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Problems of set-theoretic non-standard analysis

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Abstract. The main results in the area of set-theoretic non-standard analysis (non-standard set and class theories) obtained over the last few years are presented. It is demonstrated how a universe of a comparably simple theory (beginning with the usual Zermelo–Fraenkel set theory **ZFC**) can be extended to a universe of a more complicated *non-standard* set or class theory. The last section develops the foundations of Boolean-valued analysis as a part of set-theoretic non-standard analysis.

Contents

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Introduction

Contemporary non-standard analysis was introduced in the early 1960s when Robinson ([1], Ch. IX) demonstrated how one can utilize non-standard models of algebraic structures and of the field of real numbers in order to obtain mathematical results. (see the books [2], [3] of Uspenskii or the book [4] of Gordon, Kusraev, and Kutateladze on the early history of infinitely large and infinitesimal values, which goes back to works of Newton, Leibniz, and even earlier mathematicians).

One of the original ideas of non-standard analysis is a presentation of (standard) mathematical structures $\mathfrak{A}$ by means of their (non-standard) extensions ${}^*\mathfrak{A}$ that allows one to reduce complex questions related to a given standard structure $\mathfrak{A}$ to more elementary questions related to its non-standard extension ${}^*\mathfrak{A}$. One of the first examples of such a reduction was found by Robinson essentially simultaneously with his introduction of non-standard analysis. That was a solution of the rather old and then still open problem of the existence of an invariant subspace of any compact operator on an infinite-dimensional space ([5], Ch. 5). The reduction consisted of replacing this complex problem by a question having an obvious answer: does a matrix acting on a finite-dimensional space have an invariant subspace? This argument makes use of a (non-standard) natural number h bigger than all standard natural numbers; then the given infinite-dimensional space is represented by a (non-standard) h-dimensional vector space. Some other applications of non-standard methods of this kind can be found, for example, in [5], [6], as well as in the more recent sources [4], [7].

It became clear closer to the end of the 1960s that the model-theoretic methods used by Robinson were applicable to any mathematical structure $\mathfrak{A}$, especially if algebraic notions and the notions of distance, nearness, and measure play an essential role. One of the forms of non-standard analysis employs the construction of a *non-standard extension* ${}^*\mathfrak{A}$ of a given mathematical structure $\mathfrak{A}$, together with an associated embedding $\mathfrak{A} \to {}^*\mathfrak{A}$, denoted by $a \mapsto {}^*a$ (or $a \mapsto a^\vee$ in the Boolean-valued case). Several methods of constructing such extensions ${}^*\mathfrak{A}$ are known, for instance, by means of an *ultrapower* of $\mathfrak{A}$ that depends on the choice of an ultrafilter U on an index set.

It was soon discovered that the applicability of non-standard methods is based on a short list of general principles like elementary character and saturation, rather than on the details of the construction of a non-standard extension ${}^*\mathfrak{A}$. The principle of *elementary character* means that any object $a \in \mathfrak{A}$ has the same properties in $\mathfrak{A}$ as the object ${}^*a$ has in ${}^*\mathfrak{A}$. The principle of *saturation*, which has an uncountable cardinal κ as a parameter (depending, for instance, on the choice of an ultrafilter U), requires that any system S of strictly fewer than κ conditions $P_\alpha(x)$ on an unknown x has a solution in ${}^*\mathfrak{A}$ provided that every finite subsystem $S' \subseteq S$ of those conditions has a solution in ${}^*\mathfrak{A}$. For instance, $\aleph_1$-saturation of the non-standard extension ${}^*\mathbb{N}$ of the set $\mathbb{N}$ of natural numbers implies the existence of an infinitely large number h in ${}^*\mathbb{N}$. In the 1970s this resulted in the development of *axiomatic* forms of non-standard analysis, that is, forms which describe axiomatically a non-standard extension ${}^*\mathbf{V}$ of the class $\mathbf{V}$ of all sets. People began to study such non-standard set-theoretic axiomatic systems as Nelson's internal set theory **IST** [8], [9] and the non-standard theories of Hrbáček [10], [11] and Kawai [12], [13]. This direction, connected with the most fundamental problems of non-standard mathematics, is called *set-theoretic non-standard analysis*, in distinction from the more special *applied* (or robinsonian) non-standard analysis that deals with non-standard extensions ${}^*\mathfrak{A}$ of various mathematical structures $\mathfrak{A}$.

Some properties of *bounded* sets (that is, elements of standard sets) discovered in [14], in particular, a theorem on reduction of formulae to Σ_2^{st} form (Theorem 1.19 below), were the basis for a series of deep studies in the field of set-theoretic non-standard analysis [15]–[24], including the monograph [25].

After the publication of [25], non-standard *class* theories attracted considerable interest in 2004–05 among specialists in this area. It was soon realized that their relationships with the corresponding non-standard *set* theories were not as simple as in the case when non-standard objects are not considered. These studies resulted in the solution of some fundamental problems of set-theoretic non-standard analysis, which we consider in this article.

Problem 1. (a)[1] Can one define saturated extensions of mathematical structures 'effectively', that is, by means of an explicit construction?

(b) Do there exist absolutely saturated (that is, κ-saturated for every cardinal κ) 'effective' elementary extensions?

The interest in Question (a) is connected with the fact that the traditional methods for constructing a non-standard extension ${}^*\mathfrak{A}$ require first choosing an ultrafilter U (in the case of ultrapowers) or instead, for instance, a well-ordering of the domain of the given structure $\mathfrak{A}$, that is, of objects whose existence is based on the full axiom of choice and which cannot be represented by means of a concrete 'effective' construction. By analogy with other similar questions (for instance, with the problem of existence of sets of reals that are not Lebesgue measurable [26], [27]) it was expected that this question should be answered in the negative. However, a positive answer was found (Kanovei, Uspenskii, Shelah, [24], [28]).

We remark that Question (b) is considerably more difficult even in the absence of the effectivity requirement. Indeed, absolutely saturated extensions of this kind must be proper classes, and hence their most natural construction employs a preliminary choice of an Ord-long sequence of ultrafilters. And this needs a global

[1] This problem was communicated to one of the authors (V. G. Kanovei) by V. A. Uspenskii in the early 1990s.

choice axiom, which is beyond the framework of the Zermelo–Fraenkel set theory **ZFC**. We show in § 1 that this question is also answered in the affirmative: the theory **ZFC** is strong enough to define an absolutely saturated extension of the whole 'standard' set universe, and such an extension satisfies the axioms of the well-known non-standard set theory **BST**.

Problem 2. How can classes be adjoined to a non-standard set universe in order to obtain a structure satisfying this or that non-standard class theory, provided that the standard domain of sets is already extended by standard classes?

Here the problem is to transform the superstructure of standard classes over the domain of standard sets into a superstructure of non-standard classes over the domain of non-standard sets. In a relatively simple case of predicative class theories, the problem was solved in [15]. A solution for impredicative class theories has recently been obtained. We present it in § 2.

Problem 3. How can 'external' sets be adjoined to a non-standard set universe of 'internal' type so that the extended universe satisfies a reasonable 'external' non-standard set theory?

Unlike the case of standard set-theoretic universes, in non-standard universes we consider the *standardness* relation $\mathrm{st}\,(\cdot)$ in addition to the membership relation $\in$. Universes of 'internal' type are just elementary extensions (in the $\in$-language) of the standard universe. They usually do **not** satisfy the separation axiom in the extended st-$\in$-language, that is, st-$\in$-definable parts of sets are themselves not necessarily sets. In contrast, non-standard universes of 'external' type do satisfy the separation axiom in the st-$\in$-language, which is much more convenient for applications, but they are not elementary extensions of their standard subdomains. (See below on theories which describe universes of both types.) The natural way to transform an 'internal' set universe to an 'external' one is to adjoin all st-$\in$-definable 'non-sets', then all suitable collections of sets and 'non-sets', and so forth. We show in § 3 that this process can be carried out in a certain well-defined form, leading to a solution of Problem 3 both at the level of sets and at the level of classes.

In the last section we briefly touch upon some questions which do not belong to non-standard analysis in the narrow sense but are connected with it in methodological aspects and generally occupy a similar place in mathematics. This can be understood as a certain generalization of methods of non-standard analysis. Indeed, non-standard set-theoretic universes, and non-standard structures in general, can be viewed as a specific tool for *evaluating the truth values of sentences*. For instance, in the most elementary case of an ultrapower $\mathfrak{A}^I/U$ of a given structure $\mathfrak{A}$ modulo an ultrafilter U on a given index set I, the universe of the non-standard extension ${}^*\mathfrak{A}$ of $\mathfrak{A}$ consists of functions from I into $\mathfrak{A}$. And elements of $\mathfrak{A}$ are represented by constant functions. To evaluate the truth of a formula $\varphi(f_1,\dots,f_n)$ in ${}^*\mathfrak{A}$ (where all the f_i belong to the Cartesian power $\mathfrak{A}^I$), the *truth value*

$$[\![\varphi]\!] = \{i\in I : \mathfrak{A}\models\varphi(f_1(i),\dots,f_n(i))\}$$

is defined. It belongs to the Boolean algebra $\mathscr{P}(I)=\{X : X\subseteq I\}$. Here $\mathfrak{A}\models\cdot$ means 'holds', or 'is true' in $\mathfrak{A}$. It follows from the Łoś theorem that a sentence φ is true in ${}^*\mathfrak{A}$ if and only if its value $[\![\varphi]\!]$ belongs to U. Thus, investigation of the truth of ${}^*\mathfrak{A} \models \varphi$ is reduced to the question of what the truth value of $[\![\varphi]\!]$ is, that is, to the more elementary notion $\mathfrak{A} \models \varphi$.

This sort of *extensions with evaluations* can be defined for any Boolean or Heyting algebra, and even for more general algebras and lattices $\mathbf{B}$. Together with non-standard analysis, this general approach includes forcing in modern set theory, as well as Boolean-valued and Heyting-valued analysis. Instead of the quotient modulo an ultrafilter, the truth of a sentence φ in the extended structure is determined by the equality $[\![\varphi]\!] = \top_{\mathbf{B}}$ or the inequality $[\![\varphi]\!] \geqslant b$, where b is a chosen element in an algebra $\mathbf{B}$. The significance of this method consists in a representation of mathematical structures which enables one to estimate truth values connected with certain problems, instead of directly studying the problems themselves. Robinson's theorem mentioned above is a good example.

Problem 4. How can one systematically define 'simplifying representations' of this kind by means of reduction of given structures and relations to more elementary (but connected with the given ones) non-standard structures and relations? How can one define and classify the associated truth values?

This programme has been successfully carried out since the 1960s in non-standard analysis and since the 1970s in Boolean-valued and Heyting-valued analysis. We present in § 4 some interesting results in this direction, but without attempting a comprehensive survey of the problem (see for instance [5], [29]–[33]).

General notation.

$\operatorname{dom} f$	—	the domain of a function f;
$\operatorname{dom} X$	=	$\{x : \exists y(\langle x, y\rangle \in X)\}$ for any binary relation X;
$\operatorname{ran} f$	—	the range (all the values) of a function f;
embedding	—	(or injection) of X into Y is a bijection between X and a part of Y;
$f``X$	=	$\{f(x) : x \in \operatorname{dom} f \cap X\}$, the f-image of a set X;
$\operatorname{card} X$	—	the cardinality of a set X;
Card	—	the class of all cardinals;
Ord	—	the class of all ordinals;
$\mathscr{P}(X)$	=	$\{Y : Y \subseteq X\}$, the family of all subsets of X;
$\mathbf{V}_\alpha$	—	the αth level of the von Neumann cumulative set hierarchy; recall that $\mathbf{V}_0 = \varnothing$, and then by transfinite induction $\mathbf{V}_\alpha = \mathscr{P}\Bigl(\bigcup_{\gamma<\alpha} \mathbf{V}_\gamma\Bigr)$, while $\mathbf{V} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbf{V}_\alpha$ is the universe of all sets in **ZFC**;
$\operatorname{rk} x$	=	$\min\{\alpha \in \mathrm{Ord} : x \in \mathbf{V}_\alpha\}$, the von Neumann rank of x.

Zermelo–Fraenkel set theory **ZFC** (with the axiom of choice) will be our ground set theory. All cases of deviation from this convention (for instance, when class theories or non-standard theories are used) will be explicitly mentioned. We give [34]–[38] as references in matters of set theory, and [39] as a general reference in matters of model theory. See [40]–[46], [5], [47], [29], [32], [2], [3] regarding 'robinsonian' or model-theoretic non-standard analysis. Finally, [25] is given as a general reference on set-theoretic non-standard analysis.

The exposition in this article can be understood by any mathematician acquainted with the foundations of set theory and non-standard analysis.

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§ 1. From standard to non-standard set universe

This section is devoted to Problem 1 from the Introduction, that is, to an effective construction of saturated elementary extensions of given 'standard' mathematical structures. (This will include the most universal structure, the set-theoretic universe.) Absolutely saturated elementary extensions will also be defined on this basis. We begin (the first four subsections) with a compact exposition of some notions and results in model theory necessary for understanding more essential results in the remaining sections.

The key construction of an 'effective' saturation of a given $\in$-structure by methods of [24], [28] is presented in subsection 1d. It is used in the construction of effective and absolutely saturated extensions in subsection 1e. Then we prove in subsection 1g that this extension of the whole universe of **ZFC** satisfies the axioms of bounded set theory **BST**, one of non-standard set theories briefly discussed in subsection 1f. We end in subsection 1h with an important theorem on reduction of formulae to Σ_2^{st} form in **BST**, on which the subsequent exposition is largely based.

1a. Sets, classes, relational structures. It is a special feature of the topics discussed in this article that classes together with sets play an important role. Arguing in **ZFC**, we understand a *class* to be any collection of sets defined by a formula of the form $\{x : \varphi(x, p_1, \dots, p_n)\}$, where $p_1, \dots, p_n$ are sets called *parameters of definition*. Some collections of this type are sets, for instance $\{x : \varphi(x) \wedge x \in y\}$, by the axiom of Separation. Those classes which are not sets, for instance the class $\mathbf{V} = \{x : x = x\}$ of all sets, are *proper classes*.

This is an informal understanding, of course. However, it turns out that rather simple and reasonable statements about classes admit a natural transformation to 'legitimate' statements about sets. For instance, the sentence $\forall X \exists Y \Phi(X, Y)$ (X, Y are classes) is adequately understood in the sense that, given a formula φ, one can define a formula ψ such that $\Phi(X, Y)$ holds, where X and Y are the classes defined by the formulae φ and ψ, respectively.

On the other hand, the Gödel–Bernays class theory **GB** (see subsection 2a) can be employed for treatment of classes. This theory views classes as formal objects (while sets are just those classes which are elements of other classes). It is important here that **GB** is a conservative extension of **ZFC** in the sense that any theorem of **GB** relating only to sets is also a theorem of **ZFC**.

In general, everything considered about classes in the set universe of **ZFC** below is compatible with both the first (informal) and the second (**GB**) approach.

We shall consider *relational structures*[2] of the form $\mathfrak{A} = \langle A; R_1, \dots, R_n \rangle$, where $A = |\mathfrak{A}|$ is a set or class called the *universe* of a structure and often identified with $\mathfrak{A}$ itself (by notation like $A = \langle A; R_1, \dots, R_n \rangle$), while $R_1, \dots, R_n$ is a finite list of finitary relations on A. For instance, the structure of natural numbers can be represented in the form $\mathbb{N} = \langle \mathbb{N}; +, \times, 0, 1 \rangle$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers, $+$ denotes the ternary relation $x + y = z$, $\times$ is understood in the same way, and 0 and 1 are constants naturally presented as unary relations (on $\mathbb{N}$) with one-element truth domains.

Structures of the form $\mathfrak{A} = \langle A; \varepsilon \rangle$ with a unique binary relation ε viewed as the membership relation, are called $\in$*-structures*. We shall also consider structures of the form $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ containing some other relations in a (perhaps, empty) list $\dots$ in addition to ε.

Generally, if $\mathscr{L}$ is a language containing relational symbols $r_1, \dots, r_n$ and the arity of each r_i is equal to the arity of the relation R_i, then $\mathfrak{A} = \langle A; R_1, \dots, R_n \rangle$ is called an $\mathscr{L}$*-structure*. In this case a *relativization* (or *interpretation*) $\varphi^{\mathfrak{A}}$ is defined for each $\mathscr{L}$-formula φ by substituting R_i for any occurrence of a relational symbol r_i, $i = 1, \dots, n$, and by restricting each quantifier in φ to A. (In other words, $\exists\, x$ changes to $\exists\, x \in A$ and $\forall\, x$ changes to $\forall\, x \in A$.) The notation $\varphi^{\mathfrak{A}}$ is also used when some or all of the free variables of φ are replaced by elements of A (called *parameters* in this case).

If the universe A, and then in general also all the relations R_i of a structure $\mathfrak{A} = \langle A; R_1, \dots, R_n \rangle$, are proper classes defined (in **ZFC**) by some $\in$-formulae, then $\varphi^{\mathfrak{A}}$ also represents an $\in$-formula obtained from φ according to the definition; for example, $\exists\, x \dots$ is changed to $\exists\, x(\alpha(x) \wedge \dots)$, where $\alpha(x)$ is the $\in$-formula defining A, while $R_i(x_1, \dots, x_m)$ is replaced by $\rho_i(x_1, \dots, x_m)$, where ρ_i is the $\in$-formula defining R_i, and $m = m(i)$ is the arity of R_i.

It is clear that if φ is a closed $\mathscr{L}$-formula with parameters in A, then $\varphi^{\mathfrak{A}}$ expresses the truth of φ in $\mathfrak{A}$. For those structures having sets as universes, there is another definition of truth or *satisfiability*. It is used in the form $\mathfrak{A} \models \varphi$ (see [39], 1.3), where the formula φ itself (understood, modulo some necessary elaborations, as a sequence of certain symbols, that is, as a set) is a variable. Fortunately, in this case $\varphi^{\mathfrak{A}}$ is equivalent to $\mathfrak{A} \models \varphi$. (This assertion involves issues not evident at first glance, but we are not going to touch upon these details.)

There is a wider class of relational structures. It consists of *invariant* structures, that is, those of the form $\mathfrak{A} = \langle A; R_1, \dots, R_n; \mathsf{E} \rangle$, where E is an equivalence relation on A and the relations R_i are E-invariant independently on each argument. In this case, equality in formulae of the corresponding language is interpreted in $\mathfrak{A}$ by means of E. Ordinary relational structures, as above, correspond to the case when E is an equality.

Any invariant structure $\mathfrak{A} = \langle A; R_1, \dots, R_n; \mathsf{E} \rangle$ can be naturally converted to an ordinary one by taking the quotient modulo E and passing to the equivalence classes $[a]_{\mathsf{E}}$, $a \in A$, and accordingly, taking quotients of the relations R_i. However, this is not good for those structures $\mathfrak{A}$ having proper classes A as their universes:

[2]We use the notion of '(relational) structure' in the same sense as it is typically used for the notion of 'model' (see, for example, [39]), with the only difference that the universe of a structure can be a proper class, not necessarily a set. Most essential structures below will have proper classes as their universes.

the equivalence classes $[a]_{\mathsf{E}}$ can then be proper classes, which is inconvenient in many aspects. In some cases (see for instance Remark 1.6) one can get around this difficulty by choosing an element in each equivalence class $[a]_{\mathsf{E}}$ (a proper class), or at least a non-empty set of elements. But such a reduction to sets is impossible in other important cases. However, invariant structures are as applicable as ordinary ones from the point of view of applications to such metamathematical problems as consistency or conservativity.

1b. Elementary extensions and direct limits. Let $\mathfrak{A} = \langle A; R_1, \dots, R_n\rangle$ and $\mathfrak{A}' = \langle A'; R'_1, \dots, R'_n\rangle$ be $\mathscr{L}$-structures for one and the same language $\mathscr{L}$. An embedding $\pi\colon A \to A'$ is called *elementary* if $\varphi^{\mathfrak{A}} \Leftrightarrow (\pi\varphi)^{\mathfrak{A}'}$ holds for every closed $\mathscr{L}$-formula with parameters in A, where the formula $\pi\varphi$ is obtained from φ by substituting $\pi(x)$ for any $x \in A$ which occurs in φ as a parameter. It is also said that $\mathfrak{A}'$ is an *elementary extension* of $\mathfrak{A}$ (by means of π) in this case.

Definition 1.1 (direct limits). Suppose that λ is a limit ordinal. A *chain of elementary structures*, or simply an *elementary chain*, of length λ is a sequence of structures $\mathfrak{A}_\xi = \langle A_\xi; R^\xi_1, \dots, R^\xi_n\rangle$, $\xi < \lambda$ (n does not depend on ξ), along with a system of elementary embeddings $\pi_{\xi\eta}\colon A_\xi \to A_\eta$ ($\xi \leqslant \eta < \lambda$ and all the $\pi_{\xi\xi}$ are identity maps) satisfying $\pi_{\xi\zeta} = \pi_{\eta\zeta} \circ \pi_{\xi\eta}$.

The *direct limit* of such a chain is a structure $\mathfrak{A} = \langle A_\lambda; R^\lambda_1, \dots, R^\lambda_n\rangle$ defined as follows. Put

$$A'_\xi = A_\xi \setminus \bigcup_{\eta<\xi} \pi_{\eta\xi}\text{"} A_\eta \quad \text{and} \quad A_\lambda = \{\langle \xi, a\rangle : \xi < \lambda \wedge a \in A'_\xi\}.$$

Assume that $1 \leqslant k \leqslant n$. All the relations R^ξ_k, $\xi < \lambda$, obviously have the same arity $s = s(k) \in \mathbb{N}$. If the pairs $\langle \xi_1, a_1\rangle, \dots, \langle \xi_s, a_s\rangle$ belong to A and $\xi = \sup\{\xi_1, \dots, \xi_s\}$, then we define $R^\lambda_k(\langle \xi_1, a_1\rangle, \dots, \langle \xi_s, a_s\rangle)$ whenever $R^\xi_k(\pi_{\xi_1,\xi}(a_1), \dots, \pi_{\xi_s,\xi}(a_s))$. This accomplishes the definition of the limit structure $\mathfrak{A}$. In addition, for any $\xi < \lambda$ we define an embedding $\pi_{\xi\lambda}\colon A_\xi \to A_\lambda$ as follows. Let $a \in A_\xi$. There exists a least ordinal $\eta \leqslant \xi$ such that $a = \pi_{\eta\xi}(a')$ for some $a' \in A_\eta$. (For instance, $\eta = \xi$ for $a \in A'_\xi$.) Then, clearly, $a' \in A'_\eta$. Put $\pi_{\xi\lambda}(a) = \langle \eta, a'\rangle$.

The definition of a direct limit also applies in the case of elementary chains of length Ord (that is, $\mathfrak{A}_\xi$ is given for any ordinal ξ). In this case the universe $A_\infty = \{\langle \xi, a\rangle : \xi \in \mathrm{Ord} \wedge a \in A'_\xi\}$ of the limit structure $\mathfrak{A}_\infty$ can be a proper class even if all the A_ξ are sets. However, the limit embedding $e_{0\infty}\colon A_0 \to A_\infty$ is in any case defined as in the case $\lambda \in \mathrm{Ord}$.

We omit the proof of the following well-known theorem. The proof is essentially fairly simple, although rather laborious if all the details are accurately accounted for. See, for example, Theorem 3.1.13 in [39] for the case when each $\pi_{\xi\eta}$ is an identity map and $A = \bigcup_\xi A_\xi$. The general case is easily reducible to this case.

Theorem 1.2. *Under the assumptions of Definition 1.1, the structure $\mathfrak{A}_\lambda$ is an elementary extension of every substructure $\mathfrak{A}_\xi$ by means of the embedding $\pi_{\xi\lambda}$. If $\xi < \eta \leqslant \lambda$, then $\pi_{\xi\lambda} = \pi_{\eta\lambda} \circ \pi_{\xi\eta}$. The same holds for $\mathfrak{A}_\infty$ and chains of length* Ord.

1c. Saturated structures and extensions. The elementary extensions most important for non-standard analysis are those which adjoin to the given structure elements whose existence is in some sense compatible with the properties of the structure. Since we are mainly interested in $\in$-structures, the following principal definition looks somewhat simpler than in the general case (see, for example, Ch. 5 in [39]).

Definition 1.3. Suppose that κ is an infinite cardinal.

(i) A structure $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ is *κ-saturated* if for any set $X \subseteq A$ of cardinality $\operatorname{card} X < \kappa$ such that the ε-intersection of every finite $X' \subseteq X$ in $\mathfrak{A}$ is non-empty,[3] the ε-intersection of X itself in $\mathfrak{A}$ is non-empty.

(ii) A structure $\mathfrak{A}' = \langle A'; \varepsilon', \dots \rangle$ is *κ-saturated over a structure* $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ by means of an elementary embedding $\pi \colon A \to A'$ if for any set $X \subseteq A$ of cardinality $\operatorname{card} X < \kappa$ such that the ε-intersection of every finite $X' \subseteq X$ in $\mathfrak{A}$ is non-empty, the ε'-intersection of $\pi X = \{\pi(x) : x \in X\}$ in $\mathfrak{A}'$ is non-empty.

Thus, a κ-saturated structure is exactly a structure κ-saturated over itself. It will be shown that saturated structures can be obtained in the form of limits of increasing chains of successive saturations.

Example 1.4. Consider an $\in$-structure $\mathfrak{P} = \langle P; \in \rangle$, where $P = \mathscr{P}(\mathbb{N})$ (the set of all subsets of the set $\mathbb{N}$ of all natural numbers). Each $n \in \mathbb{N}$ is identified with the set $\{0, 1, \dots, n-1\}$, and hence $\mathbb{N} \subseteq P$.

Suppose that an $\in$-structure $\mathfrak{A} = \langle A; \varepsilon \rangle$ is $\aleph_1$-saturated over the structure $\mathfrak{P}$ by means of an elementary embedding $\pi \colon P \to A$. (For instance, $\mathfrak{A}$ can be an $\aleph_1$-saturated elementary extension of $\mathfrak{P}$.) In particular, A contains an element ${}^*\mathbb{N} = \pi(\mathbb{N})$ and elements ${}^*n = \pi(n)$, $n \in {}^*\mathbb{N}$, and we have ${}^*k \,\varepsilon\, {}^*n \,\varepsilon\, {}^*\mathbb{N}$ for all $k \in n \in \mathbb{N}$. The set $X \subseteq P$ of all sets of the form $x_n = \{k \in \mathbb{N} : k \geqslant n\}$, where $n \in \mathbb{N}$, is countable, and every finite $X' \subseteq X$ has an obvious non-empty $\in$-intersection (that is, the usual intersection). Therefore, by the choice of $\mathfrak{A}$ there exists an element $a \in A$ that ε-belongs to all the sets $\pi(x_n)$. It follows easily that $a \in {}^*\mathbb{N}$ and ${}^*n \,\varepsilon\, a$ for all $n \in \mathbb{N}$, and this is usually interpreted as the fact that a is an infinitely large element of ${}^*\mathbb{N}$ in $\mathfrak{A}$.

Theorem 1.5. *Let κ be an infinite cardinal. Every structure of the form* $\mathfrak{A} = \langle A; \varepsilon, R_1, \dots, R_n \rangle$ (*where* A is a set or class) *admits an elementary embedding*:

(i) *in a structure κ^+-saturated over* $\mathfrak{A}$;

(ii) *in a κ^+-saturated structure.*

If the universe $A = |\mathfrak{A}|$ is an infinite set of cardinality λ, then the elementary extensions in (i) *and* (ii) *can be chosen with universes of cardinality $\leqslant \lambda^\kappa$.*

Proof. It is clear that (ii) follows from (i) by means of the construction of a chain of successive κ-saturating elementary extensions that begins with $\mathfrak{A}$, has length κ^+ (the successor of the cardinal κ), and utilizes (i) at successor steps and Theorem 1.2 at limit steps.[4] (Note also that $\lambda^\kappa\,\kappa^+ \leqslant \lambda^\kappa\, 2^\kappa = (2\lambda)^\kappa = \lambda^\kappa$.) We now prove (i).

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[3] This means the existence of a common ε-element $a \in A$ of all the elements $x \in X'$.

[4] As a matter of fact a saturated extension can be obtained in the form of the usual one-step ultrapower of the given structure, but by means of a rather complex type of ultrafilters called *good* ultrafilters; see 6.1 in [39], especially Theorem 6.1.8.

Put $I = [\kappa]^{<\omega}$ (the set of all finite subsets of κ) as the index set for the operation of ultrapower. The family of all sets of the form $I_\alpha = \{i \in I : \alpha \in i\}$, $\alpha < \kappa$, satisfies the *finite intersection property*, that is, every finite family of sets of this form has a non-empty intersection. Therefore, there exists an ultrafilter $U \subseteq \mathscr{P}(I)$ containing all sets of the form I_α, $\alpha < \kappa$. Ultrafilters of this type are called *adequate* (see [25]).

To construct the *ultrapower* $\mathfrak{A}^I/U = {}^*\mathfrak{A} = \langle {}^*A; {}^*\varepsilon, {}^*R_1, \dots, {}^*R_n \rangle$ *of the structure* $\mathfrak{A}$ *modulo the ultrafilter* U, one considers the collection A^I of all functions $f \colon I \to A$. Put

$$f =_U g \quad \text{if} \quad \{i : f(i) = g(i)\} \in U$$

for all $f, g \in A^I$. It is clear that $=_U$ is an equivalence relation on A^I, hence one can define *equivalence classes* $[f]_U = \{g \in A^I : f =_U g\}$. Finally, put ${}^*A = \{[f]_U : f \in A^I\}$. Then obviously $\operatorname{card} {}^*A \leqslant \lambda^\kappa$.

Remark 1.6. If A is a proper class, then in accordance with this definition each equivalence class $[f]_U$ can be a proper class while ${}^*\mathfrak{A}$ can be a class of classes. Collections of this form are too difficult to consider in **ZFC**, for formal reasons. However, there is a satisfactory solution: cut every equivalence class $[f]_U$ to the set of all functions $g \in [f]_U$ having the least von Neumann rank $\operatorname{rk} g$ among all $g \in [f]_U$. With such a change, it is still true that $f =_U g \Leftrightarrow [f]_U = [g]_U$, but f itself does not necessarily belong to $[f]_U$.

Next, we define $[f]_U \, {}^*\varepsilon \, [g]_U$ if $\{i : f(i) \, \varepsilon \, g(i)\} \in U$ and, in general,

$${}^*R_k([f_1]_U, \dots, [f_s]_U) \quad \text{if} \quad \{i : R_k(f_1(i), \dots, f_s(i))\} \in U \qquad (1 \leqslant k \leqslant n),$$

where s is the arity of R_k, and $f_1, \dots, f_s \in A^I$. The consistency of the definitions of ${}^*\varepsilon$ and ${}^*R_k$, that is, the independence of the right-hand sides from the choice of representatives in the equivalence classes, follows easily from the fact that U is an ultrafilter. Finally, to define an elementary embedding of $\mathfrak{A}$ in ${}^*\mathfrak{A}$ we put ${}^*x = [\mathbf{f}_x]_U$ for each $x \in A$, where $\mathbf{f}_x \in A^I$ is the constant function defined by $\mathbf{f}_x(i) = x$ for all $i \in I$.

We omit the proof of the following well-known result (see 4.1.9 in [39]). For convenience, let $\mathscr{L}$ be a language containing relational symbols of corresponding arity for ε and for all the relations R_i. For any $\mathscr{L}$-formula φ, its relativizations $\varphi^{\mathfrak{A}}$ and $\varphi^{{}^*\mathfrak{A}}$ are defined as in subsection 1a.

Proposition 1.7 (Łoś theorem). *In our notation,*

$$\varphi^{{}^*\mathfrak{A}}([f_1]_U, \dots, [f_n]_U) \Leftrightarrow \{i \in I : \varphi^{\mathfrak{A}}(f_1(i), \dots, f_n(i))\} \in U$$

for any $\mathscr{L}$-formula $\varphi(x_1, \dots, x_n)$ and any collection $f_1, \dots, f_n \in A^I$.

Corollary 1.8. *The map $x \mapsto {}^*x$ is an elementary embedding of $\mathfrak{A}$ in ${}^*\mathfrak{A}$.*

In order to prove the theorem it now remains to verify that the structure ${}^*\mathfrak{A}$ is κ^+-saturated over $\mathfrak{A}$. Consider an arbitrary set $X = \{x_\alpha : \alpha < \kappa\} \subseteq A$ of cardinality $< \kappa^+$, that is, $\leqslant \kappa$. Suppose that for every finite $Y \subseteq X$ there exists an element $z_Y \in A$ satisfying $z_Y \, \varepsilon \, y$ for all $y \in Y$. If $i = \{\alpha_1, \dots, \alpha_m\} \in I$, then put $Y(i) = \{x_{\alpha_1}, \dots, x_{\alpha_m}\}$ and $f(i) = z_{Y(i)}$. We show that $[f] \, {}^*\varepsilon \, {}^*x$ holds for all $x \in X$.

Indeed, by 1.7 it suffices to prove that the set $u_\alpha = \{i \in I : f(i) \in x_\alpha\}$ belongs to U for any $\alpha < \kappa$. But this holds because of the choice of U. Indeed, it easily follows from the construction that $I_\alpha = \{i : \alpha \in i\} \subseteq u_\alpha$. However, $I_\alpha \in U$ by the choice of U, and hence u_α belongs to U. Theorem 1.5 is proved.

1d. 'Effective' saturated extensions. It is quite easy to check that any infinite κ-saturated $\in$-structure satisfying a suitable minimal list of conditions (for instance, the axioms of **ZFC** without Replacement) has a cardinality strictly bigger than κ, and hence under these conditions every *absolutely saturated* structure, that is, κ-saturated for every cardinal κ, necessarily has a proper class as the universe. At first glance, one can define absolutely saturated extensions with the help of some results in subsections 1b and 1c without much trouble: indeed, define an elementary chain $\{\mathfrak{A}_\xi\}_{\xi \in \mathrm{Ord}}$ by direct limits at limit steps (Theorem 1.2) and by saturating extensions (Theorem 1.5) with increasing and unbounded cardinal parameters at successor steps. However, the problem becomes more complicated upon closer inspection.

Indeed, Theorem 1.5 is non-effective in the sense that the extension its proof produces depends on the choice of an ultrafilter. Therefore, in order to choose a suitable ultrafilter at every step $\xi \in \mathrm{Ord}$, the construction involves the global choice axiom in the case when the choice 'function' has as domain the class of all ordinals. Fortunately, there is a different method that avoids this disadvantage. Its idea is to convert the construction of a saturating extension to an 'effective' form, that is, independent of the choice of an ultrafilter, or something like that. In addition, this method enables one to avoid using the global choice axiom, thus keeping the arguments within the framework of **ZFC**.

Construction 1.9 ('effective' saturation). Let us fix an infinite cardinal κ and a structure $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$, with the universe A being either a set or a proper class. The following iterated ultrapower construction was introduced in [39] (see 6.5) (in slightly different notation), and all the related facts cited below are taken from [39].

1) Let us order the set $\mathscr{P}(\kappa)$ (of all subsets $X \subseteq \kappa$) so that $X < Y$ whenever the least element in the symmetric difference $X \,\Delta\, Y = (X \setminus Y) \cup (Y \setminus X)$ belongs to Y. Consider the cardinal $\delta = 2^\kappa$ and order the set $\mathscr{P}(\kappa)^\delta$ (of all δ-sequences of sets $X \subseteq \kappa$) lexicographically: $s \prec t$ if $s(\xi) < t(\xi)$ (in the sense of the previous definition), where $\xi = \min\{\xi < \delta : s(\xi) \neq t(\xi)\}$.

2) Consider the set $\mathscr{S}$ of all sequences $s = \{X_\xi\}_{\xi<\delta} \in \mathscr{P}(\kappa)^\delta$ such that the set $U_s = \operatorname{ran} s = \{X_\xi : \xi < \delta\}$ is an ultrafilter (on κ). The set $\mathscr{S}$ (as well as $\mathscr{P}(\kappa)^\delta$) is linearly ordered by $\prec$. Thus, $\{U_s\}_{s \in \mathscr{S}}$ is a linear ordering (not a well-ordering), with repetitions, of the set of all ultrafilters on κ. It will be used to define an elementary extension of $\mathfrak{A}$ which contains in itself all ultrapowers of the form $\mathfrak{A}^\kappa/U$, where U is any ultrafilter on κ. What this will result in — see below.

3) Suppose that $S = \{s_1 < s_2 < \dots < s_n\} \subseteq \mathscr{S}$ is a finite set. Let U_S be the family of all sets $X \subseteq \kappa^S$ satisfying

$$U_{s_n}\xi_n \dots U_{s_2}\xi_2 U_{s_1}\xi_1 \quad (\langle \xi_1, \xi_2, \dots, \xi_n \rangle \in X),$$

where $U_s\xi\Phi(\xi)$ means that the set $\{\xi : \varphi(\xi)\}$ belongs to U_s. It is known that U_S is an ultrafilter on κ^S, and the ultrapower modulo U_S is equal to the chain of successive ultrapower extensions modulo $U_{s_1}, U_{s_2}, \dots, U_{s_n}$. Consider the collection U'_S of all sets $X' \subseteq \kappa^{\mathscr{S}}$ of the form $X' = X \uparrow \mathscr{S} = \{i \in \kappa^{\mathscr{S}} : i \restriction S \in X\}$, where $X \in U_S$. Then the union $U = \bigcup_{S \subseteq \mathscr{S} \text{ finite}} U'_S$ is an ultrafilter in the algebra $\mathbf{B}$ of all sets $X' \subseteq \kappa^{\mathscr{S}}$ of the form $X' = X \uparrow \mathscr{S}$, where $X \subseteq \kappa^S$ is an arbitrary set while $S \subseteq \mathscr{S}$ is finite. The sets that belong to the algebra $\mathbf{B}$ are called *finitely determined* in the full power $\mathscr{P}(\kappa^{\mathscr{S}})$ (with the terminology of [39]).

4) Let us continue the construction. Put $F = \bigcup_{S \subseteq \mathscr{S} \text{ finite}} F_S$, where F_S is the family of all functions $f \colon \kappa^{\mathscr{S}} \to A$ such that $i \restriction S = j \restriction S \Rightarrow f(i) = f(j)$. ($F$ is a set or proper class together with the universe A of the given structure $\mathfrak{A}$.) Still following the terminology of [39], we call the functions in F *finitely determined* functions. The *iterated ultraproduct* ${}^*\mathfrak{A} = F/U = \langle {}^*A; {}^*\varepsilon, \dots \rangle$ itself has ${}^*A = \{[f]_U : f \in F\}$ as the universe, where $[f]_U = \{g \in F : f =_D g\}$ (with the amendment made in Remark 1.6), and $f =_U g$ means that the set $I_{fg} = \{i \in \kappa^{\mathscr{S}} : f(i) = g(i)\}$ belongs to U. (Note that I_{fg} belongs to $\mathbf{B}$ for all f, $g \in F$.) It remains to define the relation: $[f]_U \, {}^*\varepsilon \, [g]_U$ if $\{i \in \kappa^{\mathscr{S}} : f(i) \in g(i)\} \in U$ (see the proof of Theorem 1.5), and define $*$-extensions ${}^*R$ of all the relations R that occur in the list $\dots$ of relations of the given structure $\mathfrak{A}$ accordingly.

5) Put ${}^*x = [\mathbf{f}_x]_U$ for each $x \in A$, where $\mathbf{f}_x \in F$ is the constant function $\mathbf{f}_x(i) = x$ for all $i \in \kappa^{\mathscr{S}}$. The collection (a set or a class together with A) F obviously does not contain all maps from $\kappa^{\mathscr{S}}$ to A, however, F and the algebra $\mathbf{B}$ are consistent with each other in the sense that the Łoś theorem (Proposition 1.7) and Corollary 1.8 still hold (see, for instance, 6.5.5 and 6.5.6 in [39]). Thus, $x \mapsto {}^*x$ is an elementary embedding of $\mathfrak{A}$ into ${}^*\mathfrak{A}$.

6) To demonstrate that the structure ${}^*\mathfrak{A}$ is κ^+-saturated over $\mathfrak{A}$ by means of the indicated embedding, consider an arbitrary set $X \subseteq A$ of cardinality $\leqslant \kappa$ satisfying the condition of non-emptiness of the ε-intersection of any finite $X' \subseteq X$. It has been shown (see the proof of Theorem 1.5) that there exists an ultrafilter on the set $I = [\kappa]^{<\omega}$ such that the corresponding ultrapower of the structure $\mathfrak{A}$ is κ^+-saturated over $\mathfrak{A}$. An ultrafilter with this property exists on κ as well, because $\operatorname{card} I = \kappa$. Then it has the form U_s for a suitable $s \in \mathscr{S}$. Thus, the ultrapower $\mathfrak{A}^\kappa/U_s$ is κ^+-saturated over $\mathfrak{A}$ by means of the natural embedding. In other words, there exists a map $h \colon \kappa \to A$ satisfying $[h]_{U_s} \, {}^*\varepsilon \, {}^*x$ for all $x \in X$, that is, by the Łoś theorem, satisfying $E_x = \{\xi < \kappa : h(\xi) \in x\} \in U_s$ for all $x \in X$.

We define a function $f \colon \kappa^{\mathscr{S}} \to A$ by the equality $f(i) = h(i(s))$ for all $i \in \kappa^{\mathscr{S}}$. Then clearly $f \in F$; more precisely, $f \in F_{\{s\}}$. For any $x \in A$ the set $E'_x = \{i \in \kappa^{\mathscr{S}} : f(i) \in x\}$ coincides with $E_x \uparrow \mathscr{S} = \{i \in \kappa^{\mathscr{S}} : i(s) \in E_x\}$, and hence we have $E'_x \in U \Leftrightarrow E_x \in U_s$ by the definition of U. Therefore, $E'_x \in U$ for any $x \in X$ by the above. Applying the Łoś theorem in the opposite direction for U (see the reference above) we obtain $[f]_U \, {}^*\varepsilon \, {}^*x$ in ${}^*\mathfrak{A}$ for all $x \in X$, as required.

Corollary 1.10. *In the notation of Construction 1.9, the structure* ${}^*\mathfrak{A} = \langle {}^*A; {}^*\varepsilon, \dots \rangle$ *is an 'effective', clearly and unambiguously defined elementary extension of the structure* $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$*, and is* κ^+*-saturated over* $\mathfrak{A}$ *by means of the embedding* $x \mapsto {}^*x$.

Remark 1.11 (bounded embeddings). An embedding $x \mapsto {}^*x$ of a structure $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ into a structure ${}^*\mathfrak{A} = \langle {}^*A; {}^*\varepsilon, \dots \rangle$ is said to be *bounded* if for each $y \in {}^*A$

there exists an $a \in A$ satisfying $y\, {}^*\varepsilon\, {}^*a$. This is a very important property of extensions of $\in$-structures. Note that extensions of types 1.9–1.10 are not necessarily bounded, yet there is a convenient sufficient condition. A structure $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ is called *almost universal* [34] if for any **set** $X \subseteq A$ there exists an element $a \in A$ satisfying $x \,\varepsilon\, a$ for all $x \in X$. In this case A is a proper class (or in other words, $a \,\varepsilon\, a$ for some a).

We assert that under the conditions of 1.9, *if* $\mathfrak{A}$ *is an almost universal structure, then* $x \mapsto {}^*x$ *is a bounded embedding of* $\mathfrak{A}$ *into* ${}^*\mathfrak{A}$, *and* ${}^*\mathfrak{A}$ *is also an almost universal structure*. Suppose that (in the notation of 1.9) $y = [f]_U$, where $f\colon \kappa^{\mathscr{S}} \to A$ is a finitely determined map (in the sense of item 4) above). Put $a = \operatorname{ran} f$. Then $f(i) \in \mathbf{f}_a(i) = a$ for all $i \in \kappa^{\mathscr{S}}$. It follows that $[f]_U = y\, {}^*\varepsilon\, {}^*a = [\mathbf{f}_a]_U$, as required. The almost universality of ${}^*\mathfrak{A}$ easily follows.

To carry out the construction of bounded elementary extensions of non-almost universal $\in$-structures, for instance, structures of the form $\langle X; \in\restriction X\rangle$, where X is any set, it suffices to modify the ultrapower construction in 1.9 so that only those (finitely determined) functions $f\colon \kappa^{\mathscr{S}} \to A$ participate for which there exists an $a \in A$ such that $\forall\, x \in \operatorname{ran} f\, (x \,\varepsilon\, a)$. See details in [25], Ch. 4.

1e. Absolutely saturated extensions. So far we have considered only those structures with saturation bounded by some cardinal. The next definition introduces an 'absolute' form of this notion. Recall that the universe of an absolutely saturated structure either is finite or is a proper class and not a set.

Definition 1.12. An $\in$-structure is *absolutely saturated* if it is κ-saturated for every cardinal κ.

Theorem 1.13. *Every structure* $\mathfrak{A} = \langle A; \varepsilon, \dots \rangle$ (A is a set or class) *almost universal in the sense of Remark* 1.11 *admits an elementary embedding in an absolutely saturated structure* ${}^*\mathfrak{A} = \langle {}^*A; {}^*\varepsilon, \dots \rangle$, *and the embedding is bounded in the sense of Remark* 1.11.

Such an extension ${}^*\mathfrak{A}$ *can be obtained by an 'effective', clear and unambiguous construction applied to* $\mathfrak{A}$.

Proof. Case 1: A (and then $\mathfrak{A}$ as well) is a set. Here the construction is quite simple: we define a sequence of $\in$-structures $\mathfrak{A}_\xi = \langle A_\xi; \varepsilon_\xi, \dots \rangle$, $\xi \in \mathrm{Ord}$, (all the A_ξ are sets) and elementary embeddings $e_{\eta\xi}\colon \mathfrak{A}_\eta \to \mathfrak{A}_\xi$, $\eta < \xi \in \mathrm{Ord}$, by transfinite induction so that $\mathfrak{A}_0 = \mathfrak{A}$, each structure $\mathfrak{A}_{\xi+1}$ is an elementary extension of $\mathfrak{A}_\xi$ given by Corollary 1.10 and is $(\operatorname{card}(\omega + \xi))^+$-saturated[5] over $\mathfrak{A}_\xi$, $e_{\xi,\xi+1}$ coincides with the corresponding elementary embedding of $\mathfrak{A}_\xi$ in $\mathfrak{A}_{\xi+1}$, $e_{\eta,\xi+1} = e_{\xi,\xi+1} \circ e_{\eta\xi}$ for all $\eta < \xi$, and finally, each $e_{\xi,\xi+1}$ is a bounded embedding in the sense of Corollary 1.10, and $\mathfrak{A}_\lambda$ is the direct limit of the sequence $\{\mathfrak{A}_\xi\}_{\xi<\lambda}$ for each limit ordinal λ.

Let ${}^*\mathfrak{A} = \langle {}^*A; {}^*\varepsilon, \dots \rangle$ be the direct limit $\mathfrak{A}_\infty$ of the whole sequence $\{\mathfrak{A}_\xi\}_{\xi\in\mathrm{Ord}}$ with the above-defined system of elementary embeddings as in Definition 1.1, with the associated elementary embedding $x \mapsto {}^*x\colon A_0 \to A_\infty = {}^*A$, that is, $e_{0\infty}$, in Definition 1.1. To see that the structure ${}^*\mathfrak{A}$ is κ-saturated for every cardinal κ,

[5]The term ω in $\omega + \xi$ is adjoined to make the cardinal infinite even if ξ is finite.

note that by definition each structure $\mathfrak{A}_{\xi+1}$ is κ-saturated over $\mathfrak{A}_\xi$ by means of the embedding $e_{\xi,\xi+1}$.

We assert finally that each $e_{0\xi}$ is a bounded (elementary) embedding of $\mathfrak{A}_0$ in $\mathfrak{A}_\xi$, and $\mathfrak{A}_\xi$ is an almost universal structure. Taking $\xi = \infty$ (that is, the limit step after all ordinals), we obtain the boundedness of the embedding $x \mapsto {}^*x$. The limit step (including the last step ∞) is rather elementary, but the step $\xi \to \xi + 1$ is ensured by the result mentioned in Remark 1.11 (italicized there), by which $e_{\xi,\xi+1}$ provides a bounded embedding of $\mathfrak{A}_\xi$ in $\mathfrak{A}_{\xi+1}$.

Case 2: A (and then $\mathfrak{A}$) is a proper class. The construction in Case 1 does not work directly, because neither **ZFC** (with an informal understanding of classes) nor even the class theory **GB** supports the construction of proper classes by transfinite induction. However, the problem can be solved by reduction to Case 1.

Put $A^\alpha = A \cap \mathbf{V}_\alpha$ (where $\mathbf{V}_\alpha$ is the αth level of the von Neumann hierarchy) and $\varepsilon^\alpha = \varepsilon \restriction A^\alpha$. Every structure of the form $\mathfrak{A}^\alpha = \langle A^\alpha; \varepsilon^\alpha, \dots \rangle$ has a *set* A^α as the universe, and hence one can define a sequence of its elementary extensions $\mathfrak{A}^\alpha_\xi = \langle A^\alpha_\xi; \varepsilon^\alpha_\xi, \dots \rangle$, $\xi \in \mathrm{Ord}$, as in Case 1.

Note that now the universes A^α_ξ are sets rather than proper classes, but nevertheless it will be more convenient technically to accept an amendment as in Remark 1.6 at all steps of the construction. Put $\kappa_\xi = \operatorname{card}(\omega + \xi)$ (see Case 1) for each $\xi \in \mathrm{Ord}$. Let $\mathscr{S}_\xi$ and U_ξ be the sets $\mathscr{S}$ and U (see 1.9) that correspond to this choice of $\kappa = \kappa_\xi$. In particular, U_ξ is an ultrafilter in the algebra $\mathbf{B}_\xi$ of all finitely determined sets $X \subseteq I_\xi = {\kappa_\xi}^{\mathscr{S}_\xi}$. Finally, let F^α_ξ denote the set of all finitely determined functions $f \colon I_\xi \to A^\alpha_\xi$. In these terms, the construction in 1.9 (item 4)) yields $A^\alpha_{\xi+1} = \{[f]^\alpha_{U_\xi} : f \in F^\alpha_\xi\}$, where $[f]^\alpha_{U_\xi} = \{g \in F^\alpha_\xi : f =_{U_\xi} g\}$.

Modifying this definition, we put $A^\alpha_{\xi+1} = \{[f]^{\mathrm{abs}}_{U_\xi} : f \in F^\alpha_\xi\}$, where $[f]^{\mathrm{abs}}_{U_\xi}$ is the set of all finitely determined functions g with domain $\operatorname{dom} g = I_\xi$ taking **arbitrary** values, satisfying $f =_{U_\xi} g$, and having the least rank $\operatorname{rk} g$ among all such functions g. (Note that $\operatorname{ran} g \subseteq A^\alpha_\xi$ does not necessarily hold for $g \in [f]^{\mathrm{abs}}_{U_\xi}$, and in general $[f]^{\mathrm{abs}}_{U_\xi}$ depends only on f and the ultrafilter U_ξ.) This defines the inductive step $\mathfrak{A}^\alpha_\xi \to \mathfrak{A}^\alpha_{\xi+1}$. The limit step remains the same (the direct limit). Finally, let $e^\alpha_{\xi\eta}$, $\alpha \in \mathrm{Ord}$, $\xi < \eta \in \mathrm{Ord}$, be the system of elementary embeddings $\mathfrak{A}^\alpha_\xi \to \mathfrak{A}^\alpha_\eta$ that appear in this transfinite inductive construction.

With such a modification we have for all $\xi < \eta \in \mathrm{Ord}$ and $\alpha < \beta \in \mathrm{Ord}$:

($*$) $A^\alpha_\xi \subseteq A^\beta_\xi$, $\mathfrak{A}^\alpha_\xi = \mathfrak{A}^\beta_\xi \restriction A^\alpha_\xi$ (that is, $\varepsilon^\alpha_\xi = \varepsilon^\beta_\xi \restriction A^\alpha_\xi$ and the same for all relations in the list $\dots$), and finally, $e^\alpha_{\xi\eta} = e^\beta_{\xi\eta} \restriction A^\alpha_\xi$.

(The simple proof by induction is omitted.) Now put $\mathfrak{A}_\xi = \bigcup_{\alpha \in \mathrm{Ord}} \mathfrak{A}^\alpha_\xi$, that is, $\mathfrak{A}_\xi = \langle A_\xi; \varepsilon_\xi, \dots \rangle$, where $A_\xi = \bigcup_{\alpha \in \mathrm{Ord}} A^\alpha_\xi$, $\varepsilon_\xi = \bigcup_{\alpha \in \mathrm{Ord}} \varepsilon^\alpha_\xi$, and the same for all relations in the list $\dots$. It follows easily from ($*$) that each structure of the form $\mathfrak{A}_{\xi+1}$ is an ultrapower of $\mathfrak{A}_\xi$ of type 1.9 modulo the ultrafilter U_ξ, while the direct limit is preserved at limit steps. This allows us to get around the difficulty connected with the inductive definition of classes (see the beginning of Case 2). The remainder of Case 2 is analogous to Case 1.

The 'effectivity' of the construction of ${}^*\mathfrak{A}$ in both cases is a consequence of the 'effectivity' of the main ingredient of the construction, that is, the definition of 'effective' saturation in 1.9. Theorem 1.13 is proved.

1f. Bounded set theory. Taking the universe $\langle \mathbf{V}; \in \rangle$ as $\mathfrak{A}$ in Theorem 1.13, we obtain an absolutely saturated elementary extension of the whole set universe $\mathbf{V}$. It is convenient to study such extensions by the axiomatic method. In other words, we first prove several general and most fundamental facts–axioms, and then derive other properties from the axioms, without any further reference to details of the construction.

Such an axiomatization of absolutely saturated elementary extensions of $\mathbf{V}$ has to include **ZFC** because of the elementary character of the extension. Another aspect is connected with the *standard kernel* of the extension, that is, the image of the universe $\mathbf{V}$ by means of the embedding considered. To adequately present the structure of such an extended universe, one adds an atomic (non-definable) standardness predicate st to the usual $\in$-language. In this extended st-$\in$-language, one can define various *non-standard set theories*. Among them *bounded set theory* **BST** is a most appropriate tool for describing absolutely saturated elementary extensions of the universe of **ZFC**. The theory **BST** contains the following axioms:

$\mathbf{ZFC}^{\mathrm{st}}$: all formulae of the form Φ^{st} with Φ an axiom of **ZFC** in the $\in$-language.

Boundedness: $\forall x \, \exists^{\mathrm{st}} y \; (x \in y)$.

Transfer: All formulae of the form $\Phi^{\mathrm{st}} \Leftrightarrow \Phi$, where Φ is a closed $\in$-formula with standard parameters.

Basic Idealization: $\forall^{\mathrm{st}} A_0 \, \forall^{\mathrm{st}} X \, \forall \psi$ (if ψ is a map from A_0 to $\mathscr{P}(X)$, then $[\forall^{\mathrm{stfin}} A \subseteq A_0 \, \exists x \in X \, \forall a \in A \; (x \in \psi(a)) \Leftrightarrow \exists x \in X \, \forall^{\mathrm{st}} a \in A_0 \; (x \in \psi(a))]$).

Standardization: All formulae of the form $\forall^{\mathrm{st}} X \, \exists^{\mathrm{st}} Y \, \forall^{\mathrm{st}} x \; (x \in Y \Leftrightarrow x \in X \wedge \Phi(x))$, where $\Phi(x)$ is an arbitrary st-$\in$-formula with any parameters.

Notational remarks. The quantifiers $\exists^{\mathrm{st}} x \ldots$ and $\forall^{\mathrm{st}} x \ldots$ are shorthand for $\exists x \; (\mathrm{st}\, x \wedge \ldots)$ and $\forall x \; (\mathrm{st}\, x \Rightarrow \ldots)$, respectively. By Φ^{st} we denote the relativization of an $\in$-formula Φ to the structure $\langle \mathbb{S}; \in \rangle$, where $\mathbb{S} = \{x : \mathrm{st}\, x\}$ is the class of all *standard sets* — thus all the quantifiers $\exists x$, $\forall x$ are replaced by $\exists^{\mathrm{st}} x$, $\forall^{\mathrm{st}} x$. Furthermore, $\forall^{\mathrm{stfin}} A$ means the following: for any standard finite A; the finiteness is understood in the ordinary **ZFC** sense. It is worth mentioning here that $\mathbf{ZFC}^{\mathrm{st}}$ and Transfer imply **ZFC** in the $\in$-language,[6] and hence all ordinary **ZFC**-based notions retain their meanings and properties. Also note that the **ZFC** schemata of Separation and Replacement are provable in **BST** only in the $\in$-language. They do not extend, generally speaking, to all st-$\in$-formulae.

We shall use $\mathbb{I}$ to denote the universe of all sets in **BST**. Transfer means that $\mathfrak{S} = \langle \mathbb{S}; \in \rangle$ is an elementary substructure of the universe $\langle \mathbb{I}; \in \rangle$. Boundedness says that the class $\mathbb{S}$ is $\in$-cofinal in the universe. Basic Idealization is obviously an axiom of the saturation type. Indeed, the second line can be rewritten as follows:

$$\forall^{\mathrm{stfin}} A \subseteq A_0 \Big(\bigcap_{a \in A} \psi(a) \neq \varnothing \Leftrightarrow \bigcap_{a \in A_0 \cap \mathbb{S}} \psi(a) \neq \varnothing \Big).$$

In **BST** one can prove (even without Basic Idealization) that every standard finite set contains only standard elements (see Lemma 1.14(iv)). Finally, Standardization represents a certain degree of completeness of the class $\mathbb{S}$ of all standard sets.

[6] And moreover, it was established in [16] that **BST** is finitely axiomatizable over **ZFC** in the $\in$-language.

To show how the **BST** axioms work, we prove several simple assertions, gathered in the next lemma. Some of them will be applied below, sometimes without an explicit reference.

Lemma 1.14 (**BST**).

(i) *The collection* $\mathbb{N}\cap\mathbb{S}=\{n\in\mathbb{N}:\operatorname{st} n\}$ *of all standard natural numbers is a proper initial segment of the set* $\mathbb{N}$, *and not a set.*

(ii) *The principle of induction in the* $\in$*-language holds in the domain* $\mathbb{N}$, *but in the* st-$\in$*-language holds only in the domain* $\mathbb{N}\cap\mathbb{S}$.

(iii) *Any standard-finite* (*that is, its length* n *belongs to* $\mathbb{N}\cap\mathbb{S}$) *sequence* $s=\langle x_1,\dots,x_n\rangle$ *of standard sets* x_n *is itself standard.*

(iv) *If* x *is a standard finite set, then* $x\subseteq\mathbb{S}$.

(v) *If* $x\in\mathbb{S}$ *is infinite, then* $x\not\subseteq\mathbb{S}$.

(vi) *If* $x\notin\mathbb{S}$, *then the union* $\widehat{x}=\bigcup_{y\in\mathbb{S},\,y\subseteq x}y$ *of all standard subsets of* x *is a proper subset of* x.

Proof. (i) Suppose on the contrary that $Y=\mathbb{N}\cap\mathbb{S}$ is a set. Put $\psi(n)=\mathbb{N}\setminus\{0,1,\dots,n\}$ for $n\in\mathbb{N}$. The left-hand side of the equivalence $[\dots]$ in Basic Idealization holds for $A_0=X=\mathbb{N}$ (every finite subset of $\mathbb{N}$ is bounded from above). Therefore, the right-hand side is also true, that is, there exists an $x\in\mathbb{N}$ such that $n<x$ for all standard n. Thus, the set Y is bounded in $\mathbb{N}$. It follows that there is a least natural number which does not belong to Y, that is, a least non-standard natural number y. Then $x=y-1$ is a standard number. Therefore, $y=x+1$ is also standard by Transfer, a contradiction.

(ii) The st-$\in$-definable collection $\{n\in\mathbb{N}:\neg\operatorname{st} n\}$ does not contain a least element by the above, hence the induction principle fails in $\mathbb{N}$ in the st-$\in$-language. As for the domain $\mathbb{N}\cap\mathbb{S}$, suppose that $Z=\{x\in\mathbb{N}:\varphi(x)\wedge\operatorname{st} x\}\neq\varnothing$. ($Z$ is not necessarily a set.) Then by Standardization there exists a standard set $X\subseteq\mathbb{N}$ such that $X\cap\mathbb{N}=Z\cap\mathbb{N}$. The least element of X will be the least in Z as well.

(iii) If $x,y\in\mathbb{S}$, then $\langle x,y\rangle\in\mathbb{S}$ by Transfer. But we have $\langle x_1,\dots,x_n,x_{n+1}\rangle=\langle\langle x_1,\dots,x_n\rangle,x_{n+1}\rangle$ by definition. This enables us to prove the assertion by induction on $n\in\mathbb{N}\cap\mathbb{S}$.

(iv) By Transfer, the number $\#x$ of elements of x belongs to the set $\mathbb{N}\cap\mathbb{S}$. Therefore, we can argue by induction on $n=\#x$. For $n=0$ the result is obvious. Suppose that $\#x=n+1$. By Transfer there is an element $a\in x\cap\mathbb{S}$, and hence the set $y=x\setminus\{a\}$ is standard, too. But $\#y=n$, therefore $y\subseteq\mathbb{S}$ by the inductive hypothesis.

(v) It suffices to apply Basic Idealization with $A_0=x$ and $\psi(x):=x$.

(vi) By Boundedness, there exists a standard set s such that $x\subseteq s$. Standardization yields a standard set $q\subseteq\mathscr{P}(s)$ such that $y\in q\Leftrightarrow y\subseteq x$ for every standard y. We conclude that $\widehat{x}=\bigcup(q\cap\mathbb{S})$. Put $\psi(a)=x\setminus a$ for $a\in q$. If $A\subseteq q$ is standard and finite, then $\bigcup A\in\mathbb{S}$ by Transfer, therefore $\bigcup A\in q$, and hence $\bigcup A\subsetneqq x$, because x is non-standard, and finally $\bigcap_{a\in A}\psi(a)\subsetneqq x$. It remains to apply Basic Idealization for ψ and $A_0=q$. Lemma 1.14 is proved.

1g. Absolutely saturated extension of the standard universe. It turns out that **BST** quite adequately describes absolutely saturated extensions of the **ZFC** universe! The universe $\mathfrak{V}=\langle\mathbf{V};\in\rangle$ of all sets is an almost universal structure in **ZFC**, and hence, by Theorem 1.13, there is a bounded elementary embedding $a \mapsto {}^*a$ of the universe $\mathfrak{V}$ in an absolutely saturated structure, say $\langle {}^*\mathbf{V}; {}^*\!\in \rangle$. The latter satisfies **ZFC** by the elementary character. In addition, we define the *relation of standardness* on ${}^*\mathbf{V}$, that is, $\operatorname{st} x$ if $\exists\, a\ (x = {}^*a)$. This results in the structure ${}^*\mathfrak{V} = \langle {}^*\mathbf{V}; {}^*\!\in, \operatorname{st}\rangle$.

Definition 1.15. A *non-standard structure*, or st-$\in$-*structure* is any structure of the form $\mathfrak{X} = \langle X; \in_X, \operatorname{st}_X \rangle$, where $\in_X$ is a binary relation and st_X a unary relation on a set or class X. An injection $\pi \colon Y \to X$ is an $\mathbb{S}$-*type embedding* of an $\in$-structure $\mathfrak{Y} = \langle Y; \in_Y \rangle$ in an st-$\in$-structure $\mathfrak{X} = \langle X; \in_X, \operatorname{st}_X \rangle$ if the image $\operatorname{ran} \pi = \{\pi(y) : y \in Y\}$ coincides with the *standard part* $(\mathbb{S})^{\mathfrak{X}} = \{x \in X : \operatorname{st}_X x\}$ of the structure $\mathfrak{X}$, and we have $y \in_Y y' \Leftrightarrow \pi(y) \in_X \pi(y')$ for all $y, y' \in Y$.

In this case the st-$\in$-structure $\mathfrak{X}$ will be called an $\mathbb{S}$-*type extension* of the structure $\mathfrak{Y}$ (by means of the embedding π).

Theorem 1.16.

(i) *Provably in* **ZFC**, *the structure* ${}^*\mathfrak{V} = \langle {}^*\mathbf{V}; {}^*\!\in, \operatorname{st}\rangle$ *satisfies* **BST** *and is an elementary* $\mathbb{S}$-*type extension of the structure* $\mathfrak{V} = \langle \mathbf{V}; \in \rangle$ (in the sense of Definition 1.15) *by means of the map* $a \mapsto {}^*a$.

(ii) **BST** *is an* st-*conservative extension of* **ZFC**, *in the sense that an* $\in$-*formula* φ *is provable in* **ZFC** *if and only if* φ^{st} *is provable in* **BST**.[7] *Therefore*, **BST** *and* **ZFC** *are equiconsistent.*

Proof. (i) The precise content of this assertion is as follows. The proof of Theorem 1.13 (with all the supporting material) can be represented as a definition of three $\in$-formulae which define the domain ${}^*\mathbf{V}$, the relation ${}^*\!\in$, and the elementary embedding $a \mapsto {}^*a$ of the universe $\mathbf{V}$ in ${}^*\mathbf{V}$, and hence also the standardness predicate st, together with a proof, in **ZFC**, of all the required properties, for instance, of the fact that the embedding is indeed elementary. The latter means that the equivalence $\varphi(x_1, \dots, x_n) \Leftrightarrow \varphi^{\langle {}^*\mathbf{V}; {}^*\in \rangle}({}^*x_1, \dots, {}^*x_n)$ holds (that is, it is established in **ZFC**, essentially by Corollary 1.8) for any $\in$-formula φ. The right-hand side of the equivalence is also an $\in$-formula, with variables $x_1, \dots, x_n$, obtained in an obvious way from φ and the $\in$-formulae which define ${}^*\mathbf{V}$, ${}^*\!\in$, and $x \mapsto {}^*x$. Moreover, the content of Theorem 1.16 is the proof of $\varphi^{{}^*\mathfrak{V}}$ (this is a certain $\in$-formula, too) in **ZFC** for any axiom φ of **BST**.

We are not going to argue in such a pedantic manner, of course. The schemata of $\mathbf{ZFC}^{\mathrm{st}}$ and Transfer in ${}^*\mathfrak{V}$ follow immediately from the fact that the universe of all sets is elementarily embeddable in $\langle {}^*\mathbf{V}; {}^*\!\in \rangle$, so that the range of this embedding $a \mapsto {}^*a$ is equal to the standard kernel of the universe ${}^*\mathfrak{V}$ by definition.

The axiom of Boundedness follows from the boundedness of this embedding.

Let us verify Standardization. Suppose that $X \in {}^*\mathbf{V}$ is standard, that is, $X = {}^*a$ for a set a in the universe $\mathbf{V}$. Define $b = \{c \in a : \Phi({}^*c)^{{}^*\mathfrak{V}}\}$. Then $Y = {}^*b$ satisfies $\forall^{\mathrm{st}} x\ (x \in Y \Leftrightarrow x \in X \wedge \Phi(x))$ in ${}^*\mathfrak{V}$.

Finally, Basic Idealization can be reduced to the following problem. Suppose that A_0 and X are sets in the universe $\mathbf{V}$ and an element $\psi(a) \in {}^*\mathbf{V}$ is associated

[7]This implies by Transfer that any $\in$-formula φ is simultaneously provable in **BST** and in **ZFC** — this is another form of conservativity. The form given in (ii) is of more general character since not all non-standard theories contain an appropriate form of Transfer.

with each $a \in A_0$ so that the ${}^*{\in}$-intersection $\bigcap_{a\in A}\psi(a)$ is non-empty for any finite $A \subseteq A_0$. And we have to prove that the ${}^*{\in}$-intersection $\bigcap_{a\in A_0}\psi(a)$ is non-empty in ${}^*\mathfrak{V}$. But this follows from the saturation properties of the extension. (In fact, the reduction of Basic Idealization to this form requires some additional arguments, in particular, arguments related to the nature of *standard finiteness* in ${}^*\mathfrak{V}$; we leave this for the reader, with a reference to [25] (Ch. 3) or the papers [21], [22].)

(ii) Suppose that φ^{st} is provable in **BST**. (The opposite direction is obvious.) Arguing in **ZFC**, consider the structure ${}^*\mathfrak{V}$. It satisfies **BST** by Theorem 1.16, and therefore φ^{st} is true in ${}^*\mathfrak{V}$, that is, φ holds in the standard part of the structure ${}^*\mathfrak{V}$. However, the standard part is $\in$-isomorphic to the universe $\mathbf{V}$ by means of the map $a \mapsto {}^*a$, and hence φ is true in $\mathbf{V}$. This is exactly a proof of φ in **ZFC**. Theorem 1.16 is proved.

Applying Theorem 1.16(i) in an arbitrary model of **ZFC**, we deduce the following result.

Corollary 1.17. *Every model of* **ZFC** *is embeddable, as the class of all standard sets, in a model of* **BST**.

Remark 1.18. Theorem 1.16(i) means that the structure ${}^*\mathfrak{V}$ is an *interpretation* of **BST** in **ZFC**, associated (provably in **ZFC**) with an $\in$-isomorphism between the standard domain of the interpretation and the ground 'standard' universe (of **ZFC** in this case).

Interpretations of this kind will be called $\mathbb{S}$-*type interpretations*.

In less formal terms, such an interpretation of **BST** in **ZFC** can be understood as an extension of the universe of **ZFC** to a universe of **BST**, where the former remains the class of all standard sets. And this is done on the basis of the axioms of **ZFC**.

1h. Bounded set theory: reduction of formulae to $\boldsymbol{\Sigma_2^{\mathrm{st}}}$ form. It is an important feature of **BST** that any st-$\in$-formula can be reduced to Σ_2^{st} form, that is, to the form $\exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b$ ($\in$-formula). In other words, however complex a given st-$\in$-formula is, it can be converted to a form equivalent in **BST** and containing exactly two occurrences of the predicate st: those via the quantifiers $\exists^{\mathrm{st}} a$, $\forall^{\mathrm{st}} b$.

Theorem 1.19 (reduction to Σ_2^{st} form). *Let* $\Phi(x_1,\dots,x_m)$ *be a parameter-free* st-$\in$-*formula. Then there is a parameter-free* $\in$-*formula* $\varphi(x_1,\dots,x_m,a,b)$ *such that the following equivalence is provable in* **BST**:

$$\forall x_1 \dots \forall x_m \, \bigl(\Phi(x_1,\dots,x_m) \Leftrightarrow \exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b\, \varphi(x_1,\dots,x_m,a,b)\bigr).$$

Proof. Elementary simplifications reduce the theorem to the case when Φ has a prenex form $\sqcap\Psi$, where $\sqcap$ is a quantifier prefix containing quantifiers of the form $\exists^{\mathrm{st}} y$, $\forall^{\mathrm{st}} y$ and $\exists y$, $\forall y$, while Ψ is an arbitrary $\in$-formula. Under this assumption the theorem is proved in two steps.

Step 1: This is the theorem in the case when the prefix $\sqcap$ contains only quantifiers of the form $\exists^{\mathrm{st}} y$, $\forall^{\mathrm{st}} y$. For instance, let $\Phi(x)$ be the formula $\forall^{\mathrm{st}} y\, \exists^{\mathrm{st}} z\, \Psi(y,z,x)$. For any u we define $\operatorname{mon} u = \cap(u \cap \mathbb{S})$ (the intersection of all standard elements of u or the *monad of* u). Let $\operatorname{Ult} u$ be the $\in$-formula saying that u is an ultrafilter.

Let $\vartheta(u)$ be the $\in$-formula $\forall\, y\, \exists\, z\, \exists\, p \in u\, \forall\, \xi \in p\, \Psi(y, z, \xi)$. The equivalence

$$\Phi(x) \;\Leftrightarrow\; \exists^{\mathrm{st}}\, u\; (\mathrm{Ult}\, u \wedge x \in \mathrm{mon}\, u \wedge \vartheta(u)) \tag{1}$$

implies the required result, since the right-hand side is easily reducible to a Σ_2^{st} form. To prove (1), consider an arbitrary x. Using Boundedness, let d be a standard set containing x. According to Standardization, there exists a standard set $u \subseteq \mathscr{P}(d)$ satisfying $p \in u \;\Leftrightarrow\; x \in p$ for all standard $p \in u$. By Transfer, u is a standard ultrafilter on d with $x \in \mathrm{mon}\, u$. It remains to verify that

$$\Phi(x) \Leftrightarrow \forall\, y\, \exists\, z\, \exists\, p \in u\, \forall\, \xi \in p\; \Psi(y, z, \xi) \tag{2}$$

for every standard ultrafilter u with $x \in \mathrm{mon}\, u$. Let u be such an ultrafilter. The right-hand side of (2) is equivalent to $\forall^{\mathrm{st}} y\, \exists^{\mathrm{st}}\, z\, \exists\, p \in u\, \forall\, \xi \in p\; \Psi(y, z, \xi)$ by Transfer. It remains to show that for any standard y, z,

$$\Psi(y, z, x) \Leftrightarrow \exists^{\mathrm{st}}\, p \in u\, \forall\, \xi \in p\, \Psi(y, z, \xi). \tag{3}$$

Note that the sets $d = \cup u$ and $p = \{\xi \in d : \Psi(y, z, \xi)\}$ are standard by Transfer. If in this case $p \in u$, then $x \in p$ and both sides of (3) hold, while otherwise both fail. This proves the equivalence (3).

Step 2: The inductive step $\exists\, y$ of the theorem for the prenex formulae of the above-mentioned form. Let us convert a formula $\Phi(x) := \exists\, w\, \exists^{\mathrm{st}}\, a\, \forall^{\mathrm{st}} b\, \varphi(x, w, a, b)$, to a Σ_2^{st} form, where φ is an $\in$-formula. The reduction is based on the following double equivalence:

$$\begin{aligned} \Phi(x) &\Leftrightarrow \exists^{\mathrm{st}}\, a\, \exists^{\mathrm{st}}\, W\, \exists\, w \in W\, \forall^{\mathrm{st}} b\, \varphi(x, w, a, b) \Leftrightarrow \\ &\Leftrightarrow \exists^{\mathrm{st}}\, a\, \exists^{\mathrm{st}}\, W\, \forall^{\mathrm{stfin}} B\, \exists\, w \in W\, \forall\, b \in B\; \varphi(x, w, a, b). \end{aligned}$$

The first equivalence here follows from Boundedness. To verify the second equivalence, we have to prove

$$\forall^{\mathrm{stfin}} B\, \exists\, w \in W\, \forall\, b \in B\; \varphi(x, w, a, b) \Leftrightarrow \exists\, w \in W\, \forall^{\mathrm{st}} b\; \varphi(x, w, a, b) \tag{4}$$

for any standard sets a, W and any x. Note that Basic Idealization is not immediately applicable; first we have to bound the quantifier on b. Assuming that x and standard sets a, W, d $(x \in d)$ are fixed, we define

$$z_b = \{\langle y, w\rangle \in d \times W : \varphi(y, w, a, b)\} \quad \text{for every } b.$$

There exists a set B_0 such that $\forall\, b'\, \exists\, b \in B_0\; (z_b = z_{b'})$. By Transfer, one can choose a standard set B_0 of this kind, and then $\forall^{\mathrm{st}} b'\, \exists^{\mathrm{st}}\, b \in B_0\; (z_b = z_{b'})$. But in this case the right-hand side of (4) is equivalent to $\exists\, w \in W\, \forall^{\mathrm{st}} b \in B_0\; \varphi(x, w, a, b)$. This enables us to reduce (4) to Basic Idealization by elementary arguments. Theorem 1.19 is proved.

Theorem 1.19 is the cornerstone of studies of the set universe of **BST**. To illustrate how it works we present the following result.

Theorem 1.20 (**BST**). (i) *Collection holds, that is, for any* st-$\in$-*formula* $\Phi(x,y)$ (*possibly with parameters*),

$$\forall X \, \exists Y \, \forall x \in X \, \big(\exists y \; \Phi(x,y) \Rightarrow \exists y \in Y \; \Phi(x,y)\big).$$

(ii) *For any* st-$\in$-*formula* $\Phi(x,y)$ (*possibly with parameters*),

$$\forall X \, \exists F \, \big(F \textit{ is a function} \wedge X = \operatorname{dom} F \wedge \forall^{\mathrm{st}} x \in X \; [\exists y \; \Phi(x,y) \Rightarrow \Phi(x, F(x))]\big).$$

Proof. To save some space we present a proof of (i) but refer to [21], [22] and the book [25] (Ch. 3) regarding (ii). It follows from Boundedness that the following is sufficient:

$$\forall^{\mathrm{st}} X \, \exists^{\mathrm{st}} Y \, \forall x \in X \, \big(\exists^{\mathrm{st}} y \; \Phi(x,y) \Rightarrow \exists^{\mathrm{st}} y \in Y \; \Phi(x,y)\big). \tag{5}$$

Furthermore, we can assume that Φ is a Σ_2^{st} formula by Theorem 1.19. Under this assumption the leftmost quantifier in Φ, say $\exists^{\mathrm{st}} a$, can be absorbed by the quantifier $\exists^{\mathrm{st}} y$. Thus, without loss of generality we can assume that Φ is even a Π_1^{st} formula, that is, $\forall^{\mathrm{st}} b \, \varphi(x,y,b)$, where ψ is an $\in$-formula.

Now comes the final simplification. If ψ has parameters, say a set p, then they belong to an appropriate standard set P, and, changing X to $X \times P$, we easily reduce the task to the parameter-free case. Thus, we have to prove

$$\forall^{\mathrm{st}} X \, \exists^{\mathrm{st}} Y \, \forall x \in X \, \big(\exists^{\mathrm{st}} y \, \forall^{\mathrm{st}} b \; \varphi(x,y,b) \Rightarrow \exists^{\mathrm{st}} y \in Y \, \forall^{\mathrm{st}} b \; \varphi(x,y,b)\big), \tag{6}$$

where φ is a parameter-free $\in$-formula.

We put $h_{yb} = \{x \in X : \varphi(x,y,b)\}$ for all y, b, and $h_y = \{h_{yb} : b \;\text{ is any set}\}$ for any y. Then $h_y \subseteq \mathscr{P}(X)$ for all y. By the $\mathbf{ZFC}^{\mathrm{st}}$ Collection (in the $\in$-language) and Transfer,[8] there exists a standard set Y such that $\forall y' \, \exists y \in Y \, (h_y = h_{y'})$. To prove that Y satisfies (6), take any $x \in X$ and any standard y such that $\forall^{\mathrm{st}} b \, \varphi(x,y,b)$. By the choice of Y and Transfer, there exists a standard set $y' \in Y$ such that $h_y = h_{y'}$. It remains to prove $\varphi(x,y',b')$ for all standard b'. But since $h_y = h_{y'}$, again by Transfer there exists a standard b satisfying $h_{yb} = h_{y'b'}$. However, $\varphi(x,y,b)$ holds by the choice of y, in other words, $x \in h_{yb}$. Thus, $x \in h_{y'b'}$, that is, $\varphi(x,y',b')$, as required. Theorem 1.20 is proved.

We end with an important corollary of metamathematical character.

Corollary 1.21. *For any parameter-free* st-$\in$-*formula* Φ *there is an* $\in$-*formula* Ψ *such that* $\Phi \Leftrightarrow \Psi \Leftrightarrow \Psi^{\mathrm{st}}$ *is provable in* **BST**.

Proof. By Theorem 1.19, the formula Φ is equivalent in **BST** to a Σ_2^{st} formula $\Theta := \exists^{\mathrm{st}} a \, \forall^{\mathrm{st}} b \; \vartheta(a,b)$, where ϑ is an $\in$-formula. The $\in$-formula $\Psi := \exists a \, \forall b \, \vartheta(a,b)$ satisfies $\Theta \Leftrightarrow \Psi$ in **BST** by Transfer.

[8] Note that the references to Transfer here and below in the proof are possible only since φ is assumed to be parameter-free.

Historical and bibliographical remarks to § 1. What is in subsections 1a–1c belongs to the classics of logic and model theory; it is comprehensively presented in the bibliography. See, for instance, [48] regarding the basic notions of logic and [39] regarding model theory. In particular, see 4.7 in [48] on the notion of interpretation (subsection 1a) and 9.10 in [48] on invariant interpretations.

On saturated structures (subsection 1b) see [39], and, for instance, [41], [44]–[46] in the context of non-standard analysis. Absolutely saturated (in the sense of subsection 1e) extensions of the whole universe of **ZFC** were first defined in [10], [11] and [8] by somewhat different methods and under different assumptions extending the framework of **ZFC** in this or that way. A construction in the framework of **ZF** with the global choice axiom (that is, in accordance with the sketch at the beginning of subsection 1d) was given in [21], [22]. The consistency of the global choice axiom with **ZF** was established in [49].

The reduction of classes to sets as in Remark 1.6 is attributed to Scott in [50]; see also 9.10 in [47], the remark after formula (13).

The problem of 'effective' construction of saturated extensions (subsection 1f) was once communicated by V. A. Uspenskii to one of the authors (V. G. Kanovei); it is mentioned as an unsolved problem in [21], [22]. In the case of extensions of the real line the solution was obtained in [24], and in Ch. 4 of [25] in the case of extensions of the whole universe that are saturated in all cardinalities. This construction is presented in subsection 1.9 above.

Bounded set theory **BST** (subsection 1f) was introduced in [14], but it is implicitly contained in the earlier papers [10], [11]. In general, the axiomatic foundations of non-standard analysis go back to studies in the 1970s, when three different axiomatic schemes were proposed: internal set theory **IST** [8], alternative set theory **AST** [51], [52], and some axiomatic theories in [10], [11]. See more on this in the preface of [25].

Theorem 1.16(i) was established in [25], Ch. 4. Part (ii) of this theorem and Corollary 1.17, deduced here as elementary consequences of Theorem 1.16(i), were originally proved by different methods in [14] essentially on the basis of similar results in [8] for **IST**. The central **BST** theorem on reduction of all st-$\in$-formulae to Σ_2^{st} form (Theorem 1.19) was proved in [14]. Theorem 1.20 along with some other **BST** theorems was also established in [14]. See more on this in [21], [22], and also in Ch. 3 of [25] in the most substantial form.

We say a few words about the relation of **BST** to the *internal set theory* **IST** of Nelson [8]. The latter is quite popular for historical reasons. The two theories differ in that the axiom Basic Idealization in **BST** is strengthened in **IST** to the following schema of full idealization (or *internalization* as in [41], [44]):

Idealization: $\forall^{\mathrm{stfin}} A\, \exists\, x\, \forall\, a \in A\ \Phi(a,x) \Leftrightarrow \exists\, x\, \forall^{\mathrm{st}} a\ \Phi(a,x)$, where $\Phi(a,x)$ is any $\in$-formula with arbitrary parameters.

Accordingly, the **BST** axiom of Boundedness is not included in **IST**, because Idealization implies the existence of a set containing all standard sets, in contradiction to Boundedness.[9] Thus, **BST** and **IST** are incompatible. On the other hand, it is easy to see that the class $\{x : \exists^{\mathrm{st}}\, y\ (x \in y)\}$ of all *bounded sets* is an interpretation of **BST** in **IST**.

[9] But bounding one or both of the variables x, a in the formulation of Idealization by standard sets, we obtain schemata still provable in **BST** (see [25], 3.2).

Even smaller classes $\mathbb{I}_\kappa = \{x : \exists^{\rm st}\, y\ (\operatorname{card} y \leqslant \kappa \wedge x \in y)\}$ can be defined inside the set universes of the theories **IST** and **BST**, where κ is any fixed standard cardinal. In such a *partially saturated* subuniverse $\mathbb{I}_\kappa$ a modified version of the **BST** theory holds, in which Boundedness is strengthened by the requirement $\operatorname{card} y \leqslant \kappa$ while Basic Idealization is accordingly weakened.

Theorem 1.19 fails in **IST**: it is proved in [14] that there is an st-$\in$-formula not equivalent in **IST** to a $\Sigma_2^{\rm st}$ formula. On the other hand, it is known from [8], [9] that all formulae with quantifiers bounded by standard sets (for instance, $\exists\, x \in X$, $\forall^{\rm st} y \in Y$, where X and Y are standard) are reducible to $\Sigma_2^{\rm st}$ form in **IST**. See more detail on this in [25], Ch. 3, where a study of partially saturated subuniverses of the form $\mathbb{I}_\kappa$ is also presented.

The differences between **BST** and **IST** relating to their axiomatics and metamathematical properties do not affect the practical possibilities of these theories as bases for 'non-standard' reasoning in various fields of mathematics, but they do lead to serious dissimilarities in deeper aspects of the foundations. For instance, Theorem 1.16(ii) still holds for **IST**, but Corollary 1.17 fails for **IST** (minimal transitive models of **ZFC** present a counterexample; see [21]). This is the reason why **IST** does not admit such an interpretation in **ZFC** as **BST** does by Theorem 1.16(i). One can easily understand why constructions similar to those given above do not lead to an interpretation of **IST** in **ZFC**. Indeed, in view of the unbounded character of Idealization, we would have to consider arbitrary subclasses (not only subsets) of the standard universe, but this is impossible in **ZFC**. On the other hand, by adding the truth predicate for all $\in$-formulae to **ZFC** along with suitable axioms, we obtain a theory in which **IST** can be interpreted. See 4.6 in [25] about such a modification.

§ 2. Adding classes to a non-standard universe

This section is devoted to non-standard class theories. We begin with a survey of the standard and non-standard Gödel–Bernays class theories **GB** and **nGB**, and Kelley–Morse class theories **KM** and **nKM**. The theories **nGB** and **nKM** are defined as the predicative and impredicative class theories over the non-standard set universe of **BST**, respectively. Metamathematical connections among the standard and non-standard class theories are shown in the diagrams on page 81.

The main result, Theorem 2.16, yields a 'realistic' interpretation of the theories **nGB** and **nKM** in **GB** and **KM**, respectively.

The method used in the proof of this theorem can be viewed as adding classes to the non-standard set universe of **BST**, thereby giving us a solution of Problem 2 in the Introduction. This looks quite similar to the addition of non-standard sets to the standard **ZFC** universe with the goal of obtaining a non-standard set universe of **BST** in § 1. But the metamathematical aspect is quite different. We recall that, by Theorem 1.16, **ZFC** is strong enough to define the required **BST**-extension. But now **BST** itself turns out not to be strong enough to support the addition of classes even in the case of **nGB**. The reason for this is quite transparent already for the pair of standard theories **ZFC** and **GB**: the latter is a conservative and equiconsistent extension of the former, but **GB** does not admit an interpretation in **ZFC**, by Theorem 2.2. Thus, the **BST** set structure needs some help here. In fact, it is sufficient to assume that the standard kernel of the **BST** universe is already extended by classes to a universe of **GB** or **KM**: this would lead to the construction of a non-standard class extension of the whole **BST** universe (see Remark 2.20).

However, the actual structure of the exposition will be the following. We first utilize the results of § 1 in order to extend the domain of sets of the **GB** (or **KM**) universe to a non-standard set universe satisfying **BST**, and then, still arguing in **GB** (or **KM**), we adjoin classes to the latter.

2a. Standard class theories. The *predicative Gödel–Bernays* class theory **GB** describes a kind of 'minimal' extension of the **ZFC** set universe by *classes*, that is, collections of sets that are themselves not necessarily sets. And sets are characterized as those classes which are elements of other classes. Classes which are not sets are called *proper classes*. Thus, 'X is a set' is expressed by the formula $\operatorname{set} X := \exists\, Y\ (X \in Y)$.

Accordingly, for any $\in$-formula φ, φ^{sets} denotes its formal relativization to sets, that is, all quantifiers $\exists\, Y \dots$ and $\forall\, Y \dots$ in φ are changed to $\exists\, Y\ (\operatorname{set} Y \wedge \dots)$ and $\forall\, Y\ (\operatorname{set} Y \Rightarrow \dots)$, respectively. formulae of the form φ^{sets} are *predicative*, while *impredicative* means not necessarily predicative. There is not any restriction on the occurrences of free variables (parameters). Lower-case letters are normally used to denote sets, while upper-case letters are used to denote arbitrary classes (including both sets and proper classes). For instance, a formula $\exists\, x\, \varphi(x)$ is understood as a shortened form of $\exists\, X\, (\operatorname{set} X \wedge \varphi(X))$.

Definition 2.1. The Gödel–Bernays class theory **GB** includes:

$\mathbf{ZFC}^{\text{sets}}$: All formulae of the form φ^{sets}, where φ is a **ZFC** axiom (in the $\in$-language).

Extensionality: $\forall\, X\, \forall\, Y\, (\forall\, x\, (x \in X \Leftrightarrow x \in Y) \Leftrightarrow X = Y)$.

Comprehension: $\exists\, X\, \forall\, x\, (x \in X \Leftrightarrow \Phi(x))$, where Φ is any predicative $\in$-formula.[10]

Intersection: $\forall\, X\, \forall\, y\ (\operatorname{set}(X \cap y))$: the intersection of a class and a set is a set.

Collection: $\forall\, P\, \forall\, d\, \exists\, r\, \forall\, x \in d\ \bigl(\exists\, y\ (\langle x, y\rangle \in P) \Rightarrow \exists\, y \in r\ (\langle x, y\rangle \in P)\bigr)$.

Let $C[x] = \{y : \langle x, y\rangle \in C\}$ (the *cross-section*) for any class C and set x.

The *Kelley–Morse class theory* **KM** is a version of **GB** in which the schemata of Comprehension and Collection are strengthened to:

Impredicative Comprehension: $\exists\, X\, \forall\, x\ (x \in X \Leftrightarrow \Phi(x))$, where $\Phi(x)$ is any (possibly impredicative) $\in$-formula.[10]

Class Collection: $\forall\, d\, \exists\, C\, \exists\, r\, \forall\, a \in d\ (\exists\, Y\ \Phi(a, Y) \Rightarrow \exists\, x \in r\ \Phi(a, C[x]))$, where $\Phi(x)$ is any (possibly impredicative) $\in$-formula.[10]

Thus, $\mathbf{ZFC}^{\text{sets}}$ postulates that the class of all standard sets satisfies **ZFC**. The Comprehension schemata assert that any collection of sets of the form $\{x : \Phi(x)\}$ (Φ is predicative in the **GB** case) is a class. Comprehension, Intersection, and Collection suffice to infer the **ZFC** schemata of Separation and Replacement. Note that the (predicative) Comprehension itself is a corollary of a finite number of its instances (see, for example, [35], Ch. II, § 6). It follows that, unlike **ZFC**, the class theory

[10]Here and below $\Phi(x)$ is allowed to contain sets and classes as *parameters*. This assumption holds below unless explicitly stated otherwise.

GB is finitely axiomatizable. (In fact, **GB** is usually formulated with a finite list of axioms.)

Theorem 2.2. 1) **GB** *is a* sets-*conservative extension of* **ZFC**, *in the sense that any* $\in$-*formula* φ *is provable in* **ZFC** *if and only if* φ^{sets} *is provable in* **GB**.[11] *Therefore, the theories* **ZFC** *and* **GB** *are equiconsistent.*

2) *However,* **GB** *has no interpretations in* **ZFC**.

3) *The theory* **KM** *is not even equiconsistent with* **ZFC**.

Proof. 1) To prove the sets-conservativity, suppose (the non-trivial direction) that φ is an $\in$-formula not provable in **ZFC**. Then there is a model $\langle M;\varepsilon\rangle$ of **ZFC** in which φ fails. Add to M all sets $X\subseteq M$ that are $\in$-definable in $\langle M;\varepsilon\rangle$ (with parameters in M allowed), except for those of the form $\widehat{x}=\{y\in M: y\,\varepsilon\, x\}$, $x\in M$; the latter are removed to preserve Extensionality. Let K be the extended domain, and define $x\,\varepsilon\, X$ whenever $x\in X\in K\setminus M$. Then $\langle K;\varepsilon\rangle$ is a model of **GB** with set universe equal to M, and thus φ^{sets} fails in $\langle K;\varepsilon\rangle$. Therefore, φ^{sets} is not provable in **GB**.[12]

2) If **GB** has an interpretation in **ZFC**, then the required properties of the interpretation are provable in a finite subtheory $\mathbf{T}\subseteq\mathbf{ZFC}$ because of finite axiomatizability of **GB**. It follows that **ZFC** is interpretable in **T**. But it is known that **ZFC** implies the consistency $\operatorname{Consis}\mathbf{T}$ of any finite subtheory **T** of it. This easily leads to a contradiction with the Gödel incompleteness theorems.

3) It is known that **KM** proves the existence of a model of **ZFC**, and even the existence of a transitive set X such that $\langle X;\in\rangle$ is a model of **ZFC**. (Indeed, Impredicative Comprehension allows us to define the *truth predicate* for all $\in$-formulae relativized to sets; see [35], §§ 1.9 and 2.7.) Thus, **KM** proves the formal consistency of **ZFC**. Therefore, **KM** is not equiconsistent with **ZFC** by the Gödel incompleteness theorems. Theorem 2.2 is proved.

According to the sets-conservativity, **GB** proves only those facts related to sets which **ZFC** proves for its set universe. This provides the grounds for a rather liberal informal treatment of classes in **ZFC** (see subsection 1a).

2b. Non-standard class theories. The ideology of non-standard class theories is based on the following two general principles: 1) the set universe satisfies a convenient non-standard set theory, for instance, **BST**; 2) Comprehension is present for all st-$\in$-formulae, not just for $\in$-formulae. Since Separation is included in **BST** only for $\in$-formulae, it follows that, unlike the case of standard theories, subclasses of sets (semisets; see below) will themselves not necessarily be sets.

Working in the st-$\in$-language, we will follow the conventions of subsection 2a concerning small and capital letters, and predicative and impredicative formulae.

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[11] A more complicated theorem of [49] asserts that **GBGC**, that is, **GB** strengthened by the Global Choice axiom $\exists\,F\,\forall\,y\neq\varnothing\ (F \text{ is a function } \wedge y\in\operatorname{dom}F\wedge f(y)\in y)$ saying that there is a global choice function, is still a sets-conservative extension of **ZFC**.

[12] Thus, every model $M\models\mathbf{ZFC}$ is embeddable, as the class of all sets, in a suitable model $K\models\mathbf{GB}$, so that all adjoined classes are definable in M, but the whole **GB**-extension is not definable in M by the assertion 2) in Theorem 2.2.

Definition 2.3. The *non-standard Gödel–Bernays class theory* **nGB** is a theory in the st-$\in$-language that includes Extensionality and Collection, as well as Comprehension as in **GB** but for all (predicative) st-$\in$-formulae Φ, together with the following axioms:

BST$^{\text{sets}}$: all **BST** axioms relativized to the set universe;

Standardization for classes: $\forall X\,\exists^{\text{st}} Y\,\forall^{\text{st}} z\ (z\in Y \Leftrightarrow z\in X)$;

Axiom $\mathbf{S}=\mathbf{S}^{\text{loc}}$: $\forall X\ (\text{st}\,X \Leftrightarrow \forall^{\text{st}} y\,\exists^{\text{st}} z\ (X\cap y = z))$.[13]

The *non-standard Kelley–Morse class theory* **nKM** is defined similarly, but

1) Comprehension and Collection are strengthened to Impredicative Comprehension and Class Collection as in **KM** but for all st-$\in$-formulae Φ, and in addition,
2) the axioms (Impredicative Comprehension)$^{\text{st}}$ and (Class Collection)$^{\text{st}}$ are added, that is, the axioms of **KM** (for $\in$-formulae Φ) relativized to the domain $\mathbf{S}=\{X : \text{st}\,X\}$ of all standard classes (and sets).

Thus, these theories describe their universes as class superstructures (as in the case of **GB** or **KM**) over the set universe of **BST**.

The next definition introduces a convenient notational system for dealing with non-standard theories of classes (and sets).

Definition 2.4 (for non-standard class theories). A set x is *internal* if it belongs to a standard set: formally, $\exists^{\text{st}} y\ (x\in y)$. A class X is *internal* if it has the form $C[x]=\{y : \langle x,y\rangle \in C\}$, where C is a standard class and x is an internal set. By *classes* we understand here all objects of the universe, while sets are classes that belong to other classes, in accordance with subsection 2a. We define

$$\begin{aligned}
\mathbb{H} &= \{x : \text{set}\,x\} && \text{(all sets)},\\
\mathbb{S} &= \{x\in\mathbb{H} : \text{st}\,x\} && \text{(standard sets)},\\
\mathbf{S} &= \{X : \text{st}\,X\} && \text{(standard classes)},\\
\mathbb{I} &= \{x : \exists^{\text{st}} y\ (x\in y)\} && \text{(all internal sets)},\\
\mathbf{E} &= \{X : X\subseteq \mathbb{I}\} && \text{(classes of internal sets)},\\
\mathbf{I} &= \{\text{all internal classes}\},\\
\mathbf{H} &= \{\text{all classes}\}.
\end{aligned}$$

Internal classes constitute an important intermediate collection of classes in **nGB** that includes, for instance, all predicatively $\in$-definable classes (see Lemma 2.5(vi)). All standard classes are internal, and on the other hand, every internal set x is an internal class as well, because $C=\{\langle x,y\rangle;\ y\in x\}$ is a standard class by Lemma 2.5(v). Thus, in reasonably axiomatizable theories we have

$$\mathbb{S}\subseteq\mathbb{I}\subseteq\mathbb{H}\subseteq\mathbf{H},\qquad \mathbf{S}\subseteq\mathbf{I}\subseteq\mathbf{E}\subseteq\mathbf{H},\qquad \mathbb{S}=\mathbf{S}\cap\mathbb{H},\quad \mathbb{I}=\mathbf{I}\cap\mathbb{H}.$$

At the same time, **BST** has the rather trivial landscape $\mathbb{S}=\mathbf{S}\subsetneqq\mathbb{I}=\mathbf{I}=\mathbb{H}=\mathbf{E}=\mathbf{H}$ (the key equality $\mathbb{I}=\mathbb{H}$ follows from Boundedness, while the axiom of Pair

[13] The right-hand side of the equivalence in parentheses can be called the *local standardness* of X. Thus, the axiom $\mathbf{S}=\mathbf{S}^{\text{loc}}$, in its non-trivial direction $\Leftarrow$, asserts that local standardness implies standardness. In principle, one can regard this axiom as a definition of standardness for proper classes, assuming that standardness in the domain of sets is given (see the proof of Theorem 2.6).

says that all objects are sets). Some amount of trivialization can be observed in **nGB**: indeed, $\mathbb{I} = \mathbb{H}$, because by $\mathbf{BST}^{\text{sets}}$ all sets are internal, and accordingly, $\mathbf{E} = \mathbf{H}$ (all classes consist of internal sets). The following lemma contains further results.

Lemma 2.5 (**nGB**).

(i) *If* $X \in \mathbf{S}$, $w \in \mathbb{S}$, $w \cap \mathbb{S} \subseteq X$, *then* $w \subseteq X$.

(ii) *Every class* $X \in \mathbf{S}$ *is equal to the union of all its standard subsets.*

(iii) *For any standard class* X, *if* $X \neq \varnothing$, *then* $X \cap \mathbb{S} \neq \varnothing$.

(iv) *For any classes* $X, Y \in \mathbf{S}$, *if* $X \cap \mathbb{S} = Y \cap \mathbb{S}$, *then* $X = Y$.

(v) *If* $\varphi(x)$ *is a predicative* $\in$*-formula with classes in* $\mathbf{S}$ (*including sets in* $\mathbb{S}$) *as parameters, then* $X = \{x : \varphi(x)\}$ *is a standard class.*

(vi) *If* $\varphi(x)$ *is a predicative formula with classes in* $\mathbf{I}$ (*including sets in* $\mathbb{I}$) *as parameters, then* $X = \{x : \varphi(x)\}$ *is an internal class.*

(vii) (Predicative Transfer) *If* Φ *is a closed predicative formula with standard classes as parameters, then* $\Phi \Leftrightarrow \Phi^{\text{st}}$.

(viii) $\mathbf{GB}^{\text{st}}$, *that is, all* **GB** *axioms in the* $\in$*-language relativized to the standard domain* $\mathbf{S}$.

(ix) (Assuming **nKM**) $\mathbf{KM}^{\text{st}}$, *that is, all* **KM** *axioms relativized to* $\mathbf{S}$.

Proof. (i) The axiom $\mathbf{S} = \mathbf{S}^{\text{loc}}$ implies that $u = w \cap X \in \mathbb{S}$. But $w \cap \mathbb{S} = u \cap \mathbb{S}$. It follows that $w = u \subseteq X$ by Transfer in $\mathbf{BST}^{\text{sets}}$. Item (ii) easily follows from $\mathbf{S} = \mathbf{S}^{\text{loc}}$. To prove (iii) and (iv), apply (ii) and Transfer in $\mathbf{BST}^{\text{sets}}$.

(v) Consider a predicative $\in$-formula $\varphi(x)$ with standard classes $C_1, \ldots, C_k$ as parameters. To prove that $C = \{x : \varphi(x)\}$ is a standard class, let us fix a standard set $y \in \mathbb{S}$ and show that $C \cap y \in \mathbb{S}$. We can assume that φ has the prenex form, that is,

$$\mathrm{Q}_1 v_1 \ldots \mathrm{Q}_n v_n \ \psi(x, C_1, \ldots, C_k, v_1, \ldots, v_n),$$

where ψ is a quantifier-free $\in$-formula in which $C_1, \ldots, C_k$ occur only to the right of $\in$.

It follows from the **nGB** Collection and the $\mathbf{BST}^{\text{sets}}$ Boundedness that there exist standard sets $r_1, \ldots, r_n$ such that

$$x \in C \Leftrightarrow \mathrm{Q}_1 v_1 \in r_1 \ldots \mathrm{Q}_n v_n \in r_n \, \psi(x, C_1, \ldots, C_k, v_1, \ldots, v_n) \tag{7}$$

holds for all $x \in y$. However, the set $r = y \cup r_1 \cup \cdots \cup r_n$ is standard, and hence for any $i = 1, \ldots, k$ the set $c_i = C_i \cap r$ belongs to $\mathbb{S}$ by $\mathbf{S} = \mathbf{S}^{\text{loc}}$. Now (7) implies that

$$x \in C \Leftrightarrow \mathrm{Q}_1 v_1 \in r_1 \ldots \mathrm{Q}_n v_n \in r_n \ \psi(x, c_1, \ldots, c_k, v_1, \ldots, v_n)$$

for every $x \in y$. Therefore, the set $C \cap y$ is definable by an $\in$-formula with standard parameters. We conclude that $C \cap y \in \mathbb{S}$ by the $\mathbf{BST}^{\text{sets}}$ Transfer.

(vi) By the definition of internal classes, we can assume that the parameters of φ belong to two types: standard classes and arbitrary sets (in $\mathbb{I}$). Thus, suppose that φ is $\varphi(p, x)$, where $p \in \mathbb{I}$ and standard parameters are not explicitly indicated. Then $C = \{\langle p, x\rangle : \varphi(p, x)\}$ is a standard class by (v). However, $C[p] = \{x : \varphi(p, x)\}$.

(vii) It suffices to check that $\exists\, x\, \varphi(x) \Rightarrow \exists^{\text{st}}\, x\, \varphi(x)$ for any $\in$-formula φ with standard parameters. But this follows immediately from (iii) and (v).

(viii) $(\mathbf{ZFC}^{\text{sets}})^{\text{st}}$ follows from $\mathbf{BST}^{\text{sets}}$. The relativized forms of Extensionality and Intersection can be easily proved with the help of previous assertions of the lemma. $(\text{Comprehension})^{\text{st}}$ is essentially (v). To prove $(\text{Collection})^{\text{st}}$, consider a standard class P and a set d. By the **nGB** Collection and the $\mathbf{BST}^{\text{sets}}$ Boundedness there exists a standard set r such that $\forall\, x \in d\ (\exists\, y\ P(x,y) \Rightarrow \exists\, y \in r\ P(x,y))$. It remains to apply Transfer in (vii).

(ix) This follows immediately from (viii), because the relativizations of the two key axioms of $\mathbf{KM}^{\text{st}}$ are included in **nKM**. Lemma 2.5 is proved.

Metamathematically, the theories **nGB** and **nKM** are connected both with the non-standard set theory **BST** and with the standard class theories **GB** and **KM**. Different aspects of the connection of **nGB** and **nKM** with **GB** and **KM** are considered below in this section. The most important relationships with **BST** are presented in the next theorem.

Theorem 2.6. **nGB** *is a* sets-*conservative extension of* **BST** *in the same sense as in Theorem* 2.2, *but for all* st-$\in$-*formulae* φ,[14] *and hence* **nGB** *is equiconsistent with* **BST** (and with **ZFC**, by the above). *However*, **nGB** *does not admit an interpretation in* **BST**, *while* **nKM** *is even not equiconsistent with* **BST**.

Proof. The 'negative' part of the theorem follows easily from Theorem 2.2, with a reference to Theorem 1.16 to carry out the passage from **BST** to **ZFC**. To prove conservativity it suffices to verify that any model $\mathfrak{M} = \langle M; \varepsilon, \text{st}\rangle \models \mathbf{BST}$ (M a set) can be extended by classes to a model $\mathfrak{K} = \langle K; \varepsilon, \text{st}\rangle \models \mathbf{nGB}$.

We obtain K by adjoining to M all sets $X \subseteq M$ that are st-$\in$-definable in $\mathfrak{M}$ (parameters in M allowed), except for sets of the form $\widehat{x} = \{y \in M : y \,\varepsilon\, x\}$, $x \in M$. And $X \in K \setminus M$ is defined as standard in accordance with the axiom $\mathbf{S} = \mathbf{S}^{\text{loc}}$.[15] The verification of all **nGB** axioms in $\mathfrak{K}$ is not much trouble. For instance, to prove Collection we apply Theorem 1.20(i) in $\mathfrak{M}$. The axiom $\mathbf{S} = \mathbf{S}^{\text{loc}}$ is provable in **BST** (the non-trivial direction uses Lemma 1.14(vi)). Therefore, $\mathbf{S} = \mathbf{S}^{\text{loc}}$ holds in the 'old' domain $\mathfrak{M}$ (while in $\mathfrak{K} \setminus \mathfrak{M}$ it holds by definition).

A few words on Standardization for classes. First of all, since Standardization and **BST** hold in $\mathfrak{M}$, the sentence $\forall\, X\, \forall^{\text{st}} y\, \exists^{\text{st}} z\ (z \cap \mathbb{S} = X \cap y \cap \mathbb{S})$ holds in $\mathfrak{K}$. It remains to get Standardization for classes from this sentence. Let X be any class. Put $Y = \bigcup\{x \in \mathbb{S} : x \cap \mathbb{S} \subseteq X\}$. Clearly, $X \cap \mathbb{S} = Y \cap \mathbb{S}$. To show that Y is a standard class, consider any $y \in \mathbb{S}$. Then $Y \cap y = \bigcup\{x \in \mathbb{S} : x \cap \mathbb{S} \subseteq X \cap y\} = z$, where $z \in \mathbb{S}$ satisfies $z \cap \mathbb{S} = X \cap y \cap \mathbb{S}$. Theorem 2.6 is proved.

2c. The axiom of impredicative Transfer in the Kelley–Morse theory. The theories **nGB** and **nKM**, as introduced above, do not contain Transfer explicitly. Of course, $\mathbf{BST}^{\text{sets}}$ contains this axiom for sets, and Lemma 2.5(vii) gives even more, but we are mostly interested in impredicative forms.

The impredicative Transfer between the standard universe (of sets and classes) $\mathbf{S}$ and the whole class universe contradicts **nGB**: for instance, **nGB** implies the negation of Intersection in **GB**. (The class of all standard natural numbers is

[14] See an even stronger result, Corollary 2.21 below, with a different proof.

[15] An equivalent definition of standardness of classes in $\mathfrak{K}$ is as follows: all sets $X \subseteq M$ that are $\in$-definable in $\mathfrak{M}$ with $\mathfrak{M}$-standard parameters are standard. The equivalence follows from 3.2.4(4) in [25].

a non-empty bounded subclass of $\mathbb{N}$ having no largest element, that is, is not a set; see Lemma 1.14(i).) Correct forms of impredicative Transfer involve internal classes.

In the next theorem, $^{\mathrm{st}}$ means relativization to standard sets and classes, as usual. Similarly, $^{\mathrm{int}}$ will denote relativization to internal classes, that is, to $\mathbf{I}$. Sets need not be mentioned, since all sets are internal classes (see above). Thus, $\varphi \Leftrightarrow \varphi^{\mathrm{int}}$ for predicative $\in$-formulae φ.

For reasons that will be clear below, the next theorem deals with the modified theories $\mathbf{nKM}^{\circ}$ and $\mathbf{nGB}^{\circ}$ obtained by the following changes in $\mathbf{nKM}$ and $\mathbf{nGB}$, respectively:

1) The schemata Impredicative Comprehension and Class Collection in $\mathbf{nKM}$ and Comprehension and Collection in $\mathbf{nGB}$ are removed,
2) $\mathbf{KM}^{\mathrm{st}}$ is added to the first theory and $\mathbf{GB}^{\mathrm{st}}$ to the second,
3) the axiom saying that $C[x] = \{y : \langle x, y\rangle \in C\}$ is a class for any set x and any standard class C is added to both theories.

It follows from Lemma 2.5 (viii), (ix) that $\mathbf{nKM} \Rightarrow \mathbf{nKM}^{\circ}$ and $\mathbf{nGB} \Rightarrow \mathbf{nGB}^{\circ}$.

Theorem 2.7. ($\mathbf{nKM}^{\circ}$) *Let Φ be a closed $\in$-formula with standard parameters (sets and classes). Then $\Phi^{\mathrm{int}} \Leftrightarrow \Phi^{\mathrm{st}}$.*

($\mathbf{nGB}^{\circ}$) *The same, but only for predicative $\in$-formulae Φ.*

The proof of this Transfer theorem is based on Lemma 2.9 below. To formulate the lemma we need one more definition.

Definition 2.8. Suppose that $\varphi(v_1, \dots, v_n)$ is an $\in$-formula with free variables v_i, and standard parameters (sets in $\mathbb{S}$ and classes in $\mathbf{S}$) not explicitly indicated. We let $^{\mathrm{st}}\{\langle x_1, \dots, x_n\rangle : \varphi(x_1, \dots, x_n)\}$ denote the unique (by $\mathbf{GB}^{\mathrm{st}}$ or $\mathbf{KM}^{\mathrm{st}}$) class $X \in \mathbf{S}$ satisfying

$$X = \{\langle x_1, \dots, x_n\rangle : \varphi(x_1, \dots, x_n)\} \quad \text{in} \quad \mathbf{S},$$

that is,

$$\langle x_1, \dots, x_n\rangle \in X \Leftrightarrow \varphi(x_1, \dots, x_n)^{\mathrm{st}} \quad \text{for all standard } x_1, \dots, x_n.$$

Lemma 2.9. ($\mathbf{nKM}^{\circ}$) *Let $\varphi(v_1, \dots, v_n)$ be an $\in$-formula with standard parameters (sets and classes). Then for any sets $x_1, \dots, x_n$*

$$\varphi(x_1, \dots, x_n)^{\mathrm{int}} \Leftrightarrow \langle x_1, \dots, x_n\rangle \in {}^{\mathrm{st}}\{\langle \xi_1, \dots, \xi_n\rangle : \varphi(\xi_1, \dots, \xi_n)\}.$$

($\mathbf{nGB}^{\circ}$) *The same for predicative $\in$-formulae φ.*

Proof. Argue by induction on the logical complexity of φ. Suppose that φ is an atomic formula $v_1 \in v_2$. Then $X = {}^{\mathrm{st}}\{\langle x, y\rangle : x \in y\} \in \mathbf{S}$. According to Boundedness in $\mathbf{BST}^{\mathrm{sets}}$, it suffices to prove for every standard z that $\forall\, x, y \in z\ (x \in y \Leftrightarrow \langle x, y\rangle \in p)$, where $p = X \cap (z \times z)$ is also a standard set by the axiom $\mathbf{S} = \mathbf{S}^{\mathrm{loc}}$. But this sentence holds in $\mathbb{S}$ by the definition of X, so it remains to use Transfer in $\mathbf{BST}^{\mathrm{sets}}$.

The step for $\neg$. The (standard) classes $X = {}^{\mathrm{st}}\{x : \varphi(x)\}$ and $Y = {}^{\mathrm{st}}\{x : \neg\,\varphi(x)\}$ (more precisely, their intersections with $\mathbb{S}$) are complementary in $\mathbb{S}$. We show that X and Y are complementary in $\mathbb{I}$, too. Suppose on the contrary that $z \in X \cap Y$. By Lemma 2.5(ii),[16] there exist standard sets x, y such that $z \in x \subseteq X$

[16] Note that (i)–(iv) in Lemma 2.5 remain true in $\mathbf{nGB}^{\circ}$.

and $z \in y \subseteq Y$. Thus, $x \cap y \neq \varnothing$, and hence by Transfer in $\mathbf{BST}^{\text{sets}}$, $x \cap y \cap \mathbb{S} \neq \varnothing$. Therefore, $X \cap Y \cap \mathbb{S} \neq \varnothing$, a contradiction.

The step for $\wedge$. Quite similar arguments show that $X \cap Y \cap \mathbb{S} = Z \cap \mathbb{S}$ implies that $X \cap Y = Z$ for all standard classes X, Y, Z.

The step for $\exists$ over classes. This is more difficult. Consider an $\in$-formula $\exists Y\ \varphi(v, Y)$ with standard parameters. Define $X = {}^{\text{st}}\{\xi : \exists Y\ \varphi(\xi, Y)\}$. We have to prove the equivalence $x \in X \Leftrightarrow (\exists Y\ \varphi(x, Y))^{\text{int}}$ for any set x. Using $\mathbf{BST}^{\text{sets}}$, take a standard set w containing x.

From left to right. Suppose that $x \in X$. By definition, $\forall \xi \in X\, \exists Y\ \varphi(\xi, Y)$ holds in $\mathbf{S}$. It follows from (Class Collection)$^{\text{st}}$ in $\mathbf{nKM}^{\circ}$ that there is a class $U \in \mathbf{S}$ such that $\forall \xi \in s\ \exists h\ \varphi(\xi, U[h])$ holds in $\mathbf{S}$, where $s = w \cap X \in \mathbb{S}$ and, we recall, $U[h] = \{x : \langle h, x\rangle \in U\}$. However, $\varphi(\xi, U[h])$, as a formula with ξ, h as free variables (and U being one more standard parameter), is syntactically simpler than the original formula $\exists Y\ \varphi(x, Y)$. Therefore, by the inductive hypothesis the standard class $P = {}^{\text{st}}\{\langle \xi, h\rangle : \xi \in w \wedge \varphi(\xi, U[h])\}$ satisfies the equivalence $\langle \xi, h\rangle \in P \Leftrightarrow \xi \in w \wedge \varphi(\xi, U[h])^{\text{int}}$.

We assert that $x \in \operatorname{dom} P$. Indeed, since $x \in X$, Lemma 2.5(ii) gives us a standard set u with $x \in u \subseteq X$. It can be assumed that $u \subseteq w$ (otherwise, replace u by $w \cap u$; the set $w \cap u$ is standard by the $\mathbf{BST}^{\text{sets}}$ Transfer). Then both $u \subseteq X \cap w$ and $X \cap w = \operatorname{dom} P$ hold in $\mathbf{S}$ by definition. Therefore, there is a standard set p such that the sentences $p \subseteq P$ and $u = \operatorname{dom} p$ hold in $\mathbf{S}$. Then $p \subseteq P$ and $u = \operatorname{dom} p$ hold in the whole set universe by Lemma 2.5(i) for the first sentence and by the $\mathbf{BST}^{\text{sets}}$ Transfer for the second. Therefore, $x \in u = \operatorname{dom} p \subseteq \operatorname{dom} P$.

Thus $x \in \operatorname{dom} P$, that is, there is a set h with $\langle x, h\rangle \in P$. We immediately obtain $\varphi(x, U[h])^{\text{int}}$, therefore $(\exists Y \varphi(x, Y))^{\text{int}}$, because $Y = U[h]$ is an internal class.

From right to left. Suppose that $\varphi(x, Y)^{\text{int}}$, where Y is an internal class. Then by definition, $Y = U[h]$, where U is a standard class (not necessarily equal to the set U considered above), while h is a set. The standard class $P = {}^{\text{st}}\{\langle \xi, \eta\rangle : \varphi(\xi, U[\eta])\}$ satisfies $\langle \xi, \eta\rangle \in P \Leftrightarrow \varphi(\xi, U[\eta])^{\text{int}}$ by the inductive hypothesis. Therefore, $\langle x, y\rangle \in P$ and $x \in \operatorname{dom} P$. By Lemma 2.5(ii) there exists a standard $p \subseteq P$ such that $x \in \operatorname{dom} p$. In $\mathbf{S}$ we define $u = \operatorname{dom} p$. Then $u \in \mathbb{S}$ and $u \subseteq X$ in $\mathbf{S}$ (since, by definition, $X = \operatorname{dom} P$ in $\mathbf{S}$). Then $u \subseteq X$ holds in the universe by Lemma 2.5(i). Finally, $u = \operatorname{dom} p$ holds in the universe by the $\mathbf{BST}^{\text{sets}}$ Transfer. Thus, $\operatorname{dom} p \subseteq X$. We have $x \in X$, as required.

The step for $\exists$ over sets, for $\mathbf{nGB}^{\circ}$. For an $\in$-formula $\exists y\ \varphi(v, y)$ with standard parameters, we have to prove that the class $X = {}^{\text{st}}\{\xi : \exists y\ \varphi(\xi, y)\}$ satisfies the equivalence $x \in X \Leftrightarrow \exists y\ \varphi(x, y)$ for every x. Arguing as in the case of a class quantifier, we put $P = {}^{\text{st}}\{\langle \xi, \eta\rangle : \xi \in w \wedge \varphi(\xi, \eta)\}$ for the direction from left to right and $P = {}^{\text{st}}\{\langle \xi, \eta\rangle : \varphi(\xi, \eta)\}$ for the opposite direction. Lemma 2.9 is proved.

Proof of Theorem 2.7. Apply Lemma 2.9 to the formula $\varphi(v) := (v = v \wedge \Phi)$.

Corollary 2.10.

($\mathbf{nKM}^{\circ}$) *The structure* $\langle \mathbf{I}; \in\rangle$ *satisfies* $\mathbf{KM}$.

($\mathbf{nGB}^{\circ}$) *The structure* $\langle \mathbf{I}; \in\rangle$ *satisfies* $\mathbf{GB}$.

Proof. To prove the $\mathbf{nKM}^\circ$-part, apply Transfer (Theorem 2.7) and note that $\langle \mathbf{S}; \in \rangle$ satisfies $\mathbf{KM}$ by the axioms of $\mathbf{KM}^{\mathrm{st}}$. As for the $\mathbf{nGB}^\circ$-part, some axioms require separate verification, since Theorem 2.7 yields only predicative Transfer in this case. In particular, to prove the (predicative) Comprehension in $\langle \mathbf{I}; \in \rangle$ we just follow the proof of Lemma 2.5(vi). To prove Collection in $\langle \mathbf{I}; \in \rangle$ we consider an internal class $P = C[p]$, where $p \in \mathbb{I}$ and C is a standard class, together with a set $d \in \mathbb{I}$. There exists a standard set s such that $p \in s$ and $d \subseteq s$. By Collection in $\mathbf{S}$ ($\mathbf{GB}^{\mathrm{st}}$ is used), there exists a standard set $c \subseteq C$ satisfying

$$\forall\, q \in s\, \forall\, x \in s\ \big(\exists\, y\ (\langle x, y\rangle \in c[q]) \Rightarrow \exists\, y \in r\ (\langle x, y\rangle \in c[q])\big) \tag{8}$$

in $\mathbb{S}$, where $r = \operatorname{ran} c$ is a standard set, too. It remains to note that both the inclusion $c \subseteq C$ and (8) still hold in $\mathbf{I}$ by Theorem 2.7. Corollary 2.10 is proved.

2d. The axiom of $\boldsymbol{\Sigma_2^{\mathrm{st}}}$ classes. The implications $\mathbf{nKM} \Rightarrow \mathbf{nKM}^\circ$ and $\mathbf{nGB} \Rightarrow \mathbf{nGB}^\circ$ mentioned above are perhaps not reversible, but they do turn out to be reversible in the presence of the following axiom, the idea for which goes back to Theorem 1.19.

Σ_2^{st}-*classes*: Any collection of the form $X = \{z : \exists^{\mathrm{st}}\, x\, \forall^{\mathrm{st}} y\ (\langle x, y, z\rangle \in I)\}$, where I is an internal class, is a class, and conversely, for any class X there exists an internal class I such that $X = \{z : \exists^{\mathrm{st}}\, x\, \forall^{\mathrm{st}} y\ (\langle x, y, z\rangle \in I)\}$.[17]

To what extent is the axiom of Σ_2^{st} classes acceptable as an axiom? The answer to this question depends on the point of view on the role of non-standard class theories. If we are going to consider a system of classes as a suitable minimal superstructure of both the set universe and of the collection of standard classes, then this will be precisely equivalent to the axiom of Σ_2^{st} classes.

Theorem 2.11.

$$\begin{aligned} &\big(\mathbf{nKM}^\circ + \Sigma_2^{\mathrm{st}}\ \textit{classes}\big) \quad \textit{All } \mathbf{nKM} \textit{ axioms hold.}\\ &\big(\mathbf{nGB}^\circ + \Sigma_2^{\mathrm{st}}\ \textit{classes}\big) \quad \textit{All } \mathbf{nGB} \textit{ axioms hold.}\end{aligned}$$

Proof. We begin with the proof of Impredicative Comprehension in the $\mathbf{KM}$-case. The result is somewhat unexpected, because a rather special Σ_2^{st} form of the class definition turns out to imply the closure property with respect to all st-$\in$-definitions. The following lemma implies the result required.

Lemma 2.12 ($\mathbf{nKM}^\circ$ + Σ_2^{st}classes). *For each parameter-free* st-$\in$-*formula* $\Phi(X_1, \dots, X_m, x_1, \dots, x_n)$ *there is a parameter-free* $\in$-*formula* $\varphi(X_1, \dots, X_m, x_1, \dots, x_n, a, b)$ *such that for all sets* x_i *and all standard classes* X_j

$$\Phi(X_1, \dots, X_m, x_1, \dots, x_n) \Leftrightarrow \exists^{\mathrm{st}}\, a\, \forall^{\mathrm{st}} b\ \varphi(X_1, \dots, X_m, x_1, \dots, x_n, a, b)^{\mathrm{int}}.$$

Proof. We follow the scheme used in the proof of Theorem 1.19. Elementary simplifications reduce the general case to the case when

($*$) Φ has the prenex form $\Pi\Psi$, where Π is a quantifier prefix containing quantifiers only of the form $\exists^{\mathrm{st}}\, Y$, $\forall^{\mathrm{st}} Y$ (over standard classes) and $\exists\, y$, $\forall\, y$ (over sets), while Ψ is an $\in$-formula (then $\Psi \Leftrightarrow \Psi^{\mathrm{int}}$, $\Phi \Leftrightarrow \Phi^{\mathrm{int}}$).

[17] The first part of this axiom is provable in $\mathbf{nGB}$, of course.

Under this assumption we prove the lemma in two steps.

Step 1: The lemma in the case when the prefix Π in $(*)$ contains only quantifiers of the form $\exists^{\rm st} Y$, $\forall^{\rm st} Y$.

For instance, let $\Phi(X, x)$ be $\forall^{\rm st} Y \exists^{\rm st} Z\ \Psi(Y, Z, x, X)$. Let $\vartheta(u, X)$ denote the $\in$-formula $\forall Y \exists Z \exists p \in u \forall \xi \in p\ \Psi(Y, Z, \xi, X)$. In the notation of the proof of Theorem 1.19, the equivalence

$$\Phi(X, x) \Leftrightarrow \exists^{\rm st} u\ (\mathrm{Ult}\, u \wedge x \in \mathrm{mon}\, u \wedge \vartheta(u, X)^{\rm int}) \tag{9}$$

implies the desired result, since the right-hand side can be transformed to a $\Sigma_2^{\rm st}$-form. To prove (9) suppose that $x \in \mathbb{I}$ and X is a standard class. By Standardization in $\mathbf{BST}^{\rm sets}$ there is a standard ultrafilter $u \in \mathbb{S}$ such that $x \in \mathrm{mon}\, u$ (see the proof of Theorem 1.19). It remains to prove that

$$\Phi(x, X) \Leftrightarrow \forall^{\rm int} Y \exists^{\rm int} Z \exists p \in u \forall \xi \in p\ \Psi(Y, Z, \xi, X) \tag{10}$$

for each standard ultrafilter u with $x \in \mathrm{mon}\, u$. The proof of (10) is similar to the corresponding argument in the proof of Theorem 1.19, and is omitted. (The Transfer involved in this argument follows from Theorem 2.7.)

Step 2: The inductive step $\exists y$ of the lemma under the assumption of $(*)$. Let us convert the formula $\Phi(x, X) := \exists y \exists^{\rm st} a \forall^{\rm st} b\ \psi(y, a, b, x, X)^{\rm int}$ to an (equivalent) $\Sigma_2^{\rm st}$ form, where ψ is an $\in$-formula while X is a variable over standard classes. We assert that

$$\Phi(x, X) \Leftrightarrow \exists^{\rm st} a \exists^{\rm st} w \forall^{\rm stfin} \beta \exists y \in w \forall b \in \beta\ \psi(y, a, b, x, X)^{\rm int},$$

which easily implies the required result. It suffices to verify that

$$\exists y \forall^{\rm st} b\ \psi(y, b)^{\rm int} \Leftrightarrow \exists^{\rm st} w \forall^{\rm stfin} \beta \exists y \in w \forall b \in \beta\ \psi(y, b)^{\rm int} \tag{11}$$

for any $\in$-formula ψ with standard parameters (sets and classes). The implication $\Rightarrow$ is easy: according to $\mathbf{BST}^{\rm sets}$, standard finite sets contain only standard elements (see 3.1.20 in [25]). Thus, we concentrate on the implication $\Leftarrow$. Suppose that the right-hand side of (11) holds for some standard w. By Transfer of Theorem 2.7, the class $C = \{\langle y, b\rangle : \psi(y, b)^{\rm int}\}$ and the map $b \mapsto w_b = \{y \in w : C(y, b)\}$ are standard classes. Using Collection in $\langle \mathbf{S}; \in\rangle$ and Transfer, we obtain a standard set s such that $\forall b \exists b' \in s\ (w_b = w_{b'})$, and then $\forall^{\rm st} b \exists^{\rm st} b' \in s\ (w_b = w_{b'})$. Then by the choice of w the right-hand side of (11) implies that

$$\forall^{\rm stfin} \beta \subseteq s \exists y \in w \forall b \in \beta\ (\langle y, b\rangle \in c),$$

where $c = C \cap (w \times s)$ (a standard set), and then $\exists y \in w \forall^{\rm st} b \in s\ (y \in w_b)$ by Basic Idealization in $\mathbf{BST}^{\rm sets}$. (We note that $y \in w_b$ is equivalent to $\langle y, b\rangle \in c$, for any $y \in w$ and $b \in s$.) Let $y \in w$ satisfy $\forall^{\rm st} b \in s\ (y \in w_b)$. Then $\forall^{\rm st} b\ (y \in w_b)$ by the choice of s, and hence we have the left-hand side of (4), as required. Lemma 2.12 is proved.

Now we prove the (predicative) Comprehension under the assumption $\mathbf{nGB}^{\circ}$. The Transfer of Theorem 2.7 holds only for predicative $\in$-formulae in this case. Accordingly, Lemma 2.12 also holds only for predicative st-$\in$-formulae, that is, in accordance with the predicative form of Transfer, the only form available in this case. (Quantifiers over standard classes do not appear in Step 1, of course.) But this still suffices for proving the (predicative) Comprehension in **nGB**.

Finally, the proof of Collection, or Class Collection in the case when $\mathbf{nKM}^\circ$ is assumed, is entirely analogous to the proof of Theorem 1.20. Theorem 2.11 is proved.

2e. Semisets and the axiom of $\boldsymbol{\Sigma_2^{\mathrm{st}}}$ semisets. For non-standard class theories let us now consider several additional axioms involving the notion of semisets. The axioms do not belong to the natural circle of ideas connected with the definition of a suitable superstructure of classes over the non-standard set universe of **BST**. However, some applications immediately reveal the need for these additional axioms.

Definition 2.13. In non-standard class theories, a *semiset* is a subclass of a set. An st-$\in$-formula is *sem-bounded* (sem from 'semiset'), if all its class quantifiers have the form $\exists^{\mathrm{sem}}$ or $\forall^{\mathrm{sem}}$ ('there is a semiset', 'for all semisets').

For instance, if $p \in \mathbb{I}$ is a function (a set) defined on a set of the form $a \times b$, where $a, b \in \mathbb{S}$, then the subclass

$$E_p = \{z : \exists^{\mathrm{st}}\, x \in a\, \forall^{\mathrm{st}} y \in b\ (z \in p(x,y))\}$$

of the set $r = \operatorname{ran} p \in \mathbb{I}$ is a semiset in **nGB**. Let $\mathbf{C}_0$ be the class of all $p \in \mathbb{I}$ of this form (codes of Σ_2^{st} semisets); $\mathbf{C}_0 \subseteq \mathbb{I}$. We introduce the axiom:

Σ_2^{st} *semisets*: for any semiset $X \subseteq \mathbb{I}$ there is a code $p \in \mathbf{C}_0$ such that $X = E_p$.

A semiset X has *standard size* if there exist a standard set x and a semiset F which is a bijection of $x \cap \mathbb{S}$ onto X.

Every set is a semiset, of course (correspondingly, predicative formulae are sem-bounded); the converse holds in standard theories by the axiom of Intersection. However, **nGB** proves the existence of a semiset (consisting of all the standard natural numbers) which is not a set, since a suitable version of Lemma 1.14(i) holds in **nGB**. As for the notion of standard size, note that any semiset of the form $x \cap \mathbb{S}$, where x is standard, is essentially x itself considered within the universe $\mathbb{S}$ of standard sets. (Infinite sets $x \in \mathbb{S}$ contain also non-standard elements.) Standard size indicates smallness: a subclass of a semiset of standard size is itself a semiset of standard size.[18]

Lemma 2.14 (**nGB**). *The axiom of Σ_2^{st} classes implies the axiom of Σ_2^{st} semisets.*

Proof. According to Σ_2^{st} classes there is an internal class I such that $X = \{z : \exists^{\mathrm{st}}\, x\, \forall^{\mathrm{st}} y\ (\langle x,y,z\rangle \in I)\}$. Collection gives us sets $a, b \in \mathbb{S}$ satisfying $X = \{z : \exists^{\mathrm{st}}\, x \in a\, \forall^{\mathrm{st}} y \in b\ (\langle x,y,z\rangle \in I)\}$. We define the required code p by $p(x,y) = \{z \in w : \langle x,y,z\rangle \in I\}$, where $w \in \mathbb{S}$ is any set satisfying $X \subseteq w$. Lemma 2.14 is proved.

The axiom of Σ_2^{st} semisets looks less restrictive in the context of the general rules for constructing non-standard universes. It implies several further corollaries. We

[18] We can also require here that non-small sets contain subsets of arbitrarily large standard size. In **nGB** + the axiom of Σ_2^{st} classes such a requirement can also be satisfied. About this requirement see 1.4 in [25] in a somewhat different setup.

introduce another pair of theories that will be useful below:

$$\mathbf{nGB}^+\!: \ \mathbf{nGB} + \Sigma_2^{\rm st}\ \textsf{semisets},\ \text{a subtheory of } \mathbf{nGB} + \Sigma_2^{\rm st}\ \textsf{classes};$$
$$\mathbf{nKM}^+\!: \ \mathbf{nKM} + \Sigma_2^{\rm st}\ \textsf{semisets},\ \text{a subtheory of } \mathbf{nKM} + \Sigma_2^{\rm st}\ \textsf{classes}.$$

The theory $\mathbf{nGB}^+$ is remarkable in that it infers a certain part of the Comprehension and Collection schemata for impredicative formulae by the next theorem, while still being a **GB**-type theory in the sense of equiconsistency by Theorem 2.16.

Theorem 2.15 ($\mathbf{nGB}^+$). *The following statements hold.*

Sem-bounded Comprehension: Comprehension for all sem-bounded st-$\in$-*formulae*;

Semiset Collection: $\forall\, d\, \exists^{\rm sem}\, C\, \exists\, r\, \forall\, a \in d\ (\exists^{\rm sem}\, Y \varphi(a, Y) \Rightarrow \exists\, p \in r\ \varphi(a, C[p]))$, *where* $\varphi(a, Y)$ *is any sem-bounded* st-$\in$-*formula and* $C[p]$ *is defined in Definition* 2.1.

Standard Size Choice: If X *is a semiset of standard size and* P *any semiset, then there is a function* (*a set*) f *such that* $X \subseteq \operatorname{dom} f$ *and for all* $x \in X$ *the following holds*: $\exists\, y\ (\langle x, y\rangle \in P) \Rightarrow \langle x, f(x) \in P$.

Saturation: If H *is a* $\cap$-*closed* (*that is*, $x, y \in H \Rightarrow x \cap y \in H$) *semiset of standard size and* $\varnothing \notin H$, *then the intersection* $\bigcap H$ *is non-empty.*

Dependent Choice: If X *is any set and* $R \subseteq X \times X$ *is a semiset with* $\operatorname{dom} R = X$, *then there is a semiset of the form of a sequence* $\{x_n\}_{n \in \mathbb{N} \cap \mathbb{S}}$, *where* $\langle x_n, x_{n+1}\rangle \in R$ *for all standard* $n \in \mathbb{N}$.

If in addition **nKM** *is assumed, then Semiset Collection holds for all, not necessarily sem-bounded*, st-$\in$-*formulae.*

Proof. Let us prove sem-bounded Comprehension. The axiom of $\Sigma_2^{\rm st}$ semisets allows us to convert any sem-bounded formula into a predicative one: just change every occurrence of, say, $\exists^{\rm sem}\, C\, \psi(C)$ into $\exists\, p \in \mathbf{C}_0\ \psi(E_p)$.

Standard Size Choice. By definition, there exist a semiset-function F and a standard set d such that F maps $d \cap \mathbb{S}$ onto X. By the principle of $\Sigma_2^{\rm st}$ semisets, the semisets X, P, F are st-$\in$-definable in the universe $\mathbb{I}$ of all sets, with sets in $\mathbb{I}$ as parameters. Applying Theorem 1.20(ii) in $\mathbb{I}$ (a **BST** universe), we obtain the required result.

Saturation. Using Standard Size Choice, we obtain a standard set d and a set-function ψ mapping $d \cap \mathbb{S}$ onto H. There is a set R such that $\bigcup H \subseteq R$. Without loss of generality we assume that $\psi(a) \subseteq R$ for all $a \in d$. It follows from the $\cap$-closure property of H that for any standard finite set $d' \subseteq d$ the intersection $\bigcap_{a \in d'} \psi(a)$ is equal to some $\psi(a')$, $a' \in d \cap \mathbb{S}$, and therefore this intersection is non-empty. (Induction on the standard finite number of elements in d' is used; see Lemma 1.14(ii).)

Dependent Choice. The plan of the proof is the same as in the proof of Standard Size Choice: the principle of $\Sigma_2^{\rm st}$ semisets enables us to obtain the result as a consequence of a related Dependent Choice theorem of **BST** proved in 3.2e of [25].

Semiset Collection. The formula $\varphi(a, E_y)$ with variables a, y is sem-bounded. Therefore, $Z = \{\langle a, y\rangle : \varphi(a, E_y)\}$ is a class by sem-bounded Comprehension (see above). By the ordinary Collection there exists a set r satisfying $\exists\, p\ \varphi(a, E_p) \Rightarrow \exists\, p \in r\ \varphi(a, E_p)$ for every $a \in d$. Now define $C = \{\langle p, x\rangle : p \in r \wedge x \in E_p\}$; then $C[p] = E_p$ for all $p \in r$.

If **nKM** is assumed, then to prove Semiset Collection for any st-$\in$-formula $\varphi(a,Y)$ we first apply Class Collection. Thus, for any given d there exist a class C' and a set r such that

$$\forall\, a \in d\ \big(\exists^{\mathrm{sem}}\, Y\ \varphi(a,Y) \Rightarrow \exists\, p \in r\ (C'[p] \text{ is a semiset} \wedge \varphi(a, C'[p]))\big).$$

By the ordinary Collection there exists a set Z such that for each $a \in d$, if $C'[p]$ is a semiset, then $C'[p] \subseteq z$ for some $z \in Z$. Now consider the semiset $C = C' \cap (d \times \cup Z)$. Theorem 2.15 is proved.

2f. Interpretation of non-standard class theories in standard ones. Here we prove the following theorem.

Theorem 2.16. *There is an* **S**-*type interpretation*[19] *of* $\mathbf{nGB} + \Sigma_2^{\mathrm{st}}$ classes *and* $\mathbf{nGB}^+$ *in* **GB** (*of* $\mathbf{nKM} + \Sigma_2^{\mathrm{st}}$ classes *and* $\mathbf{nKM}^+$ *in* **KM**, *respectively*).

The theories **nGB**, $\mathbf{nGB}^+$, *and* $\mathbf{nGB} + \Sigma_2^{\mathrm{st}}$ classes *are* st-*conservative, in the sense of Theorem* 1.16(ii), *and hence equiconsistent extensions of* **GB**. *The theories* **nKM**, $\mathbf{nKM}^+$, *and* $\mathbf{nKM} + \Sigma_2^{\mathrm{st}}$ classes *are* st-*conservative and equiconsistent extensions of* **KM**.

Proof. The second assertion of the theorem (conservativity and equiconsistency) follows easily from the first (see, for instance, the proof of Theorem 1.16(ii)). We focus on the first assertion. It suffices to consider the theory $\mathbf{nGB} + \Sigma_2^{\mathrm{st}}$ classes (because $\mathbf{nGB}^+$ is a subtheory of it by Lemma 2.14) and, respectively, the theory $\mathbf{nKM} + \Sigma_2^{\mathrm{st}}$ classes.

Arguing in **GB**, we let **K** denote the **GB** class universe, and let $\mathbf{V} = \{x : \mathrm{set}\, x\} \subseteq \mathbf{K}$ denote the class of all sets. The structure $\mathfrak{V} = \langle \mathbf{V}; \in\rangle$ satisfies **ZFC**, and hence there exists an absolutely saturated non-standard elementary extension ${}^*\mathfrak{V} = \langle {}^*\mathbf{V}; {}^*{\in}, \mathrm{st}\rangle$ by means of an embedding * (see subsection 1g).

Recall that ${}^*\mathfrak{V}$ satisfies **BST** by Theorem 1.16.

To avoid notational problems, we assume that there is no set x such that both $x \subseteq {}^*\mathbf{V}$ and $x \in {}^*\mathbf{V}$. If this is not the case, then one can replace every $x \in {}^*\mathbf{V}$ by the pair $\langle x, x\rangle$ (note that, by definition, ${}^*\mathbf{V}$ does not contain ordered pairs). The relations ${}^*{\in}$ and st and the map * have to be redefined accordingly.

Thus, we have an $\mathbb{S}$-type interpretation ${}^*\mathfrak{V}$ of **BST** in **GB**. We are going to extend it to an interpretation of the class theory **nGB** by adjoining some classes $X \subseteq {}^*\mathbf{V}$ to ${}^*\mathfrak{V}$.[20] We begin with a technical definition.

Definition 2.17. Put $\widehat{x} = \{y \in {}^*\mathbf{V} : y \,{}^*{\in}\, x\}$ for each $x \in {}^*\mathbf{V}$.

Call a class $X \subseteq {}^*\mathbf{V}$ *admissible* if it does **not** have the form $\widehat{x}$, $x \in {}^*\mathbf{V}$.[21]

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[19] The notion of **S**-type interpretation is pretty similar to Definition 1.15 and Remark 1.18, that is, this is an interpretation whose standard domain **S** is provably isomorphic to the whole **GB** universe by means of a certain concrete embedding. We use **S** instead of $\mathbb{S}$ because the whole standard domain is denoted by **S** in class theories by Definition 2.4.

[20] It is a good exercise for the reader to figure out why there is not much sense in adding all classes. Indeed, since ${}^*\mathfrak{V}$ is absolutely saturated, $\widehat{x}$ will be a proper class in the sense of the ground **GB** universe for many elements $x \in {}^*\mathbf{V}$, for instance, for any $x = {}^*a$, where $a \in \mathbf{V}$ is infinite. It takes some effort to define a class F which is a function from $\widehat{x}$ onto a cofinal subclass of the class Ord of all ordinals. Adjoining the class $\{\langle x, {}^*a\rangle : \langle x, a\rangle \in F\}$ to ${}^*\mathfrak{V}$ leads to an irreparable failure of Collection.

[21] Adding inadmissible classes to ${}^*\mathfrak{V}$ would lead to a contradiction with Extensionality.

We put ${}^*A = \bigcup_{a\in\mathbf{V},\ a\subseteq A} {}^*a$ for any *proper* class $A \in \mathbf{K}$.[22]

To agree with the notation introduced in subsection 2b, we redefine $\mathbb{I} = {}^*\mathbf{V}$.

Put $\mathbb{S} = \{x \in \mathbb{I} : \operatorname{st} x\} = \{{}^*a : a \in \mathbf{V}\}$, the standard domain of ${}^*\mathfrak{V}$.

We consider the following subclasses of $\mathbb{I} = {}^*\mathbf{V}$.

Standard classes: those of the form ${}^*A$, where A is any *proper* class in $\mathbf{K}$.

Put $\mathbf{S} = \mathbb{S} \cup \{$all admissible standard classes$\}$.

Internal classes: those of the form $I = C[x] = \{y \in {}^*\mathbf{V} : {}^*\langle x, y\rangle \in C\}$, where C is an admissible standard class, $x \in {}^*\mathbf{V}$, and ${}^*\langle x, y\rangle$ denotes the ${}^*\mathfrak{V}$-*pair*, that is, the unique set $p \in {}^*\mathbf{V}$ such that 'p is equal to $\langle x, y\rangle$ in ${}^*\mathfrak{V}$' holds. Put $\mathbf{I} = \mathbb{I} \cup \{$all admissible internal classes$\}$.

External classes: those of the form $X = E_I = \bigcup_{x\in\mathbf{V}} \bigcap_{y\in\mathbf{V}} I[{}^*x, {}^*y]$, where I is an internal class, $I[x, y] = \{z \in {}^*\mathbf{V} : {}^*\langle x, y, z\rangle \in I\}$, and ${}^*\langle x, y, z\rangle \in {}^*\mathbf{V}$ is defined in a way similar to that for ${}^*\langle x, y\rangle$ above. Put $\mathbf{E} = \mathbb{I} \cup \{$all admissible external classes$\}$.

The collections $\mathbf{S}$, $\mathbf{I}$, $\mathbf{E}$ obviously consist of sets and classes; they are considered in class theories in the same way as definable classes of sets in **ZFC** (see subsection 1a).

Definition 2.18. For $X, Y \in \mathbf{E}$ we define $X \,{}^*{\in}\, Y$ if $X \in \mathbb{I}$ and either $Y \in \mathbb{I}$ and $X {}^*{\in} Y$, or Y is an admissible class and simply $X \in Y$.[23]

We define $\operatorname{st} X$ for each class $X \in \mathbf{S}$.

We are going to prove that the structure $\mathfrak{E} = \langle \mathbf{E}; {}^*{\in}, \operatorname{st}\rangle$ satisfies the axioms of $\mathbf{nGB} + \Sigma_2^{\text{st}}$ classes and is a non-standard extension (in the sense of Definition 1.15) of the **GB** class universe by means of the map *. And if in addition **KM** is assumed, then we assert that $\mathfrak{E}$ satisfies $\mathbf{nKM}^+$. This will complete the proof of Theorem 2.16: the structure $\mathfrak{E}$ becomes a 'realistic' $\mathbf{S}$-type interpretation of the theory $\mathbf{nGB} + \Sigma_2^{\text{st}}$ classes in **GB** and of the theory $\mathbf{nKM} + \Sigma_2^{\text{st}}$ classes in **KM**.

That $\mathfrak{E}$ is a non-standard extension of $\mathbf{K}$ is an immediate corollary of (i)–(iii) in the following lemma.

Lemma 2.19 (**GB**).

(i) *If* $A \in \mathbf{K}$, *then* ${}^*A \in \mathbf{S}$.

(ii) *If* $X, Y \in \mathbf{K}$, *then* $X \in Y \Leftrightarrow {}^*X \,{}^*{\in}\, {}^*Y$.

(iii) $\mathbf{S} = \{X \in \mathbf{E} : \operatorname{st} X\} = \{{}^*A : A \in \mathbf{K}\}$ (the standard domain of $\mathfrak{E}$) *and* $\mathbb{S} = \mathbb{I} \cap \mathbf{S}$.

(iv) $\mathbb{I} = \{X \in \mathbf{E} : (\operatorname{set} X)^{\mathfrak{E}}\} = \{$*all sets in* $\mathfrak{E}\}$.

(v) $\mathbf{I} = \{$*all internal classes in* $\mathfrak{E}$ *in the sense of Definition* 2.4, *subsection* 2b$\}$.

(vi) *The axiom of* Σ_2^{st} *classes holds in* $\mathfrak{E}$.

Proof. (i) If $A = a$ is a set, then ${}^*a \in \mathbb{S} \subseteq \mathbf{S}$ by definition. If A is a proper class, then we have to check that ${}^*A$ is admissible. Suppose on the contrary that ${}^*A = \widehat{x} = \{y \in \mathbb{I} : y \,{}^*{\in}\, x\}$, $x \in \mathbb{I}$. Since ${}^*\mathfrak{V} = \langle \mathbb{I}; {}^*{\in}, \operatorname{st}\rangle$ satisfies **BST**, there exists a standard set ${}^*b \in \mathbb{I}$ $(b \in \mathbf{V})$ such that $x \subseteq {}^*b$ in ${}^*\mathfrak{V}$. Then ${}^*A = \widehat{x} \subseteq \widehat{{}^*b}$, and

[22] Thus, ${}^*A$ for proper classes is defined differently than for sets (where ${}^*a \in {}^*\mathbf{V}$ for $a \in \mathbf{V}$ is defined just by the application of *). If we still define, say, ${}^\star x = \bigcup_{a\in\mathbf{V},\ a\subseteq x} \widehat{{}^*a}$ for a *set* x, then ${}^\star x = \widehat{{}^\star x} \neq {}^*x$.

[23] Note that this definition does not change the action of ${}^*{\in}$ on $\mathbb{I} = {}^*\mathbf{V}$, because we assumed that no class $Y \subseteq \mathbb{I}$ is an element of $\mathbb{I}$.

further, $a \subseteq b$ for all $a \subseteq A$, thus $A \subseteq a$ and A is a set (in the 'standard' universe of **GB**), a contradiction to the choice of A.

(ii) Assume that $X = x$ is a set (otherwise both sides of the equivalence fail). If Y also is a set, then use the fact that * is an elementary embedding on sets in $\mathbf{V}$. If Y is a proper class and $x \in Y$, then immediately $^*x \in {^*Y}$ (consider $a = \{x\}$). Suppose that $^*x \in {^*Y}$, that is, there exists a set $a \subseteq Y$ with $^*x \in {^*a}$, that is, $^*x \mathrel{^*{\in}} {^*a}$. Then $x \in a \subseteq Y$, as required.

(iii) This is easy: the assumption that there is no set $x \in \mathbb{I} = {^*\mathbf{V}}$ which is a subset of $\mathbb{I}$ easily leads to the result.

(iv) By definition, only sets in $\mathbb{I}$ can be $^*{\in}$-elements in $\mathfrak{E}$. On the other hand, each set $x \in \mathbb{I} = {^*\mathbf{V}}$ $^*{\in}$-belongs to some $y \in \mathbb{I}$.

(v) Let $I \in \mathbf{E}$ be an internal class in $\mathfrak{E}$ in the sense of subsection 2b, that is, there exist a class $C \in \mathbf{S}$ and a set $x \in \mathbb{I}$ such that $I = \{y \in {^*\mathbf{V}} : \langle x, y\rangle \in C\}$ holds in $\mathfrak{E}$. If $I \in \mathbb{I}$, then the result is clear, since $\mathbb{I} \subseteq \mathbf{I}$. Thus, assume that $I \subseteq \mathbb{I}$ is an admissible class. If $C \in \mathbb{S}$, then C is a set in $\mathbb{I}$, and hence even $I \in \mathbb{I}$. If C is an admissible standard class, $C = {^*A}$ for a proper class $A \in \mathbf{K}$, then by definition $I = C[x]$ is an internal class, as required.

(vi) If $X \in \mathbb{I}$, then the result is clear. If X is an admissible external class, then apply the definition of external classes. Lemma 2.19 is proved.

We now proceed to the verification of the axioms. According to Theorem 2.11 and Lemma 2.19(vi), it suffices to prove that the structure $\mathfrak{E}$ satisfies $\mathbf{nGB}^\circ$, or even $\mathbf{nKM}^\circ$ if we start with **KM**.

To check $\mathbf{BST}^{\mathrm{sets}}$ note that by Lemma 2.19 the set universe of $\mathfrak{E}$ sets coincides with $\langle \mathbb{I}; {^*{\in}}, \mathrm{st}\rangle$, but the latter satisfies **BST**.

To check the schemata of $\mathbf{GB}^{\mathrm{st}}$, or $\mathbf{KM}^{\mathrm{st}}$ if **KM** is assumed, note that the standard domain $\langle \mathbf{S}; \in\rangle$ of $\mathfrak{E}$ is isomorphic, by Lemma 2.19, to the ground 'standard' universe $\mathbf{K}$ by means of the map *.

The axiom $\mathbf{S} = \mathbf{S}^{\mathrm{loc}}$. Suppose that $X = {^*A} \in \mathbf{S}$ (A is a proper class) and $y = {^*a} \in \mathbb{S}$ ($a \in \mathbf{V}$). Then $b = A \cap a$ is still a set in the **GB** universe $\mathbf{K}$ by the Intersection axiom. Thus $z = {^*b} \in \mathbb{S}$, and it is easy to see that $X \cap y = z$ in $\mathfrak{E}$. To prove the converse suppose that $X \in \mathbf{E}$ satisfies $\forall^{\mathrm{st}} y\, \exists^{\mathrm{st}} z\, (X \cap y = z)$ in $\mathfrak{E}$. It follows that for any set $a \in \mathbf{V}$ there exists a $b \in \mathbf{V}$ with $b \subseteq a$ such that $X \cap {^*a} = {^*b}$ holds in $\mathfrak{E}$. Clearly, such a $b = b_a$ is unique. In the **GB**-universe $\mathbf{K}$ we define the class $A = \bigcup_{a \in \mathbf{V}} b_a$. Then $Y = {^*A} \in \mathbf{S}$, and $Y \cap {^*a} = {^*(b_a)}$ holds in $\mathfrak{E}$ for every a. Thus $\forall^{\mathrm{st}} y\, (X \cap y = Y \cap y)$ holds in $\mathfrak{E}$, and therefore $X = Y \in \mathbf{S}$ by Boundedness in $\mathbf{BST}^{\mathrm{sets}}$.

The axiom of Extensionality in $\mathfrak{E}$ is proved by equally simple arguments.

Standardization for classes. If $X \in \mathbf{E}$, then $A = \{b \in \mathbf{V} : {^*b} \in X\}$ is a class in $\mathbf{K}$ (by Comprehension in **GB**). It follows easily that the equivalence $^*b \mathrel{^*{\in}} X \Leftrightarrow {^*b} \mathrel{^*{\in}} {^*A}$ holds for any set $b \in \mathbf{V}$. Theorem 2.16 is proved.

Remark 2.20. It is one of the possible interpretations of Theorem 2.16, in the light of the general scheme of its proof, that the non-standard set universe $\mathbb{I}$ of **BST** admits an extension by adjoining classes to a universe of **nGB** (respectively, **nKM**), under the assumption that the class $\mathbb{S}$ of all standard sets admits an extension by classes to a universe of **GB** (respectively, **KM**).

To give this result an exact meaning, one can introduce class theories in the st-$\in$-language which postulate that the set universe $\mathbb{I}$ satisfies **BST**, every proper class X satisfies $X \subseteq \mathbb{S}$, and proper classes, along with standard sets, form a class universe of **GB** (respectively, **KM**). In such a theory one can define standard, internal, and external classes as indicated in subsection 2f. Such an extended structure will interpret **nGB** (or **nKM**) in this intermediate theory, essentially by the same arguments.

However, **BST** itself is not strong enough to define an interpretation of **nGB** (see Theorem 2.6).

2g. Metamathematical corollaries. The main metamathematical results that connect the theories **GB**, **nGB**, **KM**, **nKM** with each other and with the set theories **ZFC** and **BST** are represented on Diagrams 1 and 2, where $A \longrightarrow B$ means that A is reducible to B in this or that sense.

Diagram 1 displays natural reducibilities of 'weaker' theories to 'stronger' ones. The vertical arrows marked by st indicate the interpretation by means of relativization to standard sets and classes. The arrows marked by sets indicate the interpretation by means of relativization to the set subuniverse (in a class universe). The arrows marked by $\subset$ indicate the fact that **GB** and **nGB** are subtheories of **KM** and **nKM**, respectively.

Diagram 2 presents less trivial opposite reducibilities. Crossed arrows indicate the reducibilities which fail simply because impredicative class theories are strictly stronger in the sense of equiconsistency than predicative ones (see the end of subsection 2a). The vertical arrows marked by r indicate the st-type interpretations of non-standard theories in corresponding standard structures in the sense of 1.18. The interpretations are given by Theorem 1.16 for the pair **ZFC**, **BST** (see the discussion in Remark 1.18), and by Theorem 2.16 for the two pairs of class theories.

Finally, the arrows marked by eq, cons (meaning equiconsistency, conservativity) indicate that the class theory is reducible to the set theory in the context of sets-conservativity and equiconsistency, but irreducible in the context of interpretability. This result follows from Theorems 2.2, 2.6.

$$\begin{array}{ccccc}
\mathbf{ZFC} & \xrightarrow{\text{sets}} & \mathbf{GB} & \xrightarrow{\subset} & \mathbf{KM} \\
\big\downarrow{\scriptstyle \text{st}} & & \big\downarrow{\scriptstyle \text{st}} & & \big\downarrow{\scriptstyle \text{st}} \\
\mathbf{BST} & \xrightarrow{\text{sets}} & \mathbf{nGB} & \xrightarrow{\subset} & \mathbf{nKM}
\end{array}$$

DIAGRAM 1

$$\begin{array}{ccccc}
\mathbf{ZFC} & \xleftarrow{\text{eq, cons}} & \mathbf{GB} & \not\longleftarrow & \mathbf{KM} \\
\big\uparrow{\scriptstyle \text{r}} & & \big\uparrow{\scriptstyle \text{r}} & & \big\uparrow{\scriptstyle \text{r}} \\
\mathbf{BST} & \xleftarrow{\text{eq, cons}} & \mathbf{nGB} & \not\longleftarrow & \mathbf{nKM}
\end{array}$$

DIAGRAM 2

In either of the two diagrams the theories **nGB** and **nKM** can be replaced by $\mathbf{nGB}^+$ and $\mathbf{nKM}^+$ or by $\mathbf{nGB} + \Sigma_2^{\text{st}}$ classes and $\mathbf{nKM} + \Sigma_2^{\text{st}}$ classes, respectively

We finish this subsection with the following result strengthening Theorem 2.6.

Corollary 2.21. *The theories* $\mathbf{nGB} + \Sigma_2^{\text{st}}$ classes *and* $\mathbf{nGB}^+$ *still are* sets-*conservative and hence equiconsistent extensions of* **BST**.

Proof. Consider an arbitrary st-$\in$-formula φ. Suppose (the non-trivial direction) that the theory $\mathbf{nGB} + \Sigma_2^{\text{st}}$ classes proves φ^{sets}. By Corollary 1.21 there exists an

$\in$-formula ψ such that $\varphi \Leftrightarrow \psi \Leftrightarrow \psi^{\mathrm{st}}$ is provable in **BST**, and hence $\varphi^{\mathrm{sets}} \Leftrightarrow \psi^{\mathrm{sets}}$ is provable in **nGB**. However, ψ^{sets} is an $\in$-formula, and therefore it is provable in **GB** by Theorem 2.16, and then ψ itself in **ZFC** by Theorem 2.2. Finally, both ψ and hence the given formula φ as well are provable in **BST** by Theorem 1.16(ii). Corollary 2.21 is proved.

Historical and bibliographical remarks to § 2. The history of the Gödel–Bernays class theory **GB** goes back to studies of these mathematicians, and also von Neumann, in 1920–30. The theory is sometimes called the von Neumann–Bernays–Gödel theory, **NBG**. Our abbreviation **GB** is from [35]. Its connection with **ZFC** as in Theorem 2.2 became known slightly later; see [53], [54], [36]. See also [35], Ch. II, § 6, on the **GB**-part of Theorem 2.2. See [55] on the impredicative Kelley–Morse theory **KM**.

The non-standard class theory **nGB** was introduced in [15] under the name **NCT** (non-standard class theory), with a somewhat different (but equivalent) list of axioms. The first formulation of the non-standard theory **nKM** was given in [20].

The main results of this section, those on the reducibility of non-standard class theories to standard ones, in particular, Theorem 2.16 along with its metamathematical corollaries in subsection 2g, were obtained by Andreev and Gordon [15] in the part related to the Gödel–Bernays theories, and by Hrbáček [20] in the part related to the theories of **KM**-type.[24] The Σ_2^{st} definition of external classes in subsection 2f (compare with Theorem 1.19) plays a key role in these results. Hrbáček must be given credit for realizing that such a definition is applicable not only for **GB**-type theories, where everything is clear enough, but also to the Kelley–Morse theories, where this fact initially seemed rather improbable, because it had been assumed that non-standard theories of **KM**-type must be stronger, in the sense of consistency, than the 'standard' **KM** by analogy with some results in [42] for non-standard systems of higher-order Peano arithmetic.

The notion of a 'semiset' was introduced in studies on alternative set theory **AST** (see [52], [51]). The notion of standard size appeared in studies on foundations of non-standard analysis (perhaps first in [10]). There is another related notion: a class of $\mathbb{S}$-*size* by Kawai [12], [13]: this is any class X such that there exists a class F which is a map from $\mathbb{S}$ onto X. The axiom of Σ_2^{st} classes corresponds to the axiom of chromatic classes in a different (but essentially equivalent) version of axiomatization of the predicative non-standard class theory in [15].

Problem 2.22. Add Transfer for all (impredicative) $\in$-formulae to **nGB**. Will such an extension of **nGB** still be equiconsistent with **GB**?

Problem 2.23. Recall Footnote 13. Let *locally internal* refer to any class X such that $X \cap y$ is internal for any set y. Nothing is known regarding the status of the hypothesis $\mathbf{I} = \mathbf{I}^{\mathrm{loc}}$ (all locally internal classes are internal) in non-standard class theories.

[24] To unify the exposition, the original arguments of [15], [20] are essentially modified here on the basis of a scheme developed earlier by Kanovei and Reeken for **BST** in [21], [22], [25]. Technical details of the modification, in particular, Theorems 2.7 and 2.15, were elaborated by Kanovei after reading Hrbáček's preprint [20] in August 2004.

§ 3. Adjoining external sets

This section is devoted to Problem 3 in the Introduction: how can 'external' sets be adjoined to a non-standard universe of 'internal' sets? To begin with, we specify what basically is meant here.

We recall that set universes of such non-standard theories as **BST** or **IST** (or the set universe in the class theories **nGB** and **nKM**) satisfy Transfer in the form $\Phi \Leftrightarrow \Phi^{\mathrm{st}}$ (where Φ is any $\in$-formula) and Separation only in the $\in$-language, but not in the extended st-$\in$-language. Therefore, an st-$\in$-definable part of a set is itself not necessarily a set. (Lemma 1.14 presents a simple example.) Non-standard set-theoretic universes of this type, as well as sets in them, are usually called 'internal' (with or without quotation marks) for historical reasons. Another type of 'external' universes is characterized by exactly the opposite properties, that is, Separation holds in the st-$\in$-language but Transfer does not.[25] 'External' non-standard universes usually contain subuniverses of 'internal' type: under a suitable axiomatization such is the class of all *internal* sets, that is, those which are elements of standard sets.

Here we solve the opposite problem: how can one extend an 'internal' universe, in this sense, to an 'external' one? We prove (Theorem 3.24) that the **BST** universe admits an extension to a universe of **HST**, a well-known non-standard set theory of 'external' type. (See an informal comment in the end of subsection 3e.) Moreover (Theorem 3.21), if the given **BST** universe is the class of all sets in a structure satisfying **nGB** or **nKM**, then this superstructure of classes can be extended to a suitable class superstructure over the extended set universe of **HST**.

3a. Structure of the 'external' universe of Hrbáček's theory. With the goal of extending the **BST** universe to a universe of a non-standard set theory with Separation in the st-$\in$-language, we start by outlining the structure of the latter. Let us begin with the axiomatization. Several reasonable non-standard theories with Separation in the st-$\in$-language are known. Of them, *Hrbáček's set theory* **HST** attracts the most attention. The axioms of **HST** are grouped in three parts.

The **first** part contains the axioms of Extensionality, Pair, Union, and Infinity in their usual **ZFC** forms, together with the Separation and Collection (and therefore Replacement as well) schemata for all st-$\in$-formulae. The **ZFC** axioms of Power Set, Choice, and Regularity are <u>not</u> included; in fact they contradict **HST** (see [25], Ch. 1.3). However, Regularity and Choice will be included in special weaker forms (Regularity over $\mathbb{I}$, Standard Size Choice, Dependent Choice; see below) important in the context of **HST**.

The **second** part contains the axioms for standard and internal sets. As usual, the indices $^{\mathrm{st}}$ and $^{\mathrm{int}}$ mean relativization to the classes

$$\mathbb{S} = \{x : \mathrm{st}\, x\} \qquad \text{(standard sets);}$$
$$\mathbb{I} = \{x : \exists^{\mathrm{st}}\, y\, (x \in y)\} \qquad \text{(internal sets).}$$

[25] Some other versions of the definitions of the notions of 'external' and 'internal' (set or universe), known in the foundations of non-standard analysis, are essentially equivalent to the ones given above in that they express similar ideas in the areas of their applications.

ZFC$^{\mathrm{st}}$: All formulae of the form Φ^{st}, where Φ is a **ZFC** axiom in the $\in$-language.

Transfer: All formulae of the form $\Phi^{\mathrm{st}} \Leftrightarrow \Phi^{\mathrm{int}}$, where Φ is a closed $\in$-formula with standard parameters.

Transitivity of $\mathbb{I}$: $\forall\, x \in \mathbb{I}\, \forall\, y \in x\ (y \in \mathbb{I})$.

Regularity over $\mathbb{I}$: Every non-empty set X contains an element $x \in X$ such that $x \cap X \subseteq \mathbb{I}$. (The full Regularity of **ZFC** requires $X \cap x = \varnothing$.)

Standardization: $\forall\, X\, \exists^{\mathrm{st}}\, Y\ (X \cap \mathbb{S} = Y \cap \mathbb{S})$.

These two groups of axioms organize the **HST** set universe $\mathbb{H}$ as a **ZF**-type (but without the Power Set axiom) superstructure over the transitive class $\mathbb{I} \subseteq \mathbb{H}$ of all internal sets, where the latter satisfies **BST**.

In addition to the **BST** technique restricted to the domain $\mathbb{I}$, **HST** admits the $*$-technique of model-theoretic, 'robinsonian' non-standard analysis. Namely, the class $\mathbb{WF}$ of all well-founded sets[26] satisfies **ZFC**, and there exists an st-$\in$-definable $\in$-isomorphism ${}^*\colon \mathbb{WF} \xrightarrow{\text{onto}} \mathbb{S}$. By Transfer, this isomorphism is an elementary embedding of $\langle \mathbb{WF}; \in \rangle$ into $\langle \mathbb{I}; \in \rangle$. Namely, ${}^*x \in \mathbb{S}$ is defined for each $x \in \mathbb{WF}$ by $\in$-induction in such a way that ${}^*x$ is equal to the unique standard set satisfying ${}^*x \cap \mathbb{S} = \{{}^*y : y \in x\}$. See more details in Ch. 1 of [25].

Finally, the **third** part of **HST** includes the following variations of axioms in subsection 2e relating to sets of standard size. Saturation, one of them, supports typically 'non-standard' arguments, while the two choice axioms play important role in some constructions.

Standard Size Choice: If X is a set of standard size and P is any set, then there exists a function f such that $X \subseteq \operatorname{dom} f$ and, for any $x \in X$, $\exists\, y\, (\langle x, y \rangle \in P) \Rightarrow \langle x, f(x) \rangle \in P$.

Saturation: If $H \subseteq \mathbb{I}$ is a $\cap$-closed set of standard size not containing $\varnothing$, then the intersection $\bigcap H$ is non-empty.

Dependent Choice: For any sets X and $R \subseteq X \times X$ with $\operatorname{dom} R = X$ there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that $\langle x_n, x_{n+1} \rangle \in R$ for all $n \in \mathbb{N}$.

By analogy to Definition 2.13, a *set of standard size* in **HST** is any set X such that there exist a standard set x and a bijection F (a set) from $x \cap \mathbb{S}$ onto X. We note that in **HST** all semisets are sets, because Separation is included for all st-$\in$-formulae.

As usual, $\mathbb{N}$ denotes the set of all natural numbers in Dependent Choice. Simple arguments show that $\mathbb{N} = (\mathbb{N})^{\mathbb{WF}}$, that is, natural numbers can be equivalently defined in the **ZFC** subuniverse $\mathbb{WF}$ of all well-founded sets. However, there is a bigger set ${}^*\mathbb{N} \in \mathbb{I}$ of all $\mathbb{I}$-*natural* (or *hypernatural*) numbers, and $\mathbb{N} = {}^*\mathbb{N} \cap \mathbb{S} = {}^*\mathbb{N} \cap \mathbb{WF}$ is an initial segment of ${}^*\mathbb{N}$. Generally, the pair of sets $\mathbb{N}$, ${}^*\mathbb{N}$ in **HST** corresponds to the pair $\mathbb{N} \cap \mathbb{S}$, $\mathbb{N}$ in **BST**.

Let **HST**$^+$ be the theory obtained by adding the following two axioms of mostly technical character to **HST**. On their role see below. We recall that E_p and $\mathbf{C}_0$ are defined in subsection 2e.

Σ_2^{st} *sets*: For any set $x \subseteq \mathbb{I}$ there exists a code $p \in \mathbf{C}_0$ such that $x = E_p$.

$\mathbb{I}$-*image*: For any set d there exist a set $r \in \mathbb{I}$ and a function f (also a set) defined on r such that $d \subseteq \operatorname{ran} f$.

[26] A set x is *well-founded*, or $\mathbb{WF}$ for brevity, if there exists a transitive set T such that $x \subseteq T$ and the relation $\in\restriction T$ is well-founded, that is, every non-empty set $Y \subseteq T$ contains an $\in$-minimal element.

Metamathematically, the theories **HST** and $\mathbf{HST}^+$ are related to **BST** in approximately the same way that the latter is related to **ZFC** in the framework of Theorem 1.16. In particular, **HST** and $\mathbf{HST}^+$ are equiconsistent with **BST** (Theorem 3.24 below). This reduction of **HST** and $\mathbf{HST}^+$ to **BST** is quite complicated. On the contrary, the opposite direction is rather elementary. The simple proof of the following result is left to the reader.

Proposition 3.1. *The structure* $\langle \mathbb{I}; \in, \mathrm{st}\rangle$ *is an interpretation of* **BST** *in* **HST**. *In other words*, **HST** *proves* φ^{int} *for any axiom* φ *of* **BST**.

3b. Non-standard class theories on the basis of Hrbáček's theory. The next definition introduces non-standard theories which characterize their universes as class superstructures (of Gödel–Bernays type or Kelley–Morse type) over the set universe of **HST** or $\mathbf{HST}^+$, in just the same way that the universes of **nGB** and **nKM** are class superstructures over the **BST** universe.

Definition 3.2. **GB/HST** (that is, a **GB**-type extension of **HST**) is a class theory in the st-$\in$-language containing the $\mathbf{HSTS}^{\mathrm{sets}}$ schema, that is, all **HST** axioms formally relativized to the class $\{x : \mathrm{set}\, x\}$ of all sets, and, as in Definition 2.3, Intersection, Extensionality, Collection, the schema of Comprehension for predicative st-$\in$-formulae, Standardization for classes, and the axiom $\mathbf{S} = \mathbf{S}^{\mathrm{loc}}$ for all classes $X \subseteq \mathbb{I}$.[27]

The theory **KM/HST** (that is, a **KM**-type extension of **HST**) is defined similarly, with the addition of 1) and 2) of Definition 2.3.

The theories $\mathbf{GB}/\mathbf{HST}^+$ and $\mathbf{KM}/\mathbf{HST}^+$ are defined similarly, with the only difference that $\mathbf{HSTS}^{\mathrm{sets}}$ is replaced by $(\mathbf{HST}^+)^{\mathrm{sets}}$. This is obviously just the same as adding the axioms of $\mathbb{I}$-image and Σ_2^{st} sets to **GB/HST** and **KM/HST**, respectively.

When arguing in these theories, we keep the notation introduced in Definition 2.4 for subuniverses. Note that the equalities $\mathbb{I} = \mathbb{H}$ and $\mathbf{E} = \mathbf{H}$ are not true any more. On the other hand, the following holds in **HST**: $\mathbb{S} = \mathbf{S} \subsetneqq \mathbb{I} = \mathbf{I} \subsetneqq \mathbb{H} = \mathbf{E} = \mathbf{H}$. The next easy result (the proof is omitted to save space) demonstrates the type of interrelations between the universes of the theories in Definition 3.2 and the theories **nGB**, **nKM**. Recall that $\mathbf{E}$ is the domain of all classes $X \subseteq \mathbb{I}$, that is, classes of internal sets.

Proposition 3.3. *The structure* $\mathfrak{E} = \langle \mathbf{E}; \in, \mathrm{st}\rangle$ *is an interpretation of* **nGB** *in* **GB/HST**, *that is*, **GB/HST** *proves* $\varphi^{\mathfrak{E}}$ *for any axiom* φ *of* **nGB**.[28]

In addition, $\mathfrak{E}$ *is an interpretation of* $\mathbf{nGB}^+$ *in* $\mathbf{GB}/\mathbf{HST}^+$ *and an interpretation of* $\mathbf{nKM}^+$ *in* $\mathbf{KM}/\mathbf{HST}^+$.

Our metamathematical analysis of the theories in Definition 3.2 concentrates on the theories $\mathbf{GB}/\mathbf{HST}^+$ and $\mathbf{KM}/\mathbf{HST}^+$, rather than on the pair **GB/HST** and **KM/HST**, which would seem more natural at first glance. The reason for this is the following.

[27] The axiom $\mathbf{S} = \mathbf{S}^{\mathrm{loc}}$ fails in **HST** even for sets, because there exist non-empty sets that do not intersect $\mathbb{I}$, while any standard class Y satisfies $Y \subseteq \mathbb{I}$.

[28] Note that any **GB/HST** set $x \in \mathbb{H} \setminus \mathbb{I}$, $x \subseteq \mathbb{I}$, becomes a proper class, or, more precisely, a semiset in $\mathfrak{E}$.

In extending a universe of, say, **nGB** to a universe of **GB/HST**, we necessarily force all semisets $X \subseteq \mathbb{I}$ to become sets. Thus, to obtain Comprehension in the extension we need this schema to hold already in **nGB** for semisets in this or that form. However, semisets are technically classes in **nGB**, so it seems that there is no direct method to introduce Comprehension for them. Fortunately, there is an indirect method. It axiomatically introduces a parametrization of all semisets $X \subseteq \mathbb{I}$ by sets proper in $\mathbb{I}$. In the presence of the axiom of Σ_2^{st} semisets such a parametrization is defined in **nGB** by the map $p \mapsto E_p$.

This compels us to adjoin the axiom of Σ_2^{st} semisets to **nGB**, leading to the theory $\mathbf{nGB}^+$, and, accordingly, the theory **GB/HST** with Σ_2^{st} semisets by Proposition 3.3. However, all semisets $X \subseteq \mathbb{I}$ are sets in **GB/HST**, and therefore we have to adjoin the axiom of Σ_2^{st} sets to **HST**. Clearly, there exist other suitable parametrizations, but the one defined by the map $p \mapsto E_p$ and the axiom of Σ_2^{st} semisets is obviously the most convenient.

The other extra axiom $\mathbb{I}$-image is also added for a certain reason, connected with the axiom of Class Collection in **nKM** in Proposition 3.3. The point is that one cannot directly infer Class Collection in $\mathfrak{C}$ from Class Collection in the **KM/HST** universe. Indeed, the set r given by Class Collection in the universe may have nothing in common with the domain $\mathbb{I}$. In this case, $\mathbb{I}$-image immediately straightens out the picture.

After thereby establishing the necessity of adding the axioms of $\mathbb{I}$-image and Σ_2^{st} semisets (or Σ_2^{st} sets for the theory $\mathbf{HST}^+$), we turn to the metamathematical connections of the non-standard class theories $\mathbf{GB/HST}^+$ and $\mathbf{KM/HST}^+$ with the non-standard set theory $\mathbf{HST}^+$ from the one side, and with the class theories $\mathbf{nGB}^+$ and $\mathbf{nKM}^+$ from the other side. The first aspect manifests itself as follows.

Theorem 3.4. *The theory* $\mathbf{GB/HST}^+$ *is a* sets-*conservative extension of* $\mathbf{HST}^+$ *in the same sense as in Theorem* 2.6 (*for all* st-$\in$-*formulae* φ), *and thus* $\mathbf{GB/HST}^+$ *is equiconsistent with* $\mathbf{HST}^+$, **HST**, **BST**, and **ZFC**.

However, $\mathbf{GB/HST}^+$ *is not interpretable in* $\mathbf{HST}^+$, *while* $\mathbf{KM/HST}^+$ *is not even equiconsistent with* $\mathbf{HST}^+$.

Proof. The structure $\langle \mathbb{H}; \in, \mathrm{st}\rangle$ is clearly an interpretation of the theory $\mathbf{HST}^+$ in $\mathbf{GB/HST}^+$. On the other hand, quite like the proof of Theorem 2.6, every model of $\mathbf{HST}^+$ can be extended by adding classes to a model of $\mathbf{GB/HST}^+$ (where the given model of $\mathbf{HST}^+$ remains the class of all sets). This implies the sets-conservativity and equiconsistency of $\mathbf{HST}^+$. To prove the other assertions it suffices to take on trust the fact that $\mathbf{HST}^+$ admits an interpretation in **BST** (see below). Using this assumption, we show that, for instance, $\mathbf{GB/HST}^+$ is not interpretable in $\mathbf{HST}^+$. Indeed, otherwise $\mathbf{GB/HST}^+$ is also interpretable in **BST** (by the assumption), and then **nGB** is interpretable in **BST** (by Proposition 3.3) in contradiction with Theorem 2.6. Theorem 3.4 is proved.

As far as the relations with **nGB** and **nKM** are concerned, the result in one direction follows from Proposition 3.3. The reduction in the other direction will require much more effort.

3c. Cumulative construction of sets and classes. *We argue in* **nGB** *in this subsection.* Additional axioms of the theories $\mathbf{nGB}^+$ and $\mathbf{nKM}^+$ will be introduced whenever necessary, and that will be explicitly indicated.

Our main goal will be to define an extension of the whole **nGB** universe **I** to a class universe **H**. Under certain additional assumptions the latter will satisfy the axioms of $\mathbf{GB}/\mathbf{HST}^+$ or $\mathbf{KM}/\mathbf{HST}^+$.

At the same time, the class $\mathbb{I} = \{x : \text{set}\, x\}$ of all sets in **nGB** will be extended to a set-theoretic universe $\mathbb{H}$, a part of **H**, satisfying the axioms of **HST** (or $\mathbf{HST}^+$ under certain additional assumptions). In accordance with the general structure of an **HST** universe described in subsection 3a, $\mathbb{I}$ is the lowest level of the $\in$-hierarchy of $\mathbb{H}$. The next level will consist of semisets. Then to continue the construction of an **HST** universe on the basis of $\mathbb{I}$ we have to define collections of semisets, for instance, $\{X_p : p \in Y\}$, where X, Y are semisets and $X_p = \{x : \langle p, x\rangle \in X\}$, and so on. One can hardly hope to carry out such a construction in **nGB** in a direct form, because proper semisets are not members of any other objects in **nGB**, and hence the level immediately above the semisets cannot be defined explicitly. However, the construction can be carried out in a coded form.

We employ a known method of coding of the cumulative construction of sets by means of well-founded trees. In general form the method works as follows. Suppose that T is a well-founded tree (that is, it does not contain an infinite branch), and in addition a set $f(t)$ is attached to any endpoint (a maximal element) $t \in T$. In this case one can define an object $A_{Tf}(t)$ for each $t \in T$ in such a way that $A_{Tf}(t) = f(t)$ for all endpoints t and $A_{Tf}(t) = \{A_T(s) : s \text{ is an immediate successor of } t\}$ for non-endpoints t. We put $A_{Tf} = A_{Tf}(\Lambda)$, where Λ is the root of T.

We present this construction in a form reflecting special characteristics of **nGB**.

Blanket convention 3.5. In partial correction of the conventions in subsection 2a on the use of letters, lower-case letters (including boldface ones) can denote semisets that are not necessarily sets, depending on the context.

Unless otherwise explicitly specified, the word 'sequence' will mean a set which is a sequence, while the words 'function' or 'tree' will typically denote a class which is a function or a tree.

A. On finiteness and well-orderability in relation to classes. This technical digression is necessary to give a rationale for some constructions below. We recall that a *well-ordered* set is any linearly ordered set such that any non-empty subset of it has a least element. For instance, in **ZFC** every ordinal is well-ordered by the relation $\in$, while the set $\mathbb{N} = \omega$ of all natural numbers is well-ordered by its natural order (that formally coincides with $\in$). This remains true in **BST** (since these facts are expressed by $\in$-formulae, and **BST** implies **ZFC** in the $\in$-language), and hence in **nGB** as well (because this theory is a sets-conservative extension of **BST**).

The theory **nGB** enables us to look at the issue from another angle. Indeed, there exist classes, even subclasses of sets (that is, semisets), that are not sets. In particular, $\mathbb{N}$ contains the non-empty subsemiset $\mathbb{N} \setminus \mathbb{S}$ (of all non-standard positive integers), which does not have a least element (see Lemma 1.14). This leads us to the following definition. A linearly ordered set or class is *c-well-ordered* (the prefix 'c' is from 'class') if any of its non-empty subclasses has a least element.

Proposition 3.6 (**nGB**).

(i) *The class* $\mathrm{Ord} \cap \mathbb{S}$ *of all standard ordinals and hence the semiset* $\mathbb{N} \cap \mathbb{S}$ *are c-well-ordered.*

(ii) *For any semiset* $X \subseteq \mathrm{Ord}$ *there exists a least standard ordinal strictly bigger than all ordinals in* X.

(iii) *Any sequence (a semiset)* $\langle x_1, \dots, x_n \rangle$ *of length* $n \in \mathbb{N} \cap \mathbb{S}$ *with all its terms* x_i *being sets is itself a set.*

Proof. (i) Prove that every non-empty class $X \subseteq \mathrm{Ord} \cap \mathbb{S}$ has a least element. By eStandardization, there exists a standard class Y such that $Y \cap \mathbb{S} = X \cap \mathbb{S} = X$. The least ordinal ξ contained in Y is standard by Transfer. It follows that ξ is the least element in X.

The result for $\mathbb{N} \cap \mathbb{S}$ verifies the induction on the standard natural numbers in the case when the induction scheme is described by an arbitrary st-$\in$-formula that, for instance, refers to semisets. In particular, to prove (iii) note that by definition, $\langle x_1, \dots, x_n, x_{n+1} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$, and hence if $\langle x_1, \dots, x_n \rangle$ and x_{n+1} are sets, then so is $\langle x_1, \dots, x_n, x_{n+1} \rangle$. Proposition 3.6 is proved.

B. Trees. Let Seq denote the class of all sequences $\langle a_1, \dots, a_n \rangle$ (of any sets $a_i \in \mathbb{I}$) of a standard finite length. For any $t \in \mathrm{Seq}$ and any set a, $t^\wedge a$ is the sequence in Seq obtained by adjoining a to t as the rightmost term. The notation $a^\wedge t$ is understood similarly. The expression $t' \subseteq t$ means that the sequence $t \in \mathrm{Seq}$ *extends* $t' \in \mathrm{Seq}$ (possibly $t' = t$), and $t' \subset t$ means that t is a proper extension of t' (thus, $t' \neq t$); $\langle a \rangle$ is the sequence with only the one term a, and Λ is the empty sequence.

- A *tree* is any non-empty class $T \subseteq \mathrm{Seq}$ such that $t \in T \Rightarrow t' \in T$ holds for any pair of sequences $t', t \in \mathrm{Seq}$ with $t' \subseteq t$.
- $\mathrm{Max}\, T$ is the class of all $\subseteq$-*maximal* elements $r \in T$.
- If $t \in T$, then put $\mathrm{Succ}_T(t) = \{a : t^\wedge a \in T\}$ (also a class).
- Put $\mathrm{Min}\, T = \mathrm{Succ}_T(\Lambda) = \{a : \langle a \rangle \in T\}$; then $\mathrm{Min}\, T = \varnothing \Leftrightarrow T = \{\Lambda\}$.
- $s \in T$ is called a *type*-1 *element* in T if the class $T_s = \{t \in T : s \subseteq t\}$ is a semiset (or a set, as a particular case), and a *type*-2 *element* otherwise (that is, in the case when T_s is not covered by a set).

We say that a tree T is *c-well-founded* if every non-empty class $T' \subseteq T$ contains an element that is $\subseteq$-maximal in T'. In this case, following well-known **ZFC** patterns, we would like to define a rank function on T. In view of Proposition 3.6, such a function has to assume values in $\mathrm{Ord} \cap \mathbb{S}$ rather than in Ord. In addition, since T is not necessarily a semiset, we have to reserve a symbol ∞ supposed to be bigger than all standard ordinals.

For any class $X \subseteq (\mathrm{Ord} \cap \mathbb{S}) \cup \{\infty\}$, we define $\sup^{\mathbb{S}} X = \infty$ if either $\infty \in X$ or $X \subseteq \mathrm{Ord} \cap \mathbb{S}$ is unbounded in $\mathrm{Ord} \cap \mathbb{S}$. But $\sup^{\mathbb{S}} X$ is equal to the least standard ordinal strictly bigger than all ordinals in X whenever $X \subseteq \mathrm{Ord} \cap \mathbb{S}$ is bounded in $\mathrm{Ord} \cap \mathbb{S}$, that is, is a semiset. (We refer to Proposition 3.6.)

Lemma 3.7. (**nKM**) *If* $T \subseteq \mathrm{Seq}$ *is a c-well-founded tree, then there exists a unique function* $t \mapsto |t|_T^{\mathbb{S}}$ *from* T *to* $(\mathrm{Ord} \cap \mathbb{S}) \cup \{\infty\}$ *such that* $|t|_T^{\mathbb{S}} = \sup^{\mathbb{S}}_{t^\wedge a \in T} |t^\wedge a|_T^{\mathbb{S}}$ *for each* $t \in T$.

(**nGB**+sem-bounded Comprehension) *The same holds under the extra assumption that all* $t \in T$ *except perhaps for* Λ *are type-*1 *elements in* T.

The map $t \mapsto |t|^{\mathbb{S}}_T$ is a *rank function* on the (c-well-founded) tree T. Note that $|t|^{\mathbb{S}}_T = 0$ whenever $t \in \operatorname{Max} T$. We put $|T|^{\mathbb{S}}_T = |\Lambda|^{\mathbb{S}}_T$ (the *height* of T).

Proof. We consider the **nKM** case in more detail and prove[29] that for every $s \in T$ there exists a unique class-function F_s from $T_s = \{t \in T : s \subseteq t\}$ into $(\mathrm{Ord} \cap \mathbb{S}) \cup \{\infty\}$ such that $F_s(t) = \sup^{\mathbb{S}}{}_{t^\wedge a \in T} F_s(t^\wedge a)$ for all $t \in T_s$. Indeed, the class S of all $s \in T$ satisfying this uniqueness and existence assertion is *inductive* in T in the sense that $s \in S$ holds whenever $s^\wedge a \in S$ holds for all $s^\wedge a \in T$; in particular, every $s \in \operatorname{Max} T$ belongs to S. (Indeed, put $F_s(t) = F_{s^\wedge a}(t)$ whenever $s^\wedge a \subseteq t$, and in addition, $F_s(s) = \sup^{\mathbb{S}}{}_{s^\wedge a \in T} F_{s^\wedge a}(s^\wedge a)$.) However, it follows from the c-well-foundedness of T that every inductive class $S \subseteq T$ is equal to T. (Otherwise a maximal element of $T \setminus S$ immediately leads to a contradiction.)

In the case when only **nGB** with sem-bounded Comprehension is assumed, the argument does not work, because the definition of F_s under the assumption that all the $F_{s^\wedge a}$ have been defined involves Impredicative Comprehension. However, if $s \in T$ is a type-1 element, that is, $T_s = \operatorname{dom} F_s$ is a semiset, then F_s is still a semiset by Collection. Therefore, under our assumptions the class S can be defined by a sem-bounded st-$\in$-formula (that is, all the quantifiers are bounded by semisets; see subsection 2.13). It follows that Impredicative Comprehension can be replaced by sem-bounded Comprehension. Lemma 3.7 is proved.

C. Codes. The next definition somewhat artificially replaces a pair $\langle T, f\rangle$ (as above at the beginning of subsection 3c) by f alone, now denoted by $\mathbf{X}$. The reason is that pairs of semisets do not exist in **nGB**. On the other hand, any c-well-founded tree can be recovered provided that we know its maximal elements.

Definition 3.8 (nGB). An *A-code* (from: *assembling*) is any function $\mathbf{X}\colon D \to \mathbb{I}$ (a class) defined on a class $D \subseteq \mathrm{Seq}$ of pairwise $\subseteq$-incomparable sequences and such that $T[\mathbf{X}] = \{t \in \mathrm{Seq} : \exists\, t' \in \operatorname{dom} \mathbf{x}\, (t \subseteq t')\}$ is a c-well-founded tree. (Then $\operatorname{Max} T[\mathbf{X}] = D = \operatorname{dom} \mathbf{x}$.)

An A-code $\mathbf{X}$ is *regular* if $\{\mathbf{X}(t^\wedge a) : t^\wedge a \in T[\mathbf{X}]\} \notin \mathbb{I}$ (that is, this class is not a set) for all $t \in T[\mathbf{X}]$ such that $|t|^{\mathbb{S}}_T = 1$ (that is, $t \notin \operatorname{Max} T[\mathbf{X}]$ but every $t^\wedge a \in T[\mathbf{X}]$ belongs to $\operatorname{Max} T[\mathbf{X}]$).

Denote by $\mathbf{H}$ the family of all regular codes $\mathbf{X}$ such that every element $s \in T[\mathbf{X}]$ except perhaps for Λ is a type-1 element in $T[\mathbf{X}]$ (that is, $\{a : s^\wedge a \in \mathbf{X}\}$ is a semiset). Denote by $\mathbf{h}$ the family of all codes $\mathbf{X} \in \mathbf{H}$ such that the tree $T[\mathbf{X}]$ (and then also $\mathbf{X}$ itself) is a semiset. Codes in $\mathbf{h}$ will be denoted by lower-case boldface letters $\mathbf{x}, \mathbf{y}, \ldots$.

The regularity requirement does not involve any loss of generality. Indeed, if $|t|^{\mathbb{S}}_T = 1$ and $y = \{\mathbf{X}(t^\wedge a) : t^\wedge a \in T[\mathbf{X}]\} \in \mathbb{I}$ (that is, y is a set), then a reduced code $\mathbf{Y}$ can be defined so that $T[\mathbf{Y}] = T[\mathbf{X}] \setminus \{t^\wedge a : t^\wedge a \in T[\mathbf{X}]\}$ and $\mathbf{Y}(t) = y$.

Interpretation of the cumulative coding is based on the following definition.

Definition 3.9 (nGB). Suppose that $\mathbf{X} \in \mathbf{H}$. Put $\mathsf{F}_{\mathbf{X}}(t) = \mathbf{X}(t)$ whenever $t \in \operatorname{Max} T[\mathbf{X}]$, and $\mathsf{F}_{\mathbf{X}}(t) = \{\mathsf{F}_{\mathbf{X}}(t^\wedge a) : t^\wedge a \in T[\mathbf{X}]\}$ whenever $t \notin \operatorname{Max} T[\mathbf{X}]$.

Put $\mathsf{A}_{\mathbf{X}} = \mathsf{F}_{\mathbf{X}}(\Lambda)$ (the class coded by $\mathbf{X}$).

[29] The method of this proof is called *well-founded induction.*

Lemma 3.10 (**nGB**). *Assume that* $\mathbf{X} \in \mathbf{H}$ *and* $t \in T[\mathbf{X}]$. *Then* $\mathsf{F}_{\mathbf{X}}(t)$ *is a set when* $|t|^{\mathbb{S}}_{T[\mathbf{X}]} = 0$, *is a proper class when* $|t|^{\mathbb{S}}_{T[\mathbf{X}]} \leqslant 1$, *and is undefined* (by Definition 3.9) *when* $|t|^{\mathbb{S}}_{T[\mathbf{X}]} \geqslant 2$. *In particular,* $\mathsf{A}_{\mathbf{X}}$ *is defined when* $|T[\mathbf{x}]|^{\mathbb{S}} \leqslant 1$ *and undefined otherwise.*

Proof. The values $\mathsf{F}_{\mathbf{X}}(t) = \mathbf{X}(t) \in \mathbb{I}$ are directly defined for all $t \in \operatorname{Max} T[\mathbf{X}]$. If $t \in T[\mathbf{X}]$ and $|t|^{\mathbb{S}}_{T[\mathbf{X}]} = 1$, then all sequences $t^\wedge a \in T[\mathbf{X}]$ belong to $\operatorname{Max} T[\mathbf{X}]$, and hence $\mathsf{F}_{\mathbf{X}}(t) = \{\mathsf{F}_{\mathbf{X}}(t^\wedge a) : t^\wedge a \in T[\mathbf{X}]\}$ is a correctly defined class, but not a set by the regularity of $\mathbf{X}$. If $|t|^{\mathbb{S}}_{T[\mathbf{X}]} \geqslant 2$, then Definition 3.9 yields a family containing proper classes, and that is impossible in **nGB**. Lemma 3.10 is proved.

The case $|T[\mathbf{x}]|^{\mathbb{S}} \leqslant 1$ is scrutinized in items 1) and 2) of the following example.

Example 3.11. 1) For any set $x \in \mathbb{I}$ we define an A-code ${}^{\mathrm{a}}x \in \mathbf{h}$ so that $T[{}^{\mathrm{a}}x] = \{\Lambda\}$ and ${}^{\mathrm{a}}x(\Lambda) = x$, that is, formally, ${}^{\mathrm{a}}x = \{\langle \Lambda, x\rangle\}$. In this case the function $\mathsf{F}_{{}^{\mathrm{a}}x}(\,\cdot\,)$ obviously exists, and $x = \mathsf{A}_{{}^{\mathrm{a}}x}$.

2) If X is a class but not a set, then the definition of ${}^{\mathrm{a}}X$ as in 1) does not work, since proper classes cannot be elements of other classes. In this case define an A-code ${}^{\mathrm{a}}X$ so that $T[{}^{\mathrm{a}}X] = \{\Lambda\} \cup \{\langle a\rangle : a \in X\}$ and ${}^{\mathrm{a}}X(\langle a\rangle) = a$ for all $a \in X$. Thus, $\operatorname{Max} T[{}^{\mathrm{a}}X] = \{\langle a\rangle : a \in X\}$ and $\operatorname{Min} T[{}^{\mathrm{a}}X] = \{a : a \in X\} = X$. The regularity of ${}^{\mathrm{a}}X$ follows from the assumption that X is not a set. Moreover, ${}^{\mathrm{a}}X \in \mathbf{H}$, and if X is a semiset, then ${}^{\mathrm{a}}X \in \mathbf{h}$. The class-function $\mathsf{F}_{{}^{\mathrm{a}}X}(\,\cdot\,)$ exists for any class X, and $X = \mathsf{A}_{{}^{\mathrm{a}}X}$.

Definition 3.12. Suppose that $\mathbf{X} \in \mathbf{H}$ and $b \in \operatorname{Min} T[\mathbf{X}]$. We define a code $\mathbf{Y} = \mathbf{X}\big|_b$ so that $T[\mathbf{X}\big|_b] = \{t : b^\wedge t \in T[\mathbf{X}]\}$ and $\mathbf{X}\big|_b(t) = \mathbf{X}(b^\wedge t)$ for $t \in \operatorname{Max} T[\mathbf{X}]$.

Clearly, $\mathbf{X}\big|_b \in \mathbf{H}$, even $\in \mathbf{h}$, and $\operatorname{Max} T[\mathbf{X}\big|_b] = \{t : b^\wedge t \in \operatorname{Max} T[\mathbf{X}]\}$.

D. Bisimulations. In spite of the restrictions posed by Lemma 3.10, there exists a method that enables us to adequately define in **nGB** the relations of equality and membership between $\mathsf{A}_{\mathbf{X}}$ and $\mathsf{A}_{\mathbf{Y}}$ ($\mathbf{X}, \mathbf{Y} \in \mathbf{H}$) even in the case when the sets or classes $\mathsf{A}_{\mathbf{X}}, \mathsf{A}_{\mathbf{Y}}$ themselves are not defined. The method is based on the following definition.

Definition 3.13. A class-function $j : T[\mathbf{X}] \times T[\mathbf{Y}] \to \{0, 1\}$ is a *bisimulation* for A-codes $\mathbf{X}$, $\mathbf{Y}$ if it satisfies the following.

1*. If $t \in \operatorname{Max} T[\mathbf{X}]$ and $r \in \operatorname{Max} T[\mathbf{Y}]$, then $j(t, r) = 1$ is equivalent to $\mathbf{X}(t) = \mathbf{Y}(r)$.

2*. If $t \in \operatorname{Max} T[\mathbf{X}]$ but $r \notin \operatorname{Max} T[\mathbf{Y}]$ or, conversely, $t \notin \operatorname{Max} T[\mathbf{X}]$ but $r \in \operatorname{Max} T[\mathbf{Y}]$, then $j(t, r) = 0$. (In the absence of regularity of the codes this condition would be much more cumbersome.)

3*. If $t \notin \operatorname{Max} T[\mathbf{X}]$ and $r \notin \operatorname{Max} T[\mathbf{Y}]$, then $j(r, t) = 1$ is equivalent to
   (a) $\forall r^\wedge b \in T[\mathbf{Y}]\ \exists t^\wedge a \in T[\mathbf{X}]\ (j(t^\wedge a, r^\wedge b) = 1)$ and
   (b) $\forall t^\wedge a \in T[\mathbf{X}]\ \exists r^\wedge b \in T[\mathbf{Y}]\ (j(t^\wedge a, r^\wedge b) = 1)$.

In **ZFC** a unique bisimulation $j : T[\mathbf{X}] \times T[\mathbf{Y}] \to \{0, 1\}$ exists for any pair of A-codes $\mathbf{X}$, $\mathbf{Y}$, and $j(t, r) = 1$ is equivalent to $\mathsf{F}_{\mathbf{X}}(t) = \mathsf{F}_{\mathbf{Y}}(r)$. However, unlike coded sets, bisimulations do exist in **nGB**, because their construction does not increase the $\in$-rank over $\mathbb{I}$!

Lemma 3.14 ($\mathbf{nGB}+\textsf{sem-bounded Comprehension}$). *For any A-codes* $\mathbf{X}, \mathbf{Y} \in \mathbf{H}$ *there exists a unique bisimulation* j.

Proof. For all $t \in T[\mathbf{X}]$ and $r \in T[\mathbf{Y}]$, any class-function j defined on the class $Q_{tr} = \{\langle t', r'\rangle \in T[\mathbf{X}] \times T[\mathbf{Y}] : t \subseteq t' \wedge r \subseteq r'\}$ and satisfying items 1^*, 2^*, 3^* of Definition 3.13 on Q_{tr} will be called a (t, r)-*function*.

Suppose first that the codes $\mathbf{X} = \mathbf{x}$ and $\mathbf{Y} = \mathbf{y}$ belong to $\mathbf{h}$. Then Q_{tr} is a semiset for any pair $t \in T[\mathbf{x}]$, $r \in T[\mathbf{y}]$. Denote the assertion 'for every $r \in T[\mathbf{y}]$ there exists a unique (t, r)-function j_{tr}' by $S(t)$. Under the assumption that $\mathbf{x}$, $\mathbf{y}$ belong to $\mathbf{h}$, all (t, r)-functions are semisets. Therefore, the formula which defines $S(t)$ is sem-bounded, and thus $S = \{t \in T : S(t)\}$ is a legitimate class. Moreover, S is an inductive class in the sense mentioned in the proof of Lemma 3.7. Indeed, f_{tr} can be assembled from functions $f_{t^\wedge a, r^\wedge b}$ with $t^\wedge a \in T[\mathbf{x}]$ and $r^\wedge b \in T[\mathbf{y}]$ by using conditions 1^*, 2^*, 3^* in Definition 3.13. It follows that S contains Λ. Thus, there is a unique (Λ, Λ)-function, and it is a bisimulation for the codes $\mathbf{x}$, $\mathbf{y}$.

Now the general case: one or both of the codes $\mathbf{X}$, $\mathbf{Y}$ belongs to $\mathbf{H} \setminus \mathbf{h}$. By the definition of $\mathbf{H}$, any class Q_{tr} is a semiset provided that $t \neq \Lambda$ and $r \neq \Lambda$. An obvious modification of the first part of the proof yields existence of a unique (t, r)-function j_{tr} for any pair $t \in T[\mathbf{X}] \setminus \{\Lambda\}$ and $r \in T[\mathbf{Y}] \setminus \{\Lambda\}$. Thus, a unique (Λ, Λ)-function can be assembled from functions $f_{\langle a\rangle,\langle b\rangle}$ for all one-term sequences $\langle a\rangle \in T[\mathbf{X}]$, $\langle b\rangle \in T[\mathbf{Y}]$. Lemma 3.14 is proved.

We denote by $\mathbf{j_{XY}}$ this unique bisimulation, for any $\mathbf{X}$, $\mathbf{Y} \in \mathbf{H}$.

3d. Cumulative extension of a non-standard class universe. *We continue to argue in* **nGB** *in the framework of our convention* 3.5.

The next definition introduces st-$\in$-formulae that define those relations between A-codes which adequately express membership and equality between coded objects, as if the latter really existed.

Definition 3.15.

1) $\mathbf{X} \,{}^{\mathrm{a}}{=}\, \mathbf{Y}$ is the st-$\in$-formula $\mathbf{X}, \mathbf{Y} \in \mathbf{H} \wedge \mathbf{j_{XY}}(\Lambda, \Lambda) = 1$.
2) $\mathbf{X} \,{}^{\mathrm{a}}{\in}\, \mathbf{Y}$ is the st-$\in$-formula saying that $\mathbf{X}, \mathbf{Y} \in \mathbf{H}$ and one of the following two statements holds: either a) $\mathbf{X} = {}^{\mathrm{a}}x$ and $\mathbf{Y} = {}^{\mathrm{a}}Y$ for a set x and a class Y with $x \in Y$, or b) $T[\mathbf{Y}] \neq \{\Lambda\}$ and there exists a $b \in \operatorname{Min} T[\mathbf{Y}]$ such that $\mathbf{j_{XY}}(\Lambda, \langle b\rangle) = 1$.
3) ${}^{\mathrm{a}}\mathrm{st}\, \mathbf{X}$ is the st-$\in$-formula $\exists^{\mathrm{st}} Y\ (\mathbf{X} = {}^{\mathrm{a}}Y)$.

Proposition 3.16 ($\mathbf{nGB}$+$\textsf{sem-bounded Comprehension}$). *The relations* ${}^{\mathrm{a}}{\in}$, ${}^{\mathrm{a}}\mathrm{st}$, ${}^{\mathrm{a}}{=}$ *can be expressed by sem-bounded formulae.*

Proof. If only codes in $\mathbf{h}$ are considered, then all codes are semisets by definition, therefore all relevant bisimulations are semisets as well, and the sem-boundedness of definitions here is clear.

In the general case (arbitrary codes in $\mathbf{H}$) both codes and bisimulations can be 'large' classes (non-semisets). But this obstacle can be circumvented. For instance, in the definition of $\mathbf{X} \,{}^{\mathrm{a}}{=}\, \mathbf{Y}$ in Definition 3.15, 1) the formula $\mathbf{j_{XY}}(\Lambda, \Lambda) = 1$ is equivalent to

$$\forall a \in \operatorname{Min} T_{\mathbf{X}} \, \exists b \in \operatorname{Min} T_{\mathbf{Y}} \, (\mathbf{j}_{\mathbf{X}|_a \mathbf{Y}|_b}(\Lambda, \Lambda) = 1)$$
$$\wedge \, \forall b \in \operatorname{Min} T_{\mathbf{Y}} \, \exists a \in \operatorname{Min} T_{\mathbf{X}} \, (\mathbf{j}_{\mathbf{X}|_a \mathbf{Y}|_b}(\Lambda, \Lambda) = 1)$$

in the case when $T_{\mathbf{X}} \neq \{\Lambda\}$ and $T_{\mathbf{Y}} \neq \{\Lambda\}$. However, the codes $\mathbf{X}\big|_a$, $\mathbf{Y}\big|_b$ here belong to $\mathbf{h}$ in accordance with the definition of $\mathbf{H}$. This reduces the problem to the first case, and Proposition 3.16 follows.

The relations ${}^{\mathrm{a}}{=}$ and ${}^{\mathrm{a}}{\in}$ have a rather transparent meaning in **ZFC**, where the coded sets $\mathsf{A}_{\mathbf{X}}$ always exist: $\mathbf{X} {}^{\mathrm{a}}{=} \mathbf{Y}$ is equivalent to $\mathsf{A}_{\mathbf{X}} = \mathbf{Y}$, and $\mathbf{X} {}^{\mathrm{a}}{\in} \mathbf{Y}$ is equivalent to $\mathsf{A}_{\mathbf{X}} \in \mathsf{A}_{\mathbf{Y}}$. In non-standard class theories the appeal to coded objects fails, generally speaking. However, we can consider the structures

$$\mathfrak{H} = \langle \mathbf{H}; {}^{\mathrm{a}}{\in}, {}^{\mathrm{a}}\mathrm{st}; {}^{\mathrm{a}}{=} \rangle \qquad \text{and} \qquad \mathfrak{h} = \langle \mathbf{h}; {}^{\mathrm{a}}{\in} \restriction \mathbf{h}, {}^{\mathrm{a}}\mathrm{st} \restriction \mathbf{h}; {}^{\mathrm{a}}{=} \restriction \mathbf{h} \rangle, \tag{12}$$

that is, our coded cumulative extensions, as structures of the st-$\in$-language. To save space, the fairly elementary proof of the following proposition is left to the reader.

Proposition 3.17 (**nGB**+sem-bounded Comprehension).

(i) ${}^{\mathrm{a}}{=}$ *is an equivalence relation on* $\mathfrak{H}$, *while* ${}^{\mathrm{a}}{\in}$, ${}^{\mathrm{a}}\mathrm{st}$ *are* ${}^{\mathrm{a}}{=}$*-invariant relations.*

(ii) *For any class* X, ${}^{\mathrm{a}}X \in \mathbf{h}$ *if and only if* X *is a semiset.*

(iii) *The map* $X \mapsto {}^{\mathrm{a}}X$ *is an* st-$\in$*-isomorphism in the sense that for any* X, Y

$$X \in Y \Leftrightarrow {}^{\mathrm{a}}X \; {}^{\mathrm{a}}{\in} \; {}^{\mathrm{a}}Y,$$
$$\mathrm{st}\, X \Leftrightarrow {}^{\mathrm{a}}\mathrm{st}\, {}^{\mathrm{a}}X,$$
$$X = Y \Leftrightarrow {}^{\mathrm{a}}X \; {}^{\mathrm{a}}{=} \; {}^{\mathrm{a}}Y \Leftrightarrow {}^{\mathrm{a}}X = {}^{\mathrm{a}}Y.$$

(iv) (a) *If* $T[\mathbf{Y}] \neq \{\Lambda\}$, *then* $\mathbf{X} \; {}^{\mathrm{a}}{\in} \; \mathbf{Y} \Leftrightarrow \exists b \in \operatorname{Min} T[\mathbf{Y}] \; \big(\mathbf{X} \; {}^{\mathrm{a}}{=} \; \mathbf{Y}\big|_b\big)$.

(b) *If* $\mathbf{Y} = {}^{\mathrm{a}}Y$, *then* $\mathbf{X} \; {}^{\mathrm{a}}{\in} \; \mathbf{Y} \Leftrightarrow \exists x \in Y \; (\mathbf{X} = {}^{\mathrm{a}}x)$.

(v) (a) *If* $y \in \mathbb{I}$, *then* $\mathbf{X} \; {}^{\mathrm{a}}{=} \; {}^{\mathrm{a}}y$ *is equivalent to* $\mathbf{X} = {}^{\mathrm{a}}y$.

(b) *If* $Y \notin \mathbb{I}$, *then* $\mathbf{X} \; {}^{\mathrm{a}}{=} \; {}^{\mathrm{a}}Y$ *is equivalent to* $|T_{\mathbf{X}}|^{\mathbb{S}} = 1$ *and* $Y = \{\mathbf{X}(a) : a \in \operatorname{Min} T_{\mathbf{X}}\}$.

Thus, according to (i) the structures $\mathfrak{H}$, $\mathfrak{h}$ belong to the class of *invariant* st-$\in$-structures (see the end of subsection 1a): equality is interpreted as the equivalence relation ${}^{\mathrm{a}}{=}$, and the relations ${}^{\mathrm{a}}{\in}$ and ${}^{\mathrm{a}}\mathrm{st}$ are ${}^{\mathrm{a}}{=}$-invariant. In contrast to the case considered in Remark 1.6, in this case there seems to be no possibility of adequately replacing the equivalence classes (which themselves can consist of proper classes here) by sets, or at least by 'legitimate' classes of sets.

Definition 3.18 (compare with Definition 1.15). For any invariant st-$\in$-structure $\mathfrak{X} = \langle X; \in_{\mathfrak{X}}, \mathrm{st}_{\mathfrak{X}}; =_{\mathfrak{X}} \rangle$ define the domains $(\mathbb{H})^{\mathfrak{X}}$, $(\mathbb{S})^{\mathfrak{X}}$, $(\mathbf{S})^{\mathfrak{X}}$, $(\mathbb{I})^{\mathfrak{X}}$, $(\mathbf{E})^{\mathfrak{X}}$, $(\mathbf{I})^{\mathfrak{X}}$, $(\mathbf{H})^{\mathfrak{X}}$ (parts of X) in $\mathfrak{X}$ according to Definition 2.4. Suppose that $\mathfrak{Y} = \langle Y; \in_{\mathfrak{Y}}, \mathrm{st}_{\mathfrak{Y}} \rangle$ is one more st-$\in$-structure. An injection $\pi \colon Y \to X$ is called

1) an st-$\in$-*embedding of* $\mathfrak{Y}$ *in* $\mathfrak{X}$ if the equivalences $y \in_Y y' \Leftrightarrow \pi(y) \in_X \pi(y')$ and $\mathrm{st}_Y\, y \Leftrightarrow \mathrm{st}_X\, \pi(y)$ hold for all $y, y' \in Y$;

2) an $\mathbb{I}$-*type* st-$\in$-*embedding of* $\mathfrak{Y}$ *in* $\mathfrak{X}$ if in addition to 1) the full image $\operatorname{ran}\pi = \{\pi(y) : y \in Y\}$ coincides with $(\mathbb{I})^{\mathfrak{X}}$ modulo $=_{\mathfrak{X}}$;[30]
3) an $\mathbf{E}$-*type* st-$\in$-*embedding of* $\mathfrak{Y}$ *in* $\mathfrak{X}$ if in addition to 1) the full image $\operatorname{ran}\pi$ coincides with $(\mathbf{E})^{\mathfrak{X}}$ modulo $=_{\mathfrak{X}}$.

In this case the st-$\in$-structure $\mathfrak{X}$ is called an $\mathbb{I}$-*type extension*, respectively, $\mathbf{E}$-*type extension* of $\mathfrak{Y}$ (by means of π).

The next assertion reveals the nature of the domains $(\mathbb{H})^{\mathfrak{X}}$, $(\mathbb{S})^{\mathfrak{X}}$, and so forth, in the structures $\mathfrak{h}$ and $\mathfrak{H}$. A rather routine proof based on items (iii), (iv), (v) of Proposition 3.17 still takes effort in some details. The notation $\mathbb{S}$, $\mathbb{I}$, $\mathbf{S}$, $\mathbf{I}$, $\mathbf{E}$ without relativization corresponds in Proposition 3.19 and Corollary 3.20 to Definition 2.4 in the ground class universe of **nGB**+sem-bounded Comprehension.

Proposition 3.19 (**nGB** + sem-bounded Comprehension).

(i) $(\mathbb{H})^{\mathfrak{H}}$ *coincides with* $\mathbf{h}$ *modulo* ${}^{\mathrm{a}}{=}$; *in other words,* $\mathfrak{H}$ *is an extension of the structure* $\mathfrak{h}$ *by adjoining classes to its domain* $\mathbf{h}$.

(ii) $(\mathbb{S})^{\mathfrak{H}} = (\mathbb{S})^{\mathfrak{h}} = \{{}^{\mathrm{a}}x : x \in \mathbb{S}\}$ *and* $(\mathbb{I})^{\mathfrak{H}} = (\mathbb{I})^{\mathfrak{h}} = \{{}^{\mathrm{a}}x : x \in \mathbb{I}\}$.

(iii) $(\mathbf{S})^{\mathfrak{H}} = \{{}^{\mathrm{a}}X : X \in \mathbf{S}\}$, *while* $(\mathbf{I})^{\mathfrak{H}}$ *coincides with* $\{{}^{\mathrm{a}}X : X \in \mathbf{I}\}$ *modulo* ${}^{\mathrm{a}}{=}$.

(iv) $(\mathbf{E})^{\mathfrak{H}}$ *coincides with* $\{{}^{\mathrm{a}}X : X \in \mathbf{E}\}$ *modulo* ${}^{\mathrm{a}}{=}$.

The following is an immediate corollary of Propositions 3.17 and 3.19.

Corollary 3.20 (**nGB**+sem-bounded Comprehension). *The map* $X \mapsto {}^{\mathrm{a}}X$ *is an* $\mathbf{E}$-*type* st-$\in$-*embedding of the whole class universe of* **nGB** *in* $\mathfrak{H}$.

The restricted embedding $x \mapsto {}^{\mathrm{a}}x$, $x \in \mathbb{I}$, *is an* $\mathbb{I}$-*type* st-$\in$-*embedding of the structure* $\mathfrak{I} = \langle \mathbb{I}; \in, \mathrm{st}\rangle$ *in* $\mathfrak{H}$.

3e. Properties of the cumulative extension. Here we establish the following central theorem.

Theorem 3.21.

(i) *The structure* $\mathfrak{h}$ *is an interpretation of* $\mathbf{HST}^+$ *in* $\mathbf{nGB}^+$: *the relativization* $\Phi^{\mathfrak{h}}$ *of any axiom* Φ *of* $\mathbf{HST}^+$ *is provable in* $\mathbf{nGB}^+$.

(ii) *The structure* $\mathfrak{H}$ *is an interpretation of* $\mathbf{GB}/\mathbf{HST}^+$ *in* $\mathbf{nGB}^+$, *and also is an interpretation of* $\mathbf{KM}/\mathbf{HST}^+$ *in* $\mathbf{nKM}^+$.

Proof. (i) The proof consists in a quite cumbersome verification of all the necessary assertions (see § 5.4 in [25] or Part II of [21]), and thus we cannot present the details here. Actually, most of the arguments are rather elementary. In particular, it does not take much effort to prove the relativizations of the axioms Standard Size Choice, Dependent Choice, Saturation, $\mathbb{I}$-image, Σ_2^{st} sets of $\mathbf{HST}^+$ with the help of the additional axioms of $\mathbf{nGB}^+$ with the same name and using Proposition 3.17 and the following Lemma 3.22 if necessary.

We define $\mathrm{FAM}(X) = \{X[a] : a \in \operatorname{dom} X\}$ for any class X (recall that $X[a] = \{b : \langle a, b\rangle \in X\}$). Thus, $\mathrm{FAM}(X)$ is a collection of classes, that is, not a legitimate object in our class theories. But the reader can easily see that each explicit and implicit occurrence of $\mathrm{FAM}(\,\cdot\,)$ below is only a convenient figure of speech that can be easily eliminated by reduction to the definition.

---

[30] Subsets A, B of a domain D on which an equivalence relation $\approx$ is defined, *coincide modulo* $\approx$ if $\forall a \in A\, \exists b \in B\, (a \approx b)$ and $\forall b \in B\, \exists a \in A\, (a \approx b)$.

Lemma 3.22 ($\mathbf{nGB}^+$). *For any class X with* $\mathrm{FAM}(X) \subsetneq \mathbf{h}$ *there exists a code* $\mathbf{y} \in \mathbf{H}$ *whose collection of all* ${}^*\!\in$*-elements coincides with* $\mathrm{FAM}(X)$ *modulo* ${}^{\mathrm{a}}\!\!=$. *If in addition X is a semiset, then such a code* $\mathbf{y}$ *can be chosen in* $\mathbf{h}$.

Proof. If $X = \varnothing$, then $\mathbf{y} = {}^{\mathrm{a}}\varnothing$ is the required code. Assume that $X \neq \varnothing$. Then $D = \operatorname{dom} X \neq \varnothing$. We define an A-code $\mathbf{y}$ so that $T[\mathbf{y}] = \{\Lambda\} \cup \{a^\wedge t : a \in D \wedge t \in T[X[a]]\}$ and $\mathbf{y}(a^\wedge t) = X[a](t)$ whenever $a \in D$ and $t \in \operatorname{Max} T[X[a]]$. Then obviously $\operatorname{Min} T[\mathbf{y}] = D$ and $\mathbf{y}\big|_a = X[a]$ for any $a \in D$. Therefore, the ${}^*\!\in$-elements of $\mathbf{y}$ coincide modulo ${}^{\mathrm{a}}\!\!=$ with the codes in $\mathrm{FAM}(X)$. If $\mathbf{y}$ is a regular code, then this completes the task: $\mathbf{y}$ belongs to $\mathbf{H}$. If, moreover, X is a semiset, then $\mathbf{y}$ belongs to $\mathbf{h}$.

Suppose that $\mathbf{y}$ is not regular. Then, since all the codes $\mathbf{y}\big|_a = X[a]$ with $a \in D = \operatorname{Min} T[\mathbf{y}]$ are regular, we have $T[\mathbf{y}] = \langle\lambda\rangle \cup \{\langle a\rangle : a \in D\}$, and for every $a \in \operatorname{dom} X$ there is a set x_a such that $\mathbf{y}\big|_a = {}^{\mathrm{a}}(x_a)$, that is, $\mathbf{y}(\langle a\rangle) = x_a$, and $y = \{x_a : a \in D\}$ is a set. The code ${}^{\mathrm{a}}y$ proves the lemma.

The lemma enables us to immediately prove Separation in $\mathfrak{h}$ in the st-$\in$-language, that is, one of the main reasons behind this construction. Take any code $\mathbf{X} \in \mathbf{h}$ and any st-$\in$-formula $\Phi(x)$ with codes in $\mathbf{h}$ as parameters. By Lemma 3.22, it suffices to find a semiset X such that $\mathrm{FAM}(X) \subsetneq \mathbf{h}$ and $\mathrm{FAM}(X)$ contains, modulo ${}^{\mathrm{a}}\!\!=$, all codes $\mathbf{x} \,{}^{\mathrm{a}}\!\!\in \mathbf{X}$ satisfying $\Phi(\mathbf{x})^{\mathfrak{h}}$, and nothing more.

Suppose that $\mathbf{X}$ has the form $\mathbf{X} = {}^{\mathrm{a}}Y$ for some $Y \in \mathbb{I}$, and consider the family $\mathscr{X} = \{{}^{\mathrm{a}}y : y \in Y \wedge \Phi({}^{\mathrm{a}}y)^{\mathfrak{h}}\}$ of all codes, that is, by Proposition 3.17(iv)(b), of all ${}^{\mathrm{a}}\!\!\in$-elements of ${}^{\mathrm{a}}Y$ satisfying $\Phi(\,\cdot\,)^{\mathfrak{h}}$. Note that the formula $\Phi(x)^{\mathfrak{h}}$ (with x as a free variable) is a sem-bounded combination of the relations ${}^{\mathrm{a}}\!\!\in$, ${}^{\mathrm{a}}\mathrm{st}$, ${}^{\mathrm{a}}\!\!=$, because all its quantifiers are restricted to $\mathbf{h}$, a collection that contains only semisets. Therefore, $X = \{\langle y, b\rangle : y \in Y \wedge b \in {}^{\mathrm{a}}y \wedge \Phi({}^{\mathrm{a}}y)^{\mathfrak{h}}\}$ is a class by Proposition 3.16, and hence is a semiset. Finally, $\mathscr{X} = \mathrm{FAM}(X)$ as required. If $T[\mathbf{X}] \neq \{\Lambda\}$, then in order to get the same result, consider the collection $\{\mathbf{X}\big|_a : a \in \operatorname{Min} T \wedge \Phi(\mathbf{X}\big|_a)^{\mathfrak{h}}\}$ of codes and use (iv)(a) of Proposition 3.17 instead of (iv)(b)).

Let us also check Collection in $\mathfrak{h}$. Suppose that $\mathbf{X} \in \mathbf{h}$, $T[\mathbf{X}] \neq \{\Lambda\}$, and $\Phi(x, y)$ is an st-$\in$-formula with codes in $\mathbf{h}$ as parameters. By Proposition 3.17(iv)(a) and Lemma 3.22, it suffices to find a semiset X such that $\mathrm{FAM}(X) \subsetneq \mathbf{h}$ and

$$\exists\, \mathbf{y} \in \mathbf{h}\ \Phi\big(\mathbf{X}\big|_b, \mathbf{y}\big)^{\mathfrak{h}} \Rightarrow \exists\, \mathbf{y} \in \mathrm{FAM}(X)\ \big(\Phi\big(\mathbf{X}\big|_b, \mathbf{y}\big)^{\mathfrak{h}}\big)$$

holds for every A-code $\mathbf{X}\big|_b$, where $b \in \operatorname{Min} T[\mathbf{X}]$. But the existence of such a code X follows easily from Semiset Collection.

(ii) That $(\mathbf{HST}^+)^{\mathrm{sets}}$ holds in $\mathfrak{H}$ follows from (i), because $\mathfrak{h}$ is the universe of all sets in $\mathfrak{H}$ by Proposition 3.19(i). Furthermore, to prove $\mathbf{GB}^{\mathrm{st}}$, Standardization for classes, and the axiom of $\mathbf{S} = \mathbf{S}^{\mathrm{loc}}$ for classes $X \subseteq \mathbb{I}$ in $\mathfrak{H}$ it suffices to note that by Propositions 3.17(iii) and 3.19(iv) the domain $(\mathbf{E})^{\mathfrak{H}}$ is st-$\in$-isomorphic to the ground universe of $\mathbf{nGB}^+$ (or $\mathbf{nKM}^+$), where these axioms do hold. Extensionality in $\mathfrak{H}$ is an easy exercise. It remains to verify Collection and Comprehension for predicative st-$\in$-formulae in the $\mathbf{GB}$-case, and also Impredicative Comprehension and Class Collection for all st-$\in$-formulae in the $\mathbf{KM}$-case.

Comprehension and Impredicative Comprehension. It suffices to find 'the class of all sets', that is, a code $\mathbf{Y} \in \mathbf{H}$ such that $\mathbf{x} \,{}^{\mathrm{a}}\!\!\in \mathbf{Y}$ for all codes $\mathbf{x} \in \mathbf{h}$. Indeed, if we have such a $\mathbf{Y}$, then the rest of the proof follows as in the proof of Separation in part (i). To define a code $\mathbf{Y}$ as required, we put

$$P = \{p \in \mathbf{C}_0 : E_p \in \mathbf{h}\} \quad \text{and} \quad X = \{\langle p, x\rangle : p \in P \wedge x \in E_p\}.$$

Note that being an element of $\mathbf{h}$ can obviously be expressed by a sem-bounded formula. Therefore, even in $\mathbf{nGB}^+$, P and X are well-defined classes. On the other hand, by the axiom of Σ_2^{st} semisets, each code $\mathbf{x} \in \mathbf{h}$ satisfies $\mathbf{x} = E_p = X[p]$, so that $\mathrm{FAM}(X)$ contains all codes in $\mathbf{h}$. It remains to apply Lemma 3.22.

Collection (predicative): follow the proof of Collection in part (i).

Class Collection. Assuming $\mathbf{nKM}^+$, we have to prove that

$$\forall\, d\, \exists\, C\, \exists\, r\, \forall\, a \in d\ (\exists\, Y\ \Phi(a, Y) \Rightarrow \exists\, x \in r\ \Phi(a, C[x])) \tag{13}$$

in $\mathfrak{H}$, where $\Phi(x)$ is any st-$\in$-formula with codes in $\mathbf{H}$ as parameters. Suppose that $\mathbf{d} \in \mathbf{h}$. As in the proof of Separation in part (i), there exists a semiset X such that $\mathrm{FAM}(X) = \{\boldsymbol{a} \in \mathbf{h} : \boldsymbol{a}\,{}^{\mathrm{a}}{\in}\,\mathbf{d}\}$ modulo ${}^{\mathrm{a}}{=}$. Put $\delta = \operatorname{dom} X$, so that $\mathrm{FAM}(X) = \{X[\xi] : \xi \in \delta\}$.

Applying Class Collection in the ground universe of $\mathbf{nKM}^+$, we obtain a class $W \subseteq \mathbb{I}$ and a set $w \in \mathbb{I}$ such that for any $\xi \in \delta$ there is an $x \in w$ such that $\mathbf{W}_x = W[x]$ is a code in $\mathbf{H}$ and if $(\exists\, Y\, \Phi(X[\xi], Y))^{\mathfrak{H}}$ holds, then so does $\Phi(X[\xi], \mathbf{W}_x)^{\mathfrak{H}}$. By Lemma 3.22, there is a code $\mathbf{C} \in \mathbf{H}$ which, modulo ${}^{\mathrm{a}}{=}$, ${}^{\mathrm{a}}{\in}$-contains codes of the form ${}^{\mathrm{a}}\langle {}^{\mathrm{a}}x, \mathbf{z}\rangle$,[31] with $x \in w$ and $\mathbf{z}\,{}^{\mathrm{a}}{\in}\,\mathbf{W}_x$, and nothing else. Then we have $\mathbf{W}_x = \mathbf{C}[{}^{\mathrm{a}}x]$ in $\mathfrak{H}$ for any $x \in w$.

Suppose now that $\boldsymbol{a} \in \mathbf{h}$, $\boldsymbol{a}\,{}^{\mathrm{a}}{\in}\,\mathbf{d}$, and $(\exists\, Y\, \Phi(\boldsymbol{a}, Y))^{\mathfrak{H}}$ holds. By definition, there exists a $\xi \in \delta$ such that $\boldsymbol{a} = X[\xi]$, hence there exists an $x \in w$ such that $\Phi(\boldsymbol{a}, \mathbf{W}_x)^{\mathfrak{H}}$ holds, and then $\Phi(\boldsymbol{a}, \mathbf{C}[{}^{\mathrm{a}}x])^{\mathfrak{H}}$ holds as well. It follows that $\mathbf{C}$ and $r = {}^{\mathrm{a}}w$ prove relation (13) in $\mathfrak{H}$ for $d = \mathbf{d}$. Theorem 3.21 is proved.

It remains to note that by Corollary 3.20 the interpretations of the theory $\mathbf{GB/HST}^+$ in $\mathbf{nGB}^+$ and of the theory $\mathbf{KM/HST}^+$ in $\mathbf{nKM}^+$ given by the structure $\mathfrak{H}$ according to Theorem 3.21(ii) can be characterized as $\mathbf{E}$-*type interpretations* in the sense that the map $X \mapsto {}^{\mathrm{a}}X$ defines an st-$\in$-isomorphism of the whole universe of $\mathbf{nGB}^+$ onto the domain $(\mathbf{E})^{\mathfrak{H}}$ of all classes of internal sets of $\mathfrak{H}$. It is an informal consequence of Theorem 3.21(ii) and Corollary 3.20 that the universe $\mathbf{E}$ of $\mathbf{nGB}^+$ (respectively, $\mathbf{nKM}^+$) admits an extension to a structure $\mathfrak{H}$ satisfying $\mathbf{GB/HST}^+$ (respectively, $\mathbf{KM/HST}^+$), in which the given universe remains the collection $(\mathbf{E})^{\mathfrak{H}}$ of all classes of internal sets.

By Theorem 3.21(i), the interpretation given by the structure $\mathfrak{h}$ can be characterized in the same way as an $\mathbb{I}$-*type interpretation* in the sense that the map $x \mapsto {}^{\mathrm{a}}x$ defines an st-$\in$-isomorphism of the $\mathbf{BST}$-structure $\mathfrak{I} = \{\mathbb{I}; \in, \mathrm{st}\}$ (here the class of all sets in the ground $\mathbf{nGB}^+$ universe) onto the domain $(\mathbb{I})^{\mathfrak{h}}$ of internal sets of $\mathfrak{h}$. This can be informally viewed as an extension of the class $\mathfrak{I}$ of all (internal) sets in $\mathbf{nGB}^+$ to a structure $\mathfrak{h}$ satisfying $\mathbf{HST}^+$, in which $\mathfrak{I}$ remains the class $(\mathbb{I})^{\mathfrak{h}}$ of all internal sets.

Corollary 3.23.

[31] By ${}^{\mathrm{a}}\langle {}^{\mathrm{a}}x, \mathbf{z}\rangle$ we understand a code in $\mathbf{h}$ which defines in some canonical way the ordered pair of these elements in $\mathfrak{h}$.

3f. Metamathematical consequences. As above, the interpretations obtained lead us to results related to conservativity in suitable forms.

(i) *The theory* $\mathbf{HST}^+$ *is an* $\mathbb{I}$*-conservative* (*and then equiconsistent*) *extension of* $\mathbf{BST}$ *in the sense that any* st-$\in$*-formula* φ *is provable in* $\mathbf{BST}$ *if and only if* $\varphi^{\mathbb{I}}$ *is provable in* $\mathbf{HST}^+$.

(ii) *The theory* $\mathbf{GB}/\mathbf{HST}^+$ *is an* $\mathbf{E}$*-conservative extension of* $\mathbf{nGB}^+$ *in the sense that any* st-$\in$*-formula* φ *is provable in* $\mathbf{nGB}^+$ *if and only if* $\varphi^{\mathbf{E}}$ *is provable in* $\mathbf{GB}/\mathbf{HST}^+$. *The same is true for the pair* $\mathbf{KM}/\mathbf{HST}^+$, $\mathbf{nKM}^+$.

Proof. (ii) Suppose that the relativized formula $\varphi^{\mathbf{E}}$ is provable in $\mathbf{GB}/\mathbf{HST}^+$. To prove φ we consider the structure $\mathfrak{H}$, arguing in $\mathbf{nGB}^+$. Then $\varphi^{\mathbf{E}}$ holds in $\mathfrak{H}$ by Theorem 3.21. However, $(\mathbf{E})^{\mathfrak{H}}$ is provably st-$\in$-isomorphic to the ground $\mathbf{nGB}^+$ universe by Corollary 3.20, and this completes the derivation of φ.

(i) If $\varphi^{\mathbb{I}}$ is provable in $\mathbf{HST}^+$, then $\varphi^{\mathbb{I}}$ follows from $\mathbf{GB}/\mathbf{HST}^+$ by Theorem 3.4, and then also from $\mathbf{nGB}^+$ by (ii). (The relativizations $^{\text{sets}}$ and $^{\mathbf{E}}$ change nothing for a formula relativized to $\mathbb{I}$.) Therefore, φ is provable in $\mathbf{BST}$ by Corollary 2.21.

Diagrams 3 and 4 below represent our main metamathematical results connecting the theories $\mathbf{BST}$ and $\mathbf{HST}^+$ with each other and with the corresponding non-standard class theories. In the diagrams, $A \longrightarrow B$ means that a theory A is in some way reducible to B. Diagram 3 displays natural reducibilities of a 'weaker' theory to a 'stronger' one. The vertical arrows indicate the interpretation by relativization to $\mathbb{I}$ or $\mathbf{E}$ (see Propositions 3.1 and 3.3). The arrows marked by sets indicate the obvious interpretation by relativization to the subuniverse of sets (in a given class universe). The arrows marked by $\subset$ indicate the fact that $\mathbf{nGB}^+$ and $\mathbf{GB}/\mathbf{HST}^+$ are subtheories of $\mathbf{nKM}^+$ and $\mathbf{KM}/\mathbf{HST}^+$, respectively.

$$\begin{array}{ccccc} \mathbf{BST} & \xrightarrow{\text{sets}} & \mathbf{nGB}^+ & \xrightarrow{\subset} & \mathbf{nKM}^+ \\ \downarrow{\scriptstyle\mathbb{I}} & & \downarrow{\scriptstyle\mathbf{E}} & & \downarrow{\scriptstyle\mathbf{E}} \\ \mathbf{HST}^+ & \xrightarrow{\text{sets}} & \mathbf{GB}/\mathbf{HST}^+ & \xrightarrow{\subset} & \mathbf{KM}/\mathbf{HST}^+ \end{array}$$

Diagram 3

$$\begin{array}{ccccc} \mathbf{BST} & \xleftarrow{\text{eq, cons}} & \mathbf{nGB}^+ & \not\longleftarrow & \mathbf{nKM}^+ \\ \uparrow{\scriptstyle \mathrm{r}} & & \uparrow{\scriptstyle \mathrm{r}} & & \uparrow{\scriptstyle \mathrm{r}} \\ \mathbf{HST}^+ & \xleftarrow{\text{eq, cons}} & \mathbf{GB}/\mathbf{HST}^+ & \not\longleftarrow & \mathbf{KM}/\mathbf{HST}^+ \end{array}$$

Diagram 4

Opposite reducibilities are displayed on Diagram 4. Crossed arrows indicate the reducibilities that fail because impredicative class theories are strictly stronger in the sense of equiconsistency than predicative ones. (We can refer to Theorem 3.4 in matters of the pair $\mathbf{GB}/\mathbf{HST}^+$, $\mathbf{KM}/\mathbf{HST}^+$.) The vertical arrows marked by r indicate the interpretations given by Theorem 3.21 and their consequences related to conservativity and equiconsistency by Corollary 3.23.

The arrows marked by eq, cons (from equiconsistency, conservativity) indicate the reducibility of a class theory to a set theory with respect to sets-conservativity and equiconsistency, but not with respect to interpretability. Here we refer to Theorems 2.6 and 3.4.

3g. How to get rid of the assumption of the existence of a class superstructure. We recall that Theorem 3.21(i) asserts the existence of an extension of the 'internal' **BST** universe $\mathbb{I}$ to an 'external' universe of $\mathbf{HST}^+$. The theorem assumes that the given universe $\mathbb{I}$ is the class of all sets in a wider class universe of $\mathbf{nGB}^+$. This assumption, useful and well founded in view of corresponding results on class theories, can be eliminated in the context of Theorem 3.21(i) itself. This is the goal of this short subsection.

Basically, the problem consists in the following. We have to define, on the basis of the universe $\mathbb{I}$ of **BST**, a wider universe similar to the collection of all semisets in $\mathbf{nGB}^+$. Recall that the latter was the basis for the interpretation $\mathfrak{h}$. It turns out that adding to $\mathbb{I}$ all collections of the form E_p with $p \in \mathbf{C}_0$ will be sufficient. (The notation in Definition 2.13 is here understood on the basis of the **BST** universe $\mathbb{I}$.)

More precisely, our plan is as follows. Arguing in **BST**, we let ${}^{\mathbf{e}}x$ be the function defined on the singleton $\{\langle 0,0\rangle\}$ by ${}^{\mathbf{e}}x(0,0) = x$. Then obviously ${}^{\mathbf{e}}x \in \mathbf{C}_0$ and $E_{{}^{\mathbf{e}}x} = x$ for all x. Let $\mathbf{C}$ denote the family of all codes $p \in \mathbf{C}_0$ such that either $p = {}^{\mathbf{e}}x$ for some $x \in \mathbb{I}$ or E_p is not a set (in $\mathbb{I}$). This restriction ensures that any set that already exists in $\mathbb{I}$ has a unique code. Consider in **BST** the relational structure $\mathfrak{C} = \langle \mathbf{C}; {}^{\mathbf{e}}\!\in, {}^{\mathbf{e}}\mathrm{st}, {}^{\mathbf{e}}\!= \rangle$, where ${}^{\mathbf{e}}\!\in$, ${}^{\mathbf{e}}\mathrm{st}$, ${}^{\mathbf{e}}\!=$ are relations on $\mathbf{C}$ defined as follows:

$$
\begin{aligned}
p \,{}^{\mathbf{e}}\!= q \quad & \text{whenever } p, q \in \mathbf{C} \wedge \forall z\ (z \in E_p \Leftrightarrow z \in E_q);\\
p \,{}^{\mathbf{e}}\!\in q \quad & \text{whenever } p, q \in \mathbf{C} \wedge \exists z\ (p = {}^{\mathbf{e}}z \wedge z \in E_q);\\
{}^{\mathbf{e}}\mathrm{st}\, p \quad & \text{whenever } \exists^{\mathrm{st}} z\ (p = {}^{\mathbf{e}}z), \quad (\text{this implies } p \in \mathbf{C}).
\end{aligned}
$$

Obviously, the structure $\mathfrak{C}$ can be regarded as the extension of the set universe $\mathbb{I}$ of **BST** by all the 'semisets' E_p, $p \in \mathbf{C}$; the extension itself is given by the map $x \mapsto [{}^{\mathbf{e}}x]_{{}^{\mathbf{e}}=}$. This extension is well known. In particular, it was established in Ch. 5 of [25] (see also the earlier papers [21], [22]) that $\mathfrak{C}$ is an interpretation of **EEST**, the *elementary external set theory*, in **BST**. This theory in the st-$\in$-language includes $\mathbf{BST}^{\mathrm{int}}$ together with the axioms of Extensionality, Union, Separation in the st-$\in$-language, Transitivity of $\mathbb{I}$, and the axiom $\forall x\, \exists p\, (x = E_p)$ similar to the axiom of Σ_2^{st} semisets in $\mathbf{nGB}^+$ or Σ_2^{st} sets in $\mathbf{HST}^+$. This is a rather simple result. Indeed, the key fact that st-$\in$-Separation holds in $\mathfrak{C}$ follows easily from Theorem 1.19. Thus, $\mathfrak{C}$ is an interpretation of **EEST** in **BST**.

On the other hand, the theory **EEST** is strong enough to prove all those assertions which $\mathbf{nGB}^+$ implies for the universe of semisets, including the additional axiom of Σ_2^{st} semisets as well. This allows us to carry out, in **EEST**, the construction of a cumulative extension of the universe that was carried out above on the basis of the class of all semisets in $\mathbf{nGB}^+$. This construction leads to the structure $\mathfrak{h}$ of (12) on page 92. The latter satisfies Theorem 3.21 (with **EEST** instead of $\mathbf{nGB}^+$).

The superposition of these two extensions (by $\mathfrak{C}$ and by $\mathfrak{h}$) yields an extension of the **BST** universe $\mathbb{I}$ to a universe of $\mathbf{HST}^+$ in which $\mathbb{I}$ remains the class of all internal sets. A precise statement is as follows.

Theorem 3.24. *In* **BST** *one can define an invariant* st-$\in$-*structure which is an* $\mathbb{I}$-*type interpretation* (in the sense of the existence of an appropriate embedding) *of* $\mathbf{HST}^+$ *in* **BST**, *and, provably in* **BST**, *an* $\mathbb{I}$-*type extension of the universe of all sets.*

Historical and bibliographical remarks to § 3. See Ch. 8 in [25] concerning non-standard theories with Separation in the st-$\in$-language, that is, theories of 'external' type. This includes some axiomatic schemes suggested by Hrbáček [10], [11], Kawai [12], [13], and Ballard and Hrbáček [17]. Hrbáček's set theory **HST** was introduced in final form in [21] on the basis of an earlier paper of Hrbáček [10]. See Kanovei and Reeken [25], Ch. 1, about a systematic study of the set-theoretic universe of **HST**.

Definition 3.2 and the definition contained in (12) above, Proposition 3.16, Theorem 3.21, and Corollary 3.23 were obtained by V. G. Kanovei. These results are regarded as principal in this section.

The method of extension of a **BST** universe to a wider **HST** universe (in fact, to a universe of $\mathbf{HST}^+$), with the intermediate stage connected with **EEST** as in subsection 3g, was proposed in [21], [22]. The step **BST** $\to$ **EEST** is based on the coding that employs E_p and Theorem 1.19. The coding of a cumulative construction as in subsection 3c, the basis of the step **EEST** $\to$ **HST**, has been known since the 1960s, when it was employed to interpret theories like **ZFC** without the power set axiom in the second-order arithmetic. In the context of non-standard analysis this coding system was used in [21], [22], [25], and also in [10] in a somewhat different form. The notion of 'bisimulation' is typically used in some fields of combinatorics and system theory; we borrow it from the book [56], where bisimulations are involved in investigations of non-standard theories with antifoundation axioms.

If one starts with a class theory like **nGB** instead of **BST**, then 'external' sets $X \subseteq \mathbb{I}$ already exist in the form of subclasses of sets (that is, semisets), and hence there is no need for a step analogous to **BST** $\to$ **EEST**. Therefore, the proof of our key Theorem 3.21 consists only in the coding of a cumulative extension.

§ 4. On Boolean-valued analysis

In this section we touch upon some questions of Boolean-valued analysis. Boolean-valued extensions of standard structures differ significantly from non-standard extensions in their properties and composition. In particular, the canonical embedding $x \mapsto x^\vee$ of the standard universe $\mathbf{V}$ of all sets in a Boolean-valued universe $\mathbf{V}^{(\mathbf{B})}$ is not elementary (except for trivial cases). But nevertheless such an embedding is able to 'transfer' **ZFC** axioms into $\mathbf{V}^{(\mathbf{B})}$. This enables us to derive properties of objects that belong to the standard universe $\mathbf{V}$ by considering essentially more elementary objects in the Boolean-valued universe $\mathbf{V}^{(\mathbf{B})}$. An example of this kind of reduction of standard to non-standard objects will be given in subsection 4e.

4a. Boolean-valued universes. The usual notion of relational structure $\mathfrak{A} = \langle A; R_1, \dots, R_n\rangle$ that we discussed in subsection 1a can be called 2-*valued* in the sense that the expression $R_i(a_1, \dots, a_m)$ is considered to be either true or false.

Accordingly, $[\![R_i(a_1,\dots,a_m)]\!] = \top$ or $= \bot$, where $[\![\varphi]\!]$ is an evaluation of the truth value of a formula φ.

Generalizing this approach, we consider *Boolean-valued* structures. This means that the corresponding evaluations take their values in a fixed Boolean algebra $\mathbf{B}$. Then formally every relation R_i becomes a function from $A^{r(i)}$ into $\mathbf{B}$, and the equality $[\![R_i(a_1,\dots,a_{r(i)})]\!] = b$ (where $b \in \mathbf{B}$) means that $R_i(a_1,\dots,a_{r(i)}) = b$. The values $b = \top_{\mathbf{B}}, \bot_{\mathbf{B}}$ (that is, respectively, the maximal and minimal elements in $\mathbf{B}$) correspond to the ordinary truth values of true and false. The evaluation expands naturally from atomic formulae to more complicated ones. For instance, $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$, $[\![\neg\,\varphi]\!] = \complement[\![\varphi]\!]$, where the operations $\wedge$, $\vee$, $\complement$ of the Boolean algebra $\mathbf{B}$ appear on the right-hand sides. The inductive definition of the quantifier evaluation, that is, $[\![\exists\, x\, \varphi(x)]\!] = \bigvee_{x \in A} [\![\varphi(x)]\!]$, and similarly for $\forall$, requires that the algebra $\mathbf{B}$ be complete, and completeness is usually assumed. The set A should be specified.

It was demonstrated in the Introduction that a $\mathscr{P}(I)$-valued evaluation appears naturally in connection with the definition of an ultrapower $\mathfrak{A}^I/U$. Another general method for constructing Boolean-valued structures leads to a *Boolean-valued universe* $\mathbf{V}^{(\mathbf{B})}$, where $\mathbf{B}$ is a fixed complete Boolean algebra. The universe $\mathbf{V}^{(\mathbf{B})}$ is defined as the least class such that any function mapping a set $X \subseteq \mathbf{V}^{(\mathbf{B})}$ into $\mathbf{B}$ again belongs to $\mathbf{V}^{(\mathbf{B})}$. Thus, $x \in \mathbf{V}^{(\mathbf{B})}$ whenever x is a function, $\operatorname{dom} x \subseteq \mathbf{V}^{(\mathbf{B})}$, and $\operatorname{ran} x \subseteq \mathbf{B}$. The next definition introduces evaluations of atomic formulae with parameters in $\mathbf{V}^{(\mathbf{B})}$:

$$[\![x \in y]\!]_{\mathbf{B}} = \bigvee_{z \in \operatorname{dom} y} (y(z) \cdot [\![z = x]\!]_{\mathbf{B}}),$$

$$[\![x = y]\!]_{\mathbf{B}} = \bigwedge_{z \in \operatorname{dom} x} (x(z) \to [\![z \in y]\!]_{\mathbf{B}}) \cdot \bigwedge_{z \in \operatorname{dom} y} (y(z) \to [\![z \in x]\!]_{\mathbf{B}}),$$

where $x, y \in \mathbf{V}^{(\mathbf{B})}$, and $p \to q$ is defined to be equal to $(\complement p) \vee q$ in the algebra $\mathbf{B}$. As one can see, a key idea of this definition (that goes on by induction on the von Neumann rank rk of sets x, $y \in \mathbf{V}^{(\mathbf{B})}$) is that if $x \in \operatorname{dom} y$, then $[\![x \in y]\!] \geqslant y(x)$. The evaluation $[\![\varphi]\!]_{\mathbf{B}}$ extends to more complicated formulae as described above.

Define $x \approx_{\mathbf{B}} y$ if $[\![x = y]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$. Generally speaking, $x \approx_{\mathbf{B}} y$ does not imply $x = y$. However, the definition of $\mathbf{V}^{(\mathbf{B})}$ can be modified with the help of the method of Remark 1.6 so that the implication $(x \approx_{\mathbf{B}} y) \Rightarrow (x = y)$ holds (see [50]).

The most famous applications of the Boolean-valued universe $\mathbf{V}^{(\mathbf{B})}$ are connected with proofs that various mathematical sentences are independent of **ZFC**. For instance, to prove that a sentence φ is independent of the axioms of **ZFC**, it suffices to find Boolean algebras $\mathbf{B}_1$ and $\mathbf{B}_2$ such that $[\![\varphi]\!]_{\mathbf{B}_1} = \top_{\mathbf{B}_1}$ and $[\![\neg\,\varphi]\!]_{\mathbf{B}_2} = \top_{\mathbf{B}_2}$. This approach is known as *forcing* and is based on item (i) of the following principal theorem.

Theorem 4.1. *Suppose that* $\mathbf{B}$ *is a complete Boolean algebra.*

(i) *The Boolean-valued universe* $\mathbf{V}^{(\mathbf{B})}$ *satisfies* **ZFC** *in the sense that* $[\![\varphi]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$ *for any axiom of logic and any special axiom and theorem of* **ZFC**.

(ii) (‘accessibility’) *If* $x_1,\dots,x_n \in \mathbf{V}^{(\mathbf{B})}$ *and* $[\![\exists\, y\, \varphi(x_1,\dots,x_n,y)]\!]_{\mathbf{B}} = b \in \mathbf{B}$, *then there exists an element* $y \in \mathbf{V}^{(\mathbf{B})}$ *such that* $[\![\varphi(x_1,\dots,x_n,y)]\!]_{\mathbf{B}} \geqslant b$.

The theorem remains true for the wider class of *Heyting* algebras, but in this case the law of the excluded middle does not necessarily hold in $\mathbf{V}^{(\mathbf{B})}$ with the truth value $\top_{\mathbf{B}}$.

On the other hand, Boolean-valued analysis *in the proper sense* (that is, with the exclusion of forcing) investigates those properties of elements of $\mathbf{V}^{(\mathbf{B})}$ which hold with the truth value $\top_{\mathbf{B}}$ and also the resulting properties of structures in the universe $\mathbf{V}$. We will give some examples below, by necessity omitting technical details.

4b. Embeddings of the standard universe in a Boolean-valued one. One easily defines the *canonical embedding* $x \mapsto x^{\vee}$ of the whole universe $\mathbf{V}$ in $\mathbf{V}^{(\mathbf{B})}$: put $\varnothing^{\vee} = \varnothing$ and then $y^{\vee}(x^{\vee}) = \top_{\mathbf{B}}$ whenever $x \in y$. We obtain

$$x \in y \Leftrightarrow [\![x^{\vee} \in y^{\vee}]\!]_{\mathbf{B}} = \top_{\mathbf{B}} \quad \text{and} \quad x \neq y \Leftrightarrow [\![x^{\vee} = y^{\vee}]\!]_{\mathbf{B}} = \bot_{\mathbf{B}}, \tag{14}$$

and in this sense this is an $\in$-embedding by means of which the Boolean-valued universe $\mathbf{V}^{(\mathbf{B})}$ is an extension of $\mathbf{V}$. This embedding has different properties in comparison with the embedding $x \mapsto {}^{*}x$ in non-standard analysis. For instance, $\mathbf{V}^{(\mathbf{B})}$-elements of any $y^{\vee}$ with $y \in \mathbf{V}$ are *exhausted* by elements of the form $x^{\vee}$ with $x \in y$, in the sense that

$$[\![z \in y^{\vee}]\!]_{\mathbf{B}} = \bigvee_{x \in y} [\![z = x^{\vee}]\!]_{\mathbf{B}}. \tag{15}$$

This obviously fails for $*$-embeddings, because of axioms of the Idealization type.

Generally speaking, the embedding $x \mapsto x^{\vee}$ is not elementary. Accordingly, the universes $\mathbf{V}$ and $\mathbf{V}^{(\mathbf{B})}$ are not elementarily equivalent as $\in$-structures. On the other hand, it is easy to see that

$$[\![\mathbb{N}^{\vee} \text{ is the set of all natural numbers }]\!]_{\mathbf{B}} = \top_{\mathbf{B}},$$
$$[\![\mathbb{Q}^{\vee} \text{ is the field of rational numbers }]\!]_{\mathbf{B}} = \top_{\mathbf{B}},$$

and so on. However, usually

$$[\![\mathbb{R}^{\vee} \text{ is a dense proper subfield of the field of all real numbers}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}.$$

It follows from Theorem 4.1(i) that if $\exists\, x\, \varphi(x)$ is a **ZFC** theorem, then there exists an element $x \in \mathbf{V}^{(\mathbf{B})}$ satisfying $[\![\varphi(x)]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$. This is the reason for the following convenient definition. If a set $X \in \mathbf{V}$ is unambiguously defined by a formula $\varphi(\,\cdot\,)$ in the sense that X is the only set satisfying $\varphi(X)$ in $\mathbf{V}$, and in addition $\exists!\, X\, \varphi(X)$ is a **ZFC** theorem, then let $X^{(\mathbf{B})}$ denote the unique, modulo the relation $\approx_{\mathbf{B}}$, element $X' \in \mathbf{V}^{(\mathbf{B})}$ such that $[\![\varphi(X')]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$. (It is assumed that a single formula φ that defines X has been specified among all such formulae.)

Now we can define, for instance, $n^{(\mathbf{B})}$ for any natural number n, $\mathbb{N}^{(\mathbf{B})}$, $\mathbb{R}^{(\mathbf{B})}$, and so on, so that, in particular,

$$[\![n^{(\mathbf{B})} = n^{\vee}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}, \quad [\![\mathbb{N}^{(\mathbf{B})} = \mathbb{N}^{\vee}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}, \quad [\![\mathbb{Q}^{(\mathbf{B})} = \mathbb{Q}^{\vee}]\!]_{\mathbf{B}} = \top_{\mathbf{B}},$$

and so on. However, $[\![\mathbb{R}^{\vee} \subsetneqq \mathbb{R}^{(\mathbf{B})}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$. Generally speaking, the definition of $X^{(\mathbf{B})}$ is not applicable for every $X \in \mathbf{V}$, but it is valid for many important mathematical objects X.

In particular, this is the case for complete metric spaces, in the following sense. Suppose that $\mathbb{X}$ is such a space. One easily verifies that

$$[\![\mathbb{X}^{\vee} \text{ is a metric (not necessarily complete) space}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}.^{32}$$

It follows from Theorem 4.1(ii) that there is an element $\mathbb{X}^{\sim} \in \mathbf{V}^{(\mathbf{B})}$ satisfying the equality $[\![\mathbb{X}^{\sim}$ is the completion of $\mathbb{X}^{\vee}]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$.[33] Then $\mathbb{X}^{\sim}$ is something like 'the same space $\mathbb{X}$ but considered inside $\mathbf{V}^{(\mathbf{B})}$' from the point of view of the questions considered below.

Then we can define $F^{\sim}$ for any closed set $F \subseteq \mathbb{X}$ to be the closure of $F^{\vee}$ in $\mathbb{X}^{\sim}$, $\mathscr{O}^{\sim}$ for any open set $\mathscr{O} \subseteq \mathbb{X}$ by means of the complement operation (applied two times), and $X^{\sim}$ for any Borel set $X \subseteq \mathbb{X}$ by transfinite induction with respect to the construction of X from open sets with the help of the operations of complement and countable union.

4c. Transfer theorems for a Boolean-valued universe. It was mentioned that, generally speaking, the canonical embedding $x \mapsto x^{\vee}$ is not elementary. This is why the transfer scheme is more complicated and has a less universal character in the case of Boolean-valued universes than in the case of non-standard universes. On the other hand, transfer gives mathematically meaningful results here. In the first approach the transfer scheme consists in the following.

(I) Search for a *non-standard representation* of a given mathematical structure $\mathfrak{A}$ in $\mathbf{V}$. This is understood as some mathematically more elementary structure $\mathbb{X}$ in the universe $\mathbf{V}$ (for instance, a metric space) such that the quotient of the collection $(\mathbb{X}^{\sim})^{\wedge} = \{r \in \mathbf{V}^{(\mathbf{B})} : [\![r \in \mathbb{X}^{\sim}]\!]_{\mathbf{B}} = \top\}$ of all 'reliable' $\mathbf{V}^{(\mathbf{B})}$-elements of $\mathbb{X}^{\sim}$ modulo $\approx_{\mathbf{B}}$ and equipped with a suitable structure is isomorphic to $\mathfrak{A}$. We note that before the quotient the collection $(\mathbb{X}^{\sim})^{\wedge}$ is, generally speaking, a proper class. After the quotient it becomes a set, or, more precisely, the method of cutting the equivalence classes as in Remark 1.6 converts it into a set.

(II) The original structure $\mathfrak{A}$ becomes the subject of investigation. For instance, we are going to prove that a certain sentence Φ is true in $\mathfrak{A}$. It suffices to show that Φ holds in the isomorphic model $(\mathbb{X}^{\sim})^{\wedge}$. To this purpose we choose another formula φ connected with Φ in such a way that if φ holds in $\mathbb{X}^{\sim}$ with the truth value $\top_{\mathbf{B}}$ in $\mathbf{V}^{(\mathbf{B})}$, then Φ holds in $(\mathbb{X}^{\sim})^{\wedge}$. This is the

[32]More precisely, $\mathbb{X}$ is a pair that consists of the set $\mathbb{X}$ itself along with the distance function $d\colon \mathbb{X}^2 \to \mathbb{R}_+$, which is as usual identified with the set $d = \{\langle\langle x,y\rangle, d(x,y)\rangle : x,y \in \mathbb{X}\}$. Correspondingly, $\mathbb{X}^{\vee}$ consists of $\mathbb{X}^{\vee}$ and $d^{\vee}$. Moreover, if $\mathbb{X}$ carries any additional structure, for instance, a group or ring operation, then its ${}^{\vee}$-image is accordingly adjoined to $\mathbb{X}^{\vee}$, and these operations are extended to $X^{\sim}$ whenever possible.

[33]In expansion of Footnote 32, note that if $\mathbb{X}$ carries an additional structure of topological type, then one can consider the possibility of its extension from $\mathbb{X}^{\vee}$ to $\mathbb{X}^{\sim}$ in $\mathbf{V}^{(\mathbf{B})}$. For instance, if $+$ is a continuous group operation on $\mathbb{X}$, then for rather elementary reasons $+^{\vee}$ remains a continuous group operation on $\mathbb{X}^{\vee}$ with the truth value $\top_{\mathbf{B}}$. If $\mathbb{X}$ is a Polish space, that is, a separable complete metric space, then an argument based on the Shoenfield absoluteness theorem enables us to obtain an element $+^{\sim} \in \mathbf{V}^{(\mathbf{B})}$ which, with the truth value $\top_{\mathbf{B}}$, is a continuous extension of $+^{\vee}$ to $\mathbb{X}^{\sim}$ and is a group operation on the whole space $\mathbb{X}^{\sim}$, and furthermore, the latter space remains Polish in $\mathbf{V}^{(\mathbf{B})}$ with the truth value $\top_{\mathbf{B}}$. In this case the extended operation $+^{\sim}$ is naturally adjoined to the structure $\mathbb{X}^{\sim}$, converting the latter into a Polish group in $\mathbf{V}^{(\mathbf{B})}$ with the truth value $\top_{\mathbf{B}}$.

form of transfer used here. It remains to verify $[\![\mathbb{X}^\sim \models \varphi]\!]_{\mathbf{B}} = \top_{\mathbf{B}}$ on the basis of both the properties of $X^\sim$ in $\mathbf{V}^{(\mathbf{B})}$ and the content of φ.

Thus, the first part here consists in choosing a structure $\mathbb{X}$ such that $\mathfrak{A}$ is isomorphic to $(\mathbb{X}^\sim)^\wedge$ (with a suitable choice of the Boolean algebra $\mathbf{B}$, which often depends on $\mathfrak{A}$). It is not clear that such a selection of $\mathbb{X}$ for $\mathfrak{A}$ is always possible. However, if it is possible, then a transfer technique in the spirit of (II) can greatly facilitate the study of the given structure $\mathfrak{A}$. The details here are rather specific.

The first essential example of this kind was given by Gordon in [57], where it is demonstrated that the K-spaces well known in functional analysis are isomorphic to structures of the form $(\mathbb{R}^\sim)^\wedge$ if the Boolean algebra of all idempotent elements ($x^2 = x$) of a given K-space is chosen as $\mathbf{B}$. Another example [58] will be presented below in a particular case and with a different proof of rather logical character.

4d. Non-standard representations of complete metric spaces. This subsection contains preliminaries to the indicated example. To avoid repetitions let us fix a **complete** Boolean algebra $\mathbf{B}$ and agree to drop the index $_{\mathbf{B}}$ in notation like $[\![\,\cdot\,]\!]_{\mathbf{B}}$, $\top_{\mathbf{B}}$, $\perp_{\mathbf{B}}$, $\approx_{\mathbf{B}}$.

Let $\mathbb{X}$ be a fixed complete metric space. The elements $\mathbb{X}^\vee, \mathbb{X}^\sim \in \mathbf{V}^{(\mathbf{B})}$ and the collection $(\mathbb{X}^\sim)^\wedge$ of all 'reliable elements' of $\mathbb{X}^\sim$ are then defined (see subsection 4b). Our goal will be to define a map from $(\mathbb{X}^\sim)^\wedge$ to the set $\mathbf{C}''(\mathscr{S}, \mathbb{X})$ of all functions defined and continuous on dense $\mathbf{G}_\delta$-sets in the Stone space $\mathscr{S}$ of $\mathbf{B}$ and with values in $\mathbb{X}$, a map which is a bijection modulo the equivalence $\approx$ on $(\mathbb{X}^\sim)^\wedge$ and the following equivalence on $\mathbf{C}''(\mathscr{S}, \mathbb{X})$:

$$f \equiv g \quad \text{if the set } \{s \in \operatorname{dom} f \cap \operatorname{dom} g : f(s) = g(s)\} \text{ is dense in } \mathscr{S}.$$

We recall that the *Stone space* $\mathscr{S} = \mathscr{S}^{\mathbf{B}}$ of a Boolean algebra $\mathbf{B}$ consists of all ultrafilters $s \subseteq \mathbf{B}$ with the topology generated by all sets of the form $\mathscr{S}_b = \{s \in \mathscr{S} : b \in s\}$, where $b \in \mathbf{B}$. Stone spaces of complete Boolean algebras possess some special properties, in particular, closures of open sets are again open. It is well known that any Boolean algebra $\mathbf{B}$ can be canonically identified with the family of all closed-and-open subsets of $\mathscr{S}$ by means of the map $b \mapsto \mathscr{S}_b$.[34] The operations $\wedge$, $\vee$, $\complement$ in $\mathbf{B}$ correspond to the set-theoretic operations of intersection, union, and complement on closed-and-open sets in $\mathscr{S}$.[35] For instance, $\mathscr{S}_{b\vee c} = \mathscr{S}_b \cup \mathscr{S}_c$ and $\mathscr{S}_{\complement b} = \mathscr{S} \setminus \mathscr{S}_b$. It is also clear that $\mathscr{S}_\top = \mathscr{S}$ and $\mathscr{S}_\perp = \varnothing$.

Let $\mathscr{F}$ be the family of all non-empty closed sets $F \subseteq \mathbb{X}$.

Theorem 4.2. *Let* $\mathbf{B}$, $\mathbb{X}$, $\mathscr{S}$ *be as above.*

(i) *A function* $\mathbf{f}_p \in \mathbf{C}''(\mathscr{S}, \mathbb{X})$ *can be associated with each* $p \in (\mathbb{X}^\sim)^\wedge$ *in such a way that* $p \approx q \Rightarrow \mathbf{f}_p = \mathbf{f}_q$ *and in addition*

$$[\![p \in F^\sim]\!] \geqslant b \Leftrightarrow \mathbf{f}_p``(\mathscr{S}_b \cap \operatorname{dom} \mathbf{f}_\xi) \subseteq F \tag{16}$$

for all $b \in \mathbf{B}$ *and* $F \in \mathscr{F}$.

[34] Every $\mathscr{S}_b$ is open by definition, and closed since the set $\mathscr{S} \setminus \mathscr{S}_b = \mathscr{S}_{\complement b}$ is also open. If $X \subseteq \mathscr{S}$ is closed-and-open, then there exist sets $A, B \subseteq \mathbf{B}$ such that $X = \bigcup_{b\in A} \mathscr{S}_b$ and $\mathscr{S} \setminus X = \bigcup_{b\in B} \mathscr{S}_b$. One easily verifies that $X = \mathscr{S}_a$, where $a = \bigwedge A$ in $\mathbf{B}$.

[35] This property may fail for infinite operations. For instance, the operation 'closure of $\bigcup_{a\in A} \mathscr{S}_a$' in $\mathscr{S}$ and not just the union $\bigcup_{a\in A} \mathscr{S}_a$ corresponds to the infinite operation $\bigvee A$ in $\mathbf{B}$.

(ii) *An element* $p = \mathbf{p}_f \in (\mathbb{X}^\sim)^\wedge$ *can be associated with each function* $f \in \mathbf{C}''(\mathscr{S}, \mathbb{X})$ *in such a way that* $f \equiv \mathbf{f}_p$ *and in addition* $f \equiv g \Rightarrow \mathbf{p}_f \approx \mathbf{p}_g$.

This means that the maps $p \mapsto \mathbf{f}_p$ *and* $f \mapsto \mathbf{p}_f$ *induce a bijection of* $(\mathbb{X}^\sim)^\wedge/\approx$ *onto* $\mathbf{C}''(\mathscr{S}, \mathbb{X})/\equiv$ *and the inverse bijection of* $\mathbf{C}''(\mathscr{S}, \mathbb{X})/\equiv$ *onto* $(\mathbb{X}^\sim)^\wedge/\approx$.

Proof. (i) Put $\mathscr{F}_k = \{F \in \mathscr{F} :$ diameter of $F \leqslant k^{-1}\}$. If $F \in \mathscr{F}_k$, then it is easy to see that $F^\sim$ is a closed set of diameter $\leqslant k^{-1}$ in $\mathbb{X}^\sim$ in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$. Every set of the form $G_k = \bigvee_{F \in \mathscr{F}_k} \mathscr{S}_{[\![p \in F^\sim]\!]}$ is dense in $\mathscr{S}$. Indeed, consider an arbitrary $b \in \mathbf{B}$, $b \neq \bot$. We recall that $F^\vee$ is dense in $\mathbb{X}^\sim$ in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$, hence there exist a $c \in \mathbf{B}$ with $c \leqslant b$ and a point $x \in \mathbb{X}$ such that, with truth value $\geqslant c$ in $\mathbf{V}^{(\mathbf{B})}$, the distance between $x^\vee$ and p in $\mathbb{X}^\sim$ is not bigger than $(2k)^{-1}$. It follows that $[\![\xi \in F^\sim]\!] \geqslant c$, where $F \in \mathscr{F}_k$ is a closed ball with diameter k^{-1} and centre at x. Therefore, $\mathscr{S}_c \subseteq G_k$ as required.

Thus, $S_p = \bigcap_k G_k$ is a dense $\mathbf{G}_\delta$-set in $\mathscr{S}$.

By definition, if $s \in S_p$, then the set $\mathscr{F}_p(s) = \{F \in \mathscr{F} : [\![p \in F^\sim]\!] \in s\}$ has a non-empty intersection with each $\mathscr{F}_k$. On the other hand, if R and Q are disjoint closed sets in $\mathbb{X}$, then it is easy to show that $[\![R^\sim \cap Q^\sim \neq \varnothing]\!] = \bot$. Thus, the truth values $[\![p \in R^\sim]\!]$ and $[\![p \in Q^\sim]\!]$ are incompatible in $\mathbf{B}$ and hence cannot simultaneously belong to s. It follows that $\mathscr{F}_p(s)$ is a Cauchy filter (for $s \in S_p$). Therefore, the intersection $\bigcap \mathscr{F}_p(s)$ contains a single point, denoted by $\mathbf{f}_p(s)$ below. Thus, a continuous function $\mathbf{f}_p \colon S_p \to \mathbb{X}$ is defined.

It remains to prove (16) for each $b \in \mathbf{B}$ and $F \in \mathscr{F}$. Suppose that $[\![p \in F^\sim]\!] \ngeqslant b$. Then $b' = [\![p \notin F^\sim]\!] \wedge b \neq \bot$. By the definition of $\cdot^\sim$, there exist a point $x \in \mathbb{X}$ and a number k such that 1) the distance between x and F in $\mathbb{X}$ is bigger than $\frac{2}{k}$, and 2) $b'' = b' \wedge [\![$the distance between $x^\vee$ and p in $\mathbb{X}^\sim$ is $< \frac{1}{k}]\!] \neq \bot$. Furthermore, by the construction there is a set $F_1 \in \mathscr{F}_k$ such that $c = b'' \wedge [\![p \in {F_1}^\sim]\!] \neq \bot$. Clearly, F_1 is included in the $\frac{2}{k}$-neighbourhood of x, and hence $F_1 \cap F = \varnothing$. However, we still have by definition that $\mathbf{f}_p(s) \in F_1$ for all $s \in \mathscr{S}_c \cap S_p$; thus, the right-hand side of the equivalence in the lemma fails. (Indeed, $\varnothing \neq \mathscr{S}_c \cap S_p \subseteq \mathscr{S}_b$.)

To prove the converse, suppose that $\mathbf{f}_p(s) \notin F$ for at least one $s \in \mathscr{S}_b \cap S_p$. Since $\mathbf{f}_p$ is continuous and F is closed, there exist $b' \in \mathbf{B}$, $x \in \mathbb{X}$, and a number k such that $\bot \neq b' \leqslant b$, the distance between x and F in $\mathbb{X}$ is bigger than $\frac{2}{k}$, and $\mathbf{f}_p(s)$ belongs to the $\frac{1}{k}$-neighbourhood of x for any $s \in \mathscr{S}_{b'} \cap S_p$. Once again take a set $F_1 \in \mathscr{F}_k$ such that $c = b' \wedge [\![p \in {F_1}^\sim]\!] \neq \bot$. By construction, $\mathbf{f}_p(s) \in F_1$ for all $s \in \mathscr{S}_c \cap G_p$, and hence $F_1 \cap F = \varnothing$. Therefore, $[\![p \notin F^\sim]\!] \geqslant c$, so that $[\![p \notin F^\sim]\!] \ngeqslant b$.

(ii) Suppose that $G = \operatorname{dom} f = \bigcap_k G_k$, where each G_k is an open dense subset of $\mathscr{S}$. We carry out an essentially equivalent construction of a convergent filter of closed sets. Namely, we define an element $C_f \in \mathbf{V}^{(\mathbf{B})}$ as follows:

$$C_f(R^\sim) = \bigvee B_f(R), \quad \text{where} \quad B_f(R) = \{b \in \mathbf{B} : f\text{"}(\mathscr{S}_b \cap \operatorname{dom} f) \subseteq R\}$$

for every set $R \in \mathscr{F}$. We assert that $[\![\Phi]\!] = \top$, where Φ is the formula "C_f consists of closed subsets of $\mathbb{X}^\sim$, is closed with respect to $\cap$, and contains a set F of diameter $\leqslant k^{-1}$ for every $k \in \mathbb{N}^\vee$."

Indeed, suppose that, for example, $[\![C_f$ is not $\cap$-closed$]\!] = b > \bot$. There exist sets $R, Q \in \mathscr{F}$ such that $[\![R^\sim, Q^\sim \in C_f \wedge F^\sim \notin C_f]\!] = b' > \bot$, where $F = R \cap Q$. By definition, $b \leqslant \bigvee B_f(R)$ and $b \leqslant \bigvee B_f(Q)$, and hence there exist elements

$a_1 \in B_f(R)$ and $a_2 \in B_f(Q)$ satisfying $b'' = b' \wedge a_1 \wedge a_2 > \bot$. However, again by definition, the element $a_3 = a_1 \wedge a_2$ belongs to $B_f(F)$, and therefore $[\![F^\sim \in C_f]\!] \geqslant a_3$. Finally, a_1 and b are compatible in $\mathbf{B}$ because $b \wedge a_3 \geqslant b'' > \bot$, a contradiction.

It follows from $[\![\Phi]\!] = \top$ that there exists an element $p \in \mathbf{V}^{(\mathbf{B})}$ satisfying $[\![p \in \mathbb{X}^\sim]\!] = \top$ and $[\![R^\sim \in C_f \Rightarrow p \in R^\sim]\!] = \top$ for all $R \in \mathscr{F}$. We conclude that $[\![p \in R^\sim]\!] = \bigvee B_f(R) = C_f(R^\sim)$ in $\mathbf{B}$ for each $R \in \mathscr{F}$.

Now we have to prove that $\mathbf{f}_p \equiv f$. We assert that f and $\mathbf{f}_p$ coincide on the intersection $G \cap S_p$ of two dense $\mathbf{G}_\delta$-sets $G = \operatorname{dom} f$ and $S_p = \operatorname{dom} \mathbf{f}_p$. Suppose not. Then there exist a $b \in \mathbf{B}$ with $b \neq \bot$ and a pair of disjoint sets $R, Q \in \mathscr{F}$ such that $\mathbf{f}_p``(\mathscr{S}_b \cap S_p) \subseteq R$ and $f``(\mathscr{S}_b \cap G) \subseteq Q$. Then $[\![p \in R^\sim]\!] \geqslant b$ by (16). On the other hand, we have $b \in B_f(Q)$ by definition, and hence $b \leqslant C_f(Q^\sim)$ in $\mathbf{B}$, so that $[\![p \in Q^\sim]\!] \geqslant [\![Q^\sim \in C_f]\!] \geqslant b$. Thus, $[\![R^\sim \cap Q^\sim \neq \varnothing]\!] \geqslant b > \bot$, a contradiction.

Finally, $f \equiv g$ implies that $B_f(R) = B_g(R)$ for each $R \in \mathscr{F}$ by the continuity of the functions, and therefore $f \equiv g$ implies that $\mathbf{p}_f \approx \mathbf{p}_g$. Theorem 4.2 is proved.

In some cases Theorem 4.2 can be formulated more precisely with respect to the class of the function f that represents the element $p \in (\mathbb{X}^\sim)^\wedge$.

1) Suppose, for instance, that $\mathbb{X}$ is a **compact** (metric) space. Then for any k there exists a **finite** covering $\mathbb{X} = R^k_1 \cup \dots \cup R^k_{n(k)}$ by sets $R_i \in \mathscr{F}$ of diameter $\leqslant k^{-1}$. Then $\top = [\![p \in \mathbb{X}^\sim]\!] = [\![p \in {R^k_1}^\sim]\!] \vee \dots \vee [\![p \in {R^k_{n(k)}}^\sim]\!]$, hence for any point $s \in \mathscr{S}$ there exists a suitable number $1 \leqslant i \leqslant n(k)$ such that $[\![p \in {R^k_i}^\sim]\!] \in s$. This is just as above (in the proof of (i)), with the addition that $G_k = \mathscr{S}$ for all k, that is, $S_p = \mathscr{S}$. Thus, in this case Theorem 4.2 can be strengthened by the replacement of $\mathbf{C}''(\mathscr{S}, \mathbb{X})$ with the (much smaller) set $\mathbf{C}(\mathscr{S}, \mathbb{X})$ of all everywhere defined continuous functions $\mathscr{S} \to \mathbb{X}$.

2) Now suppose that the space $\mathbb{X}$ is only $\boldsymbol{\sigma}$**-compact**, that is, $\mathbb{X} = \bigcup_k F_k$, where each F_k is compact. It can be assumed that every set F_k is a closed ball $F_k = \{x \in \mathbb{X} : \rho(x, x_k) \leqslant \varepsilon_k\}$, where $x_k \in \mathbb{X}$ and $\varepsilon_k > 0$. Let $p \in (\mathbb{X}^\sim)^\wedge$. One can show[36] that $\bigvee_k [\![p \in {F_k}^\sim]\!] = \top$, and then the set $S_p = \bigcup_k \mathscr{S}_{[\![p \in {F_k}^\sim]\!]}$ is open and dense in $\mathscr{S}$. In this case a continuous function $\mathbf{f}_p \colon S_p \to \mathbb{X}$ can be defined separately on each closed-and-open set $\mathscr{S}_{[\![p \in {F_k}^\sim]\!]}$ as in the case of a compact space. Thus, in the σ-compact case Theorem 4.2 can be strengthened by replacing $\mathbf{C}''(\mathscr{S}, \mathbb{X})$ by the set $\mathbf{C}'(\mathscr{S}, \mathbb{X})$ of all functions defined and continuous on dense open subsets of $\mathscr{S}$.

3) In the case of a locally compact but not separable space $\mathbb{X}$ the argument outlined in Footnote 36 meets difficulties. However, the difficulties can be circumvented under the assumption that the local compactness of $\mathbb{X}$ is uniform in the sense that for some $r > 0$ all balls of radius r in $\mathbb{X}$ are compact. Note that this condition holds when $\mathbb{X}$ carries an appropriate algebraic structure, for instance, the structure of a topological group. Locally compact spaces that do not have this property of uniformity take us into a special area of general topology.

4) As far as non-metric spaces are concerned, the papers [31], [32], [59] contain some results similar to Theorem 4.2 and, in some aspects, to its modifications 1) and 2), but for uniform spaces in the sense of Bourbaki [60].

[36] This is not immediately transparent. To prove the equality $[\![\mathbb{X}^\sim = \bigcup_{k \in \mathbb{N}^\vee} {F_k}^\sim]\!] = \top$ (implying that $\bigvee_k [\![p \in {F_k}^\sim]\!] = \top$) we must express the evaluated formula in the form of a Π^1_1-formula and then use the Shoenfield absoluteness theorem.

4e. An example: rings of functions. In the case considered in subsection 4d, the map $p \mapsto \mathbf{f}_p$ can also be an algebraic isomorphism when the space $\mathbb{X}$ carries some algebraic structure.

As an example we consider the case when a given complete metric space $\mathbb{X}$ is a topological ring (both operations are continuous as functions of two variables). *Assume for simplicity that* $\mathbb{X}$ *is separable.* Then the operations on $\mathbb{X}$ can be extended in $\mathbf{V}^{(\mathbf{B})}$ to $\mathbb{X}^\sim$ (see Footnote 33) so that $\mathbb{X}^\sim$ remains a topological ring in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$. This enables us to define a ring structure on the quotient $(\mathbb{X}^\sim)^\wedge/{\approx}$ in the ground universe: $[\xi_1]_\approx + [\xi_2]_\approx = [\xi]_\approx$, where ξ is a unique, modulo $\approx$, element of $(\mathbb{X}^\sim)^\wedge$ satisfying $[\![\xi_1 + \xi_2 = \xi]\!] = \top$ (this is well defined, by Theorem 4.1(i)). The same holds for the product operation. On the other hand, the componentwise operations, that is, for example, $(f+g)(x) = f(x) + g(x)$, convert the quotient $\mathbf{C}''(\mathscr{S}, \mathbb{X})/\equiv$ into a ring.

Theorem 4.3. *In this case the map* $p \mapsto \mathbf{f}_p$ *is not only a bijection but also a ring isomorphism between* $(\mathbb{X}^\sim)^\wedge/\approx$ *and* $\mathbf{C}''(\mathscr{S}, \mathbb{X})/\equiv$.

If the space $\mathbb{X}$ *is compact* (*respectively, locally compact*), *then* $\mathbf{C}''(\mathscr{S}, \mathbb{X})$ *can be replaced by* $\mathbf{C}(\mathscr{S}, \mathbb{X})$ (*respectively, by* $\mathbf{C}'(\mathscr{S}, \mathbb{X})$).

Proof. We only check that the map preserves the additive operation. Suppose that $p, q, r \in (\mathbb{X}^\sim)^\wedge$ and $[\![p = q + r]\!] = \top$. We have to prove that the corresponding functions $\mathbf{f}_p$, $\mathbf{f}_q$, $\mathbf{f}_r \in \mathbf{C}''(\mathscr{S}, \mathbb{X})$ satisfy $\mathbf{f}_p(s) = \mathbf{f}_q(s) + \mathbf{f}_r(s)$ (in the ring $\mathbb{X}$) for all s in the dense open set $D = \operatorname{dom} \mathbf{f}_p \cap \operatorname{dom} \mathbf{f}_q \cap \operatorname{dom} \mathbf{f}_r$. Suppose not. Since the functions and the operation $+$ are continuous, there exist two disjoint sets $R, S \in \mathscr{F}$, another pair of sets $F, F' \in \mathscr{F}$, and a closed-and-open set of the form $\mathscr{S}_b$, $b \in \mathbf{B}$, such that $\mathbf{f}_p(s) \in S$, $\mathbf{f}_q(s) \in F$, and $\mathbf{f}_r(s) \in F'$ for all $s \in D \cap \mathscr{S}_p$, and in addition, $x + y \in R$ for all $x \in F$ and $y \in F'$. Then the elements $c_1 = [\![p \in S^\sim]\!]$, $c_2 = [\![q \in F^\sim]\!]$, $c_3 = [\![r \in F'^\sim]\!]$ of the Boolean algebra $\mathbf{B}$ belong to s. But s is an ultrafilter, and therefore $d = c_1 \wedge c_2 \wedge c_3 > \bot$. Thus, $[\![q \in F^\sim \wedge r \in F'^\sim]\!] \geqslant d > \bot$ in $\mathbf{B}$.

On the other hand, it is not hard to see that the indicated relation between the sets R, F, F' is absolute in the sense that $[\![\forall\, x \in F^\sim\, \forall\, y \in F'^\sim\, (x + y \in R^\sim)]\!] = \top$. This implies that $[\![p \in R^\sim]\!] \geqslant d > \bot$, in contradiction to the fact that $[\![p \in S^\sim]\!] = c_1 \geqslant d$, because $[\![R^\sim \cap S^\sim = \varnothing]\!] = \top$. Theorem 4.3 is proved.

The theorem just proved is applicable, for example, in the case when $\mathbb{X} = \mathbb{C} =$ the field of complex numbers[37] and $\mathfrak{A} =$ the ring $\mathbf{C}'(\mathscr{S}, \mathbb{C})/\equiv$ in the scheme of subsection 4c. Furthermore, there exists a class of formulae which admit a transfer of type (II) in subsection 4c by purely syntactical reasons. We recall that a *Horn formula* is any formula of the form ΠA, where Π is a quantifier prefix and A is a conjunction of formulae of the form $(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \Rightarrow \alpha$ with α, α_k atomic formulae of the given language.

Lemma 4.4. *Let* Φ *be a closed Horn formula in the language of ring theory, with parameters in* $(\mathbb{C}^\sim)^\wedge$. *If* $[\![\Phi$ *holds in* $\mathbb{C}^\sim]\!] = \top$, *then* Φ *holds in* $(\mathbb{C}^\sim)^\wedge$.

[37] It is important that $\mathbb{C}^\sim$ is the field of complex numbers in $\mathbf{V}^{(\mathbf{B})}$ in the sense that $[\![\mathbb{C}^\sim$ is the field of complex numbers$]\!] = \top$. An analogous assertion is true for the real number field $\mathbb{R}$ as well. One easily proves a statement similar to Corollary 4.5 for the algebra $\mathbf{C}'(\mathscr{S}, \mathbb{R})$ and algebraic equations of odd degree.

Proof. We use induction on the complexity of Φ. The inductive steps for the quantifier $\forall$ and conjunction are trivial. The inductive step for $\exists$ is easily verifiable with the help of Theorem 4.1(ii). It remains to consider the case when Φ has the form $(\alpha_1 \wedge \dots \wedge \alpha_n) \Rightarrow \alpha$, where α and α_k are atomic formulae of the language of ring theory. Suppose that $[\![\mathbb{C}^\sim \models \Phi]\!] = \top$. Then $[\![\mathbb{C}^\sim \models \alpha]\!] \geqslant [\![\mathbb{C}^\sim \models \alpha_1 \wedge \dots \wedge \alpha_n]\!]$. Assuming that all of the α_k are true in $(\mathbb{C}^\sim)^\wedge$, we prove this for α. Note that α_k is an atomic formula, for instance, $r + s = t$, where r, s, t belong to $(\mathbb{C}^\sim)^\wedge$. By the definition of truth in $(\mathbb{C}^\sim)^\wedge$, we have $[\![\mathbb{C}^\sim \models \alpha_k]\!] = \top$, and hence $[\![\mathbb{C}^\sim \models \alpha_1 \wedge \dots \wedge \alpha_n]\!] = \top$. It follows that $[\![\mathbb{C}^\sim \models \alpha]\!] = \top$, that is, the atomic formula α is true in $(\mathbb{C}^\sim)^\wedge$, as required. Lemma 4.4 is proved.

Corollary 4.5. *Suppose that Φ is a closed Horn formula in the language of ring theory, true in the field $\mathbb{C}$ of complex numbers. Then under our assumptions Φ is true in $\mathbf{C}'(\mathscr{S},\mathbb{C})/{\equiv}$. In particular, any algebraic equation of degree $\geqslant 1$ has a solution in $\mathbf{C}'(\mathscr{S},\mathbb{C})/{\equiv}$.*

Obviously, the Horn property plays a key role in the elimination of the general step for the disjunction in the proof of Lemma 4.4. We are not aware of any general results on such a transfer for non-Horn formulae. Nevertheless a 'non-Horn' transfer is possible for some examples mentioned in [29], [59]. The point is that if $[\![\mathbb{C}^\sim \models \alpha_1 \vee \dots \vee \alpha_n]\!] = \top$ in the notation of the proof of Lemma 4.4, then there exists a 'partition of unity' $\top = b_1 \vee \dots \vee b_n$ satisfying $[\![\mathbb{C}^\sim \models \alpha_k]\!] = b_k$ $\forall\, k$. In other words, $[\![\mathbb{C}^\sim \models \alpha_k]\!] = \top$ in the sense of the Boolean subalgebras $\mathbf{B}_k = \{b \in \mathbf{B} : b \leqslant b_k\}$. In the indicated examples one is able to glue together the results obtained for different indices k, leading to the required transfer theorem.

4f. Non-standard representation of a standard group homomorphism. The principal element of the transfer scheme in subsection 4c is obviously the construction of a non-standard representation of a given mathematical structure $\mathfrak{A}$ in the form $(\mathbb{X}^\sim)^\wedge$ for an appropriate more elementary structure $\mathbb{X}$. A series of interesting problems in this direction was considered in [59], [32]. Here we discuss one of them.

Suppose that $\mathbb{A}$ and $\mathbb{X}$ are complete metric spaces, and $\mathbf{B}$ is a complete Boolean algebra with Stone space $\mathscr{S}$. Assume that a function $f_a \colon S_a \to \mathbb{X}$, defined and continuous on a dense $\mathbf{G}_\delta$-set $S_a \subseteq \mathscr{S}$, is associated with each $a \in \mathbb{A}$. In other words, $f_a \in \mathbf{C}''(\mathscr{S},\mathbb{X})$. By Theorem 4.2, to each $a \in \mathbb{A}$ there corresponds an element $p_a = p_{\mathbf{f}_a} \in (\mathbb{X}^\sim)^\wedge$ such that

$$[\![p_a \in F^\sim]\!] \geqslant b \Leftrightarrow f_a``(\mathscr{S}_p \cap S_a) \subseteq F \quad \text{for all } b \in \mathbf{B} \text{ and } F \in \mathscr{F}. \tag{17}$$

($\mathscr{F}$ = all non-empty closed sets $F \subseteq \mathbb{X}$, as above.) Thus, $a \mapsto p_a$ maps $\mathbb{A}$ to $(\mathbb{X}^\sim)^\wedge$. In this case there exists a $t \in \mathbf{V}^{(\mathbf{B})}$ such that "t is a function from $\mathbb{A}^\vee$ to $\mathbb{X}^\sim$ and $t(a^\vee) = p_a$" holds in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$ for every $a \in \mathbb{A}$.

Is it possible to extend t in $\mathbf{V}^{(\mathbf{B})}$ (with truth value $\top$) to a continuous function $t' \colon \mathbb{A}^\sim \to \mathbb{X}^\sim$? This problem can be solved in general form, but we consider it under the following simplifying assumptions.

1) $\mathbb{A}$ is a continuous Abelian group, and the metric ρ of the space $\mathbb{X}$ is invariant: $\rho(a,b) = \rho(a+c, b+c)$ for all $a,b,c \in A$.

2) $\mathbb{X} = \mathbb{C}$, the complex numbers. The sets $S_a = \operatorname{dom} f_a$ can be assumed to be open and dense in $\mathscr{S}$ by the σ-compactness of $\mathbb{C}$, so that $f_a \in \mathbf{C}'(\mathscr{S}, \mathbb{X})$. (See Remark 2) at the end of subsection 4d).
3) The map $a \mapsto f_a$ is a homomorphism in the sense that for any pair $a, b \in \mathbb{A}$ the equality $f_a(s) + f_b(s) = f_{a+b}(s)$ holds for all s in some open dense set $D = D_{ab} \subseteq \mathscr{S}$.

We put $\|a\| = \rho(\mathbf{0}, a)$, where $\mathbf{0}$ is the neutral element of the group $\mathbb{A}$, so that $\rho(a,b) = \|a - b\|$. As usual, $|z|$ is the absolute value of $z \in \mathbb{C}$.

Theorem 4.6. *In this notation and under these conditions the following is a necessary and sufficient condition for the existence of an element $t' \in \mathbf{V}^{(\mathbf{B})}$ such that "t' is a continuous homomorphism $\mathbb{A}^\sim \to \mathbb{C}^\sim$ and an extension of t" holds in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$:*

for every $k \geqslant 1$ there is a function $g_k \colon D_k \to \mathbb{N}$ defined and continuous on an open dense set $D_k \subseteq \mathscr{S}$ and such that for all $s \in \mathscr{S}$ and $a \in \mathbb{A}$, $|f_a(s)| < k^{-1}$ holds whenever $s \in S_a \cap D_k$ and $\|a\| < g_k(s)^{-1}$.

This condition is obviously weaker than the convergence of $f_a(s)$ to 0 as $a \to \mathbf{0}$ uniformly with respect to s (which is obtained for g_k equal to suitable constants), but it is stronger than the pointwise convergence 'almost everywhere', that is, on some open dense set.

Proof. Sufficiency. The rather easy proof that "t is a homomorphism from $\mathbb{A}^\vee$ to $\mathbb{C}^\sim$" holds in $\mathbf{V}^{(\mathbf{B})}$ with truth value $\top$ is left to the reader. The invariance of the metric of $\mathbb{A}$ can be transferred to $\mathbf{V}^{(\mathbf{B})}$, hence the task is reduced to the proof of "t is continuous at $\mathbf{0}$" in $\mathbf{V}^{(\mathbf{B})}$, which will be carried out now.

Fix a number $k \geqslant 1$. By Theorem 4.2(ii), a function g_k as in the theorem generates an element $\nu_k = \mathbf{p}_{g_k} \in \mathbf{V}^{(\mathbf{B})}$ such that $[\![\nu_k \in \mathbb{N}^\vee]\!] = \top$ and $\mathbf{f}_{\nu_k} \equiv g_k$. It follows that

$$[\![\nu_k = n]\!] \geqslant b \Leftrightarrow \forall\, s \in \mathscr{S}_b \cap D_k \ (g_k(s) = n) \tag{18}$$

for all $b \in \mathbb{B}$ and $n \in \mathbb{N}$. We assert that $[\![\forall\, a \in \mathbb{A}^\vee \ (\|a\| < \nu_k^{-1} \Rightarrow |p_a| < (k^\vee)^{-1})]\!] = \top$. Suppose on the contrary that $a \in \mathbb{A}$ satisfies $[\![\, \|a^\vee\| < \nu_k^{-1} \wedge |p_a| \geqslant (k^\vee)^{-1}]\!] = b > \bot$. There exists an n with $c = b \cap [\![\nu_k = n]\!] > \bot$, and then $[\![\, \|a^\vee\| < (n^\vee)^{-1} \wedge |p_a| \geqslant (k^\vee)^{-1}]\!] = c$. Then $g_k(s) = n$ for all $s \in D_k \cap \mathscr{S}_c$ by (18), hence $|f_a(s)| < k^{-1}$ for all $s \in D_k \cap S_a \cap \mathscr{S}_c$ by the assumptions of the theorem. It follows that $[\![\, |p_a| < (k^\vee)^{-1}]\!] \geqslant c$ by (17), and this contradicts the choice of b.

Necessity. Suppose that $t' \in \mathbf{V}^{(\mathbf{B})}$ and

$$[\![t' \text{ is a continuous homomorphism } \mathbb{A}^\sim \to \mathbb{C}^\sim \text{ and an extension of } t]\!] = \top.$$

In particular, $[\![t' \text{ is continuous at } \mathbf{0}]\!] = \top$. Therefore, for every k there is an element $\nu_k \in \mathbf{V}^{(\mathbf{B})}$ such that the truth value of "$\nu_k \in \mathbb{N}^\vee$ and $\forall\, a \in \mathbb{A}^\sim (\|a\| < \nu_k{}^{-1} \Rightarrow |t'(a)| < (k^\vee)^{-1})$" is equal to $\top$. It remains to put $g_k = \mathbf{f}_{\nu_k}$ (Theorem 4.2(i)). Theorem 4.6 is proved.

Another approach to the same problem was suggested in [31]. It turns out that if the complete Boolean algebra of projections in an appropriate Banach space is taken in the role of $\mathbf{B}$, then the condition of Bochner integrability becomes sufficient for continuous extendibility in the spirit of Theorem 4.6 (Theorem 16 in [31]; see also the earlier paper [58]). A locally compact Abelian group is taken as $\mathfrak{A}$.

Historical and bibliographical remarks to § 4. The origins of forcing were set up in Cohen's investigations [35] on the continuum hypothesis. A systematic exposition of a general version of forcing in the form of a Boolean-valued universe (including Theorem 4.1) was probably first given in [34], [50], where it is mentioned that such a generalization is based on earlier studies of Scott, Solovay [61], and Vopěnka [62] (see also [63] and [64]). In the publications [34], [50] the notation $x^\vee$ and $X^\wedge$ was introduced, and in [29] $X^\wedge$ is re-denoted as $X\downarrow$ (the 'descent' of X).

Theorem 4.2 together with the modifications for compact and σ-compact spaces (and even in a more general form for uniform spaces in the sense of Bourbaki [60] instead of metric spaces) appeared in the papers [58], [65], [59] of Lyubetskii and Gordon (and some preliminary ideas in [66]). Further attempts to strengthen the result to continuous functions on open dense sets in the case of complete uniform topological spaces may require additional efforts. To some degree, a predecessor of all these results is the representation of real numbers in random, Cohen-generic, and certain other models of set theory as the values taken on the extending real by continuous functions.

The definition of $X^\sim$ for complete topological spaces and other notions and results related to $X^\sim$ were presented with complete proofs in [58].

On Horn formulae see, for example, 6.2 in [39]. There is a certain similarity between Lemma 4.4 and Theorem 6.2.2 in 6.2 in [39] on the stability of Horn formulae.

Theorem 4.6 was proved by the authors.

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