $\mathrm{BP}(\Sigma^1_2) \not\Longrightarrow \mathrm{LM}(\Delta^1_2)$ is concerned, we took the method of iterating the dominating forcing (as in subsection 8C) from [10], 9.3.5 (where an incorrect reference to [26] is again given). However, Stern [92] earlier proved even that $\mathrm{BP}(\Sigma^1_\infty) \not\Longrightarrow \mathrm{LM}(\Delta^1_2)$ by using a more complicated model than that considered in subsection 8C.

We could not find the result $\mathrm{BP}(\Delta^1_2) \wedge \mathrm{LM}(\Delta^1_2) \not\Longrightarrow \mathrm{BP}(\Sigma^1_2)$ in an explicit form. However, the proof given above is a certain combination of known methods; see, for instance, [10].

The assertion $\mathrm{LM}(\Delta^1_2) \not\Longrightarrow \mathrm{BP}(\Delta^1_2)$ was established in [69]. We note that the symmetric claim $\mathrm{BP}(\Delta^1_2) \not\Longrightarrow \mathrm{LM}(\Delta^1_2)$ (which is weaker, of course, than the assertion $\mathrm{BP}(\Sigma^1_2) \not\Longrightarrow \mathrm{LM}(\Delta^1_2)$ proved above in this section) was probably known in 1968, because it follows immediately from Theorem T 3323 in [74] (given there with a reference to Kunen and Solovay) claiming that the addition of arbitrarily many Cohen points to the constructible universe $\mathbf{L}$ gives a model admitting a non-measurable Δ^1_2 set.

§ 9. Further results

This concluding section contains a survey of some results on the regularity properties of projective sets, touching on problems such as the role of the existence axiom for an inaccessible cardinal, the regularity properties of point sets of at least third projective level, and a few less traditional regularity properties. For lack of space we must restrict ourselves here to a survey of the main results, without proofs but with references to the original sources.

9A. Do we need an inaccessible cardinal? We saw that the existence axiom for a (strongly) inaccessible cardinal is used in the proof of consistency of the hypothesis that all projective sets satisfy each of the three regularity properties. This axiom is a fairly strong statement, which is certainly unprovable in **ZFC** (and even implies the consistency of **ZFC**), so we face the natural question of whether or not this hypothesis is really necessary to achieve the indicated result.

The perfect kernel property admits the most elementary answer. Indeed, if every uncountable Π^1_1 set contains a perfect subset, then, by Theorem 3.3, the set $\mathbf{L}[a]\cap\mathbb{N}^\omega$ is countable, that is, $\omega_1^{\mathbf{L}[a]} < \omega_1$ for each $a \in \mathbb{N}^\omega$. This readily implies that $\aleph_1$ is an inaccessible cardinal in any universe of the form $\mathbf{L}[a]$, $a \in \mathbb{N}^\omega$, and, in particular, in $\mathbf{L}$. In other words, if the assumption $\forall\, a \in \mathbb{N}^\omega\ (\omega_1^{\mathbf{L}[a]} < \omega_1)$ does not contradict **ZFC**, then the existence of a strongly inaccessible cardinal is also consistent (as noted by Lyubetskii in [17]). Thus, as far as the proof of consistency for the perfect kernel property is concerned (even for Π^1_1 sets), the use of inaccessible cardinals is quite necessary.

The question of the role of inaccessible cardinals for the LM and BP problems turned out to be much more complicated. The answer is different for measure and category.

Theorem 9.1 (Shelah [84] and Raisonnier [79]). *If* **ZFC** *is consistent, then so is* $\mathbf{ZFC} + \mathrm{BP}(\Sigma^1_\infty)$ *and even* $\mathbf{ZFC} + \mathrm{BP}(\mathbf{ROD})$.

At the same time, each of the two assumptions $\mathrm{LM}(\Sigma^1_3)$ *and* $\mathrm{LM}(\Sigma^1_2) \wedge \mathrm{BP}(\Sigma^1_3)$ *implies that* $\aleph_1$ *is an inaccessible cardinal in* $\mathbf{L}$ *and in* $\mathbf{L}[a]$ *for any* $a \in \mathbb{N}^\omega$, *and thus implies* $\mathrm{PK}(\Pi^1_1)$ *by Theorem* 3.3(i).

Thus, the axiom of an inaccessible cardinal is necessary for the positive solution of the measurability problem for projective sets (even for Σ^1_3 sets), but it can be avoided for the positive solution of the Baire property problem. However, as proved by Stern [94], the consistency of LM(OD) (for ordinal definable sets!) does not need inaccessible cardinals.

Martin's axiom **MA** (see [11], Chap. 6) substantially strengthens some forms of the hypotheses LM and BP. For instance, under the assumption that $\mathbf{MA}_{\aleph_1}$ holds, the inaccessibility of $\aleph_1$ in $\mathbf{L}[a]$, $a \in \mathbb{N}^\omega$, follows already from $\mathrm{LM}(\Delta^1_3)$, as well as from $\mathrm{BP}(\Delta^1_3)$ (see [33]). (Without $\mathbf{MA}_{\aleph_1}$, this is not true by Theorems 9.1 and 9.2.) However, the consistency of the theory $\mathbf{ZFC} + \mathbf{MA}_{\aleph_1} + \mathrm{LM}(\Delta^1_3)$ (or the theory in which the last summand is replaced by $\mathrm{BP}(\Delta^1_3)$) actually implies the consistency of the existence even of such a large cardinal as a weakly compact cardinal [21], [27].

9B. Problems of third and fourth projective levels. The relations among the hypotheses $\mathrm{LM}(\Sigma^1_2)$, $\mathrm{BP}(\Sigma^1_2)$, $\mathrm{LM}(\Delta^1_2)$, and $\mathrm{BP}(\Delta^1_2)$ in **ZFC** are completely established by Corollary 3.5 and Theorem 5.4 (with respect to the provable connections) and Theorem 8.1 (with respect to unprovable connections). Such a complete picture is lacking for the third projective level. However, the following facts have been established.

Theorem 9.2. (1) (Shelah [84]) *The consistency of* **ZFC** *implies the consistency of* $\mathbf{ZFC} + \mathrm{BP}(\Sigma^1_3)$.

(2) (Judah [32]) $\mathrm{LM}(\Delta^1_3) \not\Longrightarrow \mathrm{BP}(\Delta^1_3)$.

(3) (Judah [31], [32]) $\mathrm{BP}(\Delta^1_3) \not\Longrightarrow \mathrm{LM}(\Delta^1_3)$, *and even* $\mathrm{BP}(\Delta^1_3) \not\Longrightarrow \mathrm{LM}(\Delta^1_2)$.

In this theorem the symbol $A \not\Longrightarrow B$ means that the implication $A \Longrightarrow B$ is not provable in **ZFC** (under the assumption that this theory is consistent).

Let us make a few comments on the theorem.

(1) The most elementary model for $\mathrm{BP}(\Sigma^1_3)$ is described in [10], p. 470. This is an iterated extension of $\mathbf{L}$ with finite base and of length $\omega_1^{\mathbf{L}}$, where the ε-random forcing works at limit steps and the Cohen and random forcings alternate at successor steps. Another model for $\mathrm{BP}(\Sigma^1_3)$ was presented in [27]; in this model, Martin's axiom $\mathbf{MA}_{\aleph_1}$ holds for a rather large class of partially ordered sets.

(2) This part of the theorem was characterized in [34], [6] as a joint result of Bagaria and Judah. The model described in [10], Theorem 9.4.19, is as follows. We begin with a model for $\mathbf{MA} + (\omega_1 = \omega_1^{\mathbf{L}})$ and then apply the forcing described in subsection 8F (adjoining $\aleph_1$ random points). A more elementary model (which, however, uses a measurable cardinal) was indicated in [33].

(3) In essence, one can use here the model in subsection 8C (iteration of the dominating forcing) because, as shown in [10], Theorem 9.4.7, one has not only $\mathrm{BP}(\Sigma^1_2)$ but also $\mathrm{BP}(\Delta^1_3)$ in this model.

The first non-trivial result concerning the fourth projective level and connected with regularity properties, namely, $\mathrm{LM}(\Delta^1_4) \not\Longrightarrow \mathrm{BP}(\Delta^1_4)$, was obtained by Judah and Spinas [34]. (The proof uses the axiom of an inaccessible cardinal, which is unavoidable here because the assertion $\mathrm{LM}(\Sigma^1_3)$ already implies the existence of an inaccessible cardinal by Theorem 9.1.) The converse implication is underivable, as is quite clear, because it follows from Theorem 9.1 that even $\mathrm{BP}(\Delta^1_\infty) \not\Longrightarrow \mathrm{LM}(\Sigma^1_3)$. For subsequent studies in this direction, see [6].

There are no noteworthy results for the fifth level and higher levels.

9C. Some new σ-CAC ideals and other regularity properties. Besides measure and category for Polish topologies, non-Polish CAC-topologies give another regular way to obtain σ-CAC ideals. To keep the terminology uniform, we mean by a σ-CAC *topology* a topology which is both second countable (that is, any family of non-empty pairwise disjoint open sets is at most countable) and Baire (that is, non-empty open sets are not meagre). This class includes, for example, the Polish topology of $\mathbb{N}^\omega$ or any other Polish space, in particular, the usual topology of the real line.

A rich source of σ-CAC topologies is provided by CAC forcings. Let us consider, for example, the dominating forcing $\mathbf{D}$. Corresponding to any 'condition' $p = \langle s, f\rangle \in \mathbf{D}$ is the set

$$U_p = U^s_f = \{x \in \mathbb{N}^\omega : s \subset x \wedge \forall\, n\ (x(n) \geqslant f(n))\}.$$

The family of all sets of the form U_p, $p \in \mathbf{D}$, is closed under finite unions, and hence it can be taken as a base of a topology on $\mathbb{N}^\omega$; we denote this topology by $\mathfrak{T}_{\mathbf{D}}$. The second countability axiom for $\mathfrak{T}_{\mathbf{D}}$ follows from the CAC for $\mathbf{D}$. Moreover, $\mathfrak{T}_{\mathbf{D}}$ is Baire, because the intersection of any decreasing system of sets $U^{s_0}_{f_0} \supseteq U^{s_1}_{f_1} \supseteq \cdots \supseteq U^{s_n}_{f_n} \supseteq \cdots$ such that $\operatorname{lh} s_n \to \infty$ contains the point $x = \bigcup_n s_n$. Thus, the ideal $\mathscr{J}_{\mathbf{D}}$ of all $\mathfrak{T}_{\mathbf{D}}$-meagre Borel sets $X \subseteq \mathbb{N}^\omega$ is a σ-CAC ideal.

Lemma 9.3. $\operatorname{cod} \mathscr{J}_{\mathbf{D}} = \{c \in \mathbf{BC} : \mathbf{B}_c \in \mathscr{J}_{\mathbf{D}}\}$ *is a* $\varSigma^1_2$ *set.*

Proof. Regarding the double sequences ('matrices') of the form $\{\langle s_{kn}, f_{kn}\rangle\}_{k,n\in\mathbb{N}}$ as points of the corresponding space, we have

$$\mathbf{B}_c \in \mathscr{J}_{\mathbf{D}} \iff \exists\,\{\langle s_{kn}, f_{kn}\rangle\}\ \biggl(\forall\, k,n\ (\langle s_{kn}, f_{kn}\rangle \in \mathbf{D}) \wedge \forall\, k\ \biggl(\mathbf{B}_c \cap \bigcup_n U^{s_{kn}}_{f_{kn}} = \varnothing\biggr)$$
$$\forall\, k\ \forall\,\langle s, f\rangle\ (\langle s, f\rangle \in \mathbf{D} \implies \exists\, n\ (U^{s_{kn}}_{f_{kn}} \cap U^s_f \neq \varnothing))\biggr).$$

Here the formulae $\langle s_{kn}, f_{kn}\rangle \in \mathbf{D}$ and $U^{s_{kn}}_{f_{kn}} \cap U^s_f \neq \varnothing$ are arithmetic, and the formula $\mathbf{B}_c \cap \bigcup_n U^{s_{kn}}_{f_{kn}} = \varnothing$ can readily be transformed to the $\varPi^1_1$ form by using the formula σ of Proposition 1.11(iv). Thus, the result becomes $\varSigma^1_2$, as was to be proved. $\square$

Therefore, $\mathscr{J}_{\mathbf{D}}$ is an $\mathbf{L}$-absolute σ-CAC ideal. This means that Theorem 3.9 can be applied to $\mathscr{J}_{\mathbf{D}}$. Hence,

$$\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Sigma^1_2) \iff \forall\, a \in \mathbb{N}^\omega\ (\mathbf{Rand}_{\mathscr{J}_{\mathbf{D}}}\, \mathbf{L}[a] \text{ is not an } \mathscr{I}\text{-full set});$$
$$\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Delta^1_2) \iff \forall\, a \in \mathbb{N}^\omega\ \forall\, c \in \mathbf{L}[a] \cap \mathbf{BC}\ (\mathbf{B}_c \notin \mathscr{J}_{\mathbf{D}} \Longrightarrow \mathbf{Rand}_{\mathscr{J}_{\mathbf{D}}}\, \mathbf{L}[a] \cap \mathbf{B}_c \neq \varnothing).$$

One can easily see that the proof of Lemma 3.2(1) is applicable to the ideal $\mathscr{J}_{\mathbf{D}}$, and hence the second equivalence can be rewritten as follows:

$$\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Delta^1_2) \iff \forall\, a \in \mathbb{N}^\omega\ (\mathbf{Rand}_{\mathscr{J}_{\mathbf{D}}}\, \mathbf{L}[a] \neq \varnothing).$$

The meaning and the content of the property of $\mathscr{J}_{\mathbf{D}}$-measurability and of the hypotheses $\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Sigma^1_2)$ and $\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Delta^1_2)$ are by no means as clear as those for the ideals $\mathscr{I}_{\mathrm{cat}}$ and $\mathscr{I}_{\boldsymbol{\lambda}}$. This makes the following results [14] all the more interesting:

(i) $\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Delta^1_2)$ is equivalent to $\mathrm{BP}(\Sigma^1_2)$;

(ii) $\mathrm{M}_{\mathscr{J}_{\mathbf{D}}}(\Sigma^1_2)$ is equivalent to $\forall\, a \in \mathbb{N}^\omega\ (\omega_1^{\mathbf{L}[a]} < \omega_1^{\mathbf{L}})$, that is, is equivalent to $\mathrm{PK}(\Pi^1_1)$ by Theorem 3.3(i).

Thus, from a given forcing $\mathbf{D}$ we have constructed an $\mathbf{L}$-absolute σ-CAC ideal $\mathcal{J}_{\mathbf{D}}$. From this ideal, Definition 4.8 constructs a forcing $\mathbb{P}_{\mathcal{J}_{\mathbf{D}}}$ having the same generic extensions as the original forcing $\mathbf{D}$, and this holds for the same reasons as in subsection 4D for the Cohen forcing.

As shown in [10], 3.4.B and 3.7, this construction of a σ-CAC ideal can be modified in such a way that it becomes applicable to some other CAC forcings $\mathbb{P}$ which need not be related to any topology. The key idea can be explained as follows. Suppose that t is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}^{\mathbf{V}} (t \in \mathbb{N}^\omega)$. Let us define $\mathcal{I}_{\mathbb{P}t}$ to be the set of all Borel sets of the form $\mathbf{B}_p$ such that $\Vdash_{\mathbb{P}}^{\mathbf{V}} (t \notin \mathbf{B}_p)$. If $\mathbb{P}$ is a CAC forcing, then $\mathcal{I}_{\mathbb{P}t}$ is a σ-CAC ideal. Another modification of this construction is applicable to some non-CAC forcings, for example, like the Sacks forcing. For the corresponding regularity properties (not necessarily related directly to some ideal), see [14].

Another approach to the construction of σ-CAC ideals was suggested in [81].

Let us say in conclusion a few words about another two regularity properties arising from topological considerations. A set $X \subseteq \mathbb{N}^\omega$ is said to be $\mathbf{K}_\sigma$-*regular* if it either is σ-compact or contains a superperfect subset.[36] (This can be compared with the perfect kernel property.) A set $X \subseteq [\mathbb{N}]^\omega = \{x \subseteq \mathbb{N} : \operatorname{card} x = \aleph_0\}$ is called a *Ramsey set* if there is an infinite set $z \subseteq \mathbb{N}$ such that $[z]^\omega \subseteq X$ or $[z]^\omega \cap X = \varnothing$. (This property arises in the study of some questions in infinitary combinatorics and model theory.)

As in the case of the properties PK, LM, and BP, every Σ^1_1 set is a $\mathbf{K}_\sigma$-regular Ramsey set (see, for instance, [30], [42]), whereas undecidable problems occur at the second projective level. For this direction of research, see the papers [26], [28]. For example, here is one of the results obtained: $\mathrm{BP}(\Sigma^1_2)$ implies the $\mathbf{K}_\sigma$-regularity of all Σ^1_2 sets, but the converse fails.

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