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Internal Approach to External Sets and Universes *

Part 1
Bounded set theory

Abstract. A problem which enthusiasts of **IST**, Nelson's internal set theory, usually face is how to treat external sets in the internal universe which does not contain them directly. To solve this problem, we consider **BST**, *bounded set theory*, a modification of **IST** which is, briefly, a theory for the family of those **IST** sets which are members of standard sets.

We show that **BST** is strong enough to incorporate external sets in the internal universe in a way sufficient to develop the most advanced applications of nonstandard methods. In particular, we define in **BST** an enlargement of the **BST** universe which satisfies the axioms of **HST**, an external theory close to a theory introduced by Hrbaček.

HST includes Replacement and Saturation for all formulas but contradicts the Power Set and Choice axioms (either of them is incompatible with Replacement plus Saturation), therefore to get an external universe which satisfies all of **ZFC** minus Regularity one has to pay by a restriction of Saturation. We prove that **HST** admits a system of subuniverses which model **ZFC** (minus Regularity but with Power Set and Choice) and Saturation in a form restricted by a fixed but arbitrary standard cardinal.

Thus the proposed system of set theoretic foundations for nonstandard mathematics, based on the simple and natural axioms of the internal theory **BST**, provides the treatment of external sets sufficient to carry out elaborate external constructions.

This article [†] is the first in the series of three articles devoted to set theoretic

foundations of nonstandard mathematics, to be published by *Studia Logica*. This research was accomplished as a single paper, too long, indeed, to be published in this Journal as a single paper.

The following Preface and Introduction present the problems which motivated this research, and the results and conclusions, related both to this first part and the two next parts.

Preface

Since Kreisel [19] initiated consideration of axiomatic systems for nonstandard analysis, several approaches to this matter have been suggested.

First of all, this is **RZ**, the theory of Robinson and Zakon [27] (see also Keisler [17]), which axiomatizes nonstandard extensions [1] of mathematical structures in the same sense as, say, the list of axioms for linearly ordered sets axiomatizes the class of all linearly ordered sets.

We consider, however, the other approach which intends to axiomatize the universe of all sets, "the universe of discourse" as it is called sometimes, in a nonstandard way rather than to describe nonstandard structures in the standard universe of **ZFC**. This approach also splits in two principal directions which we call here *internal* and *external*. Both of them have something in common: both assume that the "universe of discourse" includes a proper part, the *class of all standard sets*, which we denote by $\mathbb{S}$ and which is informally identified with the universe of all sets considered by "classical", non–nonstandard mathematics. But they differ from each other in the answer to the question of how $\mathbb{S}$ relates to the "universe of discourse".

The internal approach sees the "universe of discourse" as an elementary extension of $\mathbb{S}$ which obeys a certain amount of Saturation (called Idealization, or weak Saturation, see footnote 10), and such that, with respect to $\mathbb{S}$, no new bounded (= parts of sets) collections of standard sets can be defined. The universe of a theory of this type is usually called *internal universe*; we

[1] We refer the reader to Henson and Keisler [8], Hurd and Loeb [11], Keisler, Kunen, Miller, and Leth [18], Lindstrøm [20], Luxemburg [22], Stroyan and Bayod [28] on matters of nonstandard mathematical structures.

shall denote it by I. Formally, $\mathbb{S}$ is distinguished in the "universe of discourse" by the *predicate of standardness* st which, therefore, becomes the second basic predicate (together with the membership $\in$).

This approach was realized in Nelson's *internal set theory* **IST**. Its axioms are rather simple, easy to use, and well grounded philosophically, and have successfully demonstrated the ability to work in various branches of nonstandard mathematics.

There is, however, a problem considered sometimes as fatal for **IST** as a base for the development of nonstandard mathematics. Indeed, it seems that **IST** fails to handle a very important type of nonstandard mathematical objects, therefore fails to serve as a system of foundations for nonstandard mathematics in all its totality.

Problem [2]

The **IST** *universe does not contain external sets*[3] *directly.* For example, the "set" of all standard natural numbers and the "set" of all real numbers infinitely close to 0 are *not* sets in **IST**. ⊣

This problem is fixed in the framework of the alternative *external* approach originated by Hrbaček [9, 10]. The internal universe I is assumed as above, but it does not exhaust the "universe of discourse". In particular, the latter contains *external sets* which do not exist in the internal universe. An external universe has also to satisfy a form of *Saturation*, e.g. the *standard size* Saturation which says that any standard size[4] external family of internal sets with the finite intersection property has nonempty intersection.

But this advantage is paid for by some other problems.

First of all the external universe cannot satisfy all of **ZFC** : Regularity fails since the set of all nonstandard I-natural numbers does not contain an $\in$-minimal element.

All other axioms, including Choice, can be saved; Kawaï [16] introduced a theory which contains the standard size Saturation and provides **ZFC** minus the Regularity axiom (but with a weak form of Regularity, an axiom

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[2] "... mathematics in **IST** looks more like traditional mathematics than mathematics in **RZ** does. For this reason, it is easier for a classical mathematician to read works in **IST** than in **RZ**. However, because the external sets are missing, developments such as the Loeb measure construction and hyperfinite descriptive set theory cannot be carried out in their full generality in **IST** ." This is from Keisler [17].

[3] In this setting, "external sets" are the mathematical objects naturally defined so that the standardness predicate occurs in the definition.

[4] To be of standard size means to be an image of the set of all standard elements of a standard set.

which says that the universe of all sets is wellfounded over the internal subuniverse), so that the standardness predicate may occur in the schemata of Comprehension and Replacement, and proved that the theory is a conservative extension of **ZFC**. Kawaï's theory can be criticized, indeed, from the other point of view. It includes **IST** as the theory of the internal subuniverse, and therefore admits sets which contain all standard sets, which contradicts the idea that nonstandard sets should not be much larger than standard ones.

This is the background for a more essential and visible defect, important especially for those who are inclined to treat nonstandard methods as shortcuts for "standard" reasoning, something like a higher level language which has some advantages with respect to the basic low level language but which can be in principle reduced to the basic language, see e.g. Nelson [24]. A necessary condition for such a treatment is that the new language does not generate really new properties of the "old" objects. This can be formalized as follows.

Reduction property [5]

We say that a nonstandard set theory T containing the standardness predicate st in the language is *reducible* to **ZFC** if

1) T proves **ZFC** in the standard universe $\mathbb{S} = \{x : \text{st}\, x\}$, and

2) for any formula $\Phi(x_1, ..., x_n)$ in the language of T there exists an $\in$-formula $\widehat{\Phi}(x_1, ..., x_n)$ such that

$$\forall^{\text{st}} x_1 \, ... \, \forall^{\text{st}} x_n \left[\, \Phi(x_1, ..., x_n) \longleftrightarrow \widehat{\Phi}^{\text{st}}(x_1, ..., x_n) \, \right]$$

is a theorem of T, where $\widehat{\Phi}^{\text{st}}$ denotes the relativization of $\widehat{\Phi}$ to $\mathbb{S}$. ⊣

If this property holds then one gains a sort of "vision" of a universe of T from the much more customary universe $\mathbb{S}$, the **ZFC** universe of "standard" mathematics, at least as long as one is interested in the properties and behaviour of standard sets only. Such a "materialization" of formal reasoning is not available on the base of conservativity.

But unfortunately neither **IST** nor, therefore, any theory which includes **IST** satisfies this property. [6] Thus, if the Reduction property is seen as

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[5] This is *not* conservativity; this property deals with *truth*, – well, it postulates that the truth of the formulas with standard parameters in the universe of T can be in fact determined in the **ZFC** universe of all standard sets, – which is close to mathematics, while the conservativity deals with *provability*, more in custom in logic.

[6] Kanovei [13], [14] found a sentence Φ in the st-$\in$-language which is not provably

indispensable, Kawaï's system does not work properly. But this is not the only problem.

Model enlargement property [7]

We say that a nonstandard set theory T has this property with respect to **ZFC** if every countable model M of **ZFC** can be enlarged to a model M' of T containing M as the class of all standard sets.   ⊣

If this does not hold then T "knows" something about the standard universe which **ZFC** itself does not know. This "something" may not be a 1st order property, however one has in this case an evidence that the nonstandard tools of T are more than another way of presentation of **ZFC**.

Neither **IST** nor Kawaï's theories satisfy the Model enlargement property; we shall see that the minimal transitive **ZFC** model M is not the standard part of a model of **IST**. (This is an easy consequence of the following: first, every $x \in M$ is $\in$-definable in M; second, **IST** provides a truth definition for $\in$-formulas in the standard subuniverse.)

Another idea was realized by Hrbaček [9, 10]: every internal set should be an element of a standard set. The corresponding internal theory, *bounded set theory* **BST** (the "internal" part of Hrbaček's theories NS_1 and **NZFC**) was explicitly formulated in Kanovei [12]. It will be demonstrated below that, unlike **IST**, **BST** satisfies both the Reduction and the Model enlargement property. [8]

At the external level Hrbaček's approach faces another problem: it is the case that the Replacement axiom in the external universe contradicts both Power Set and Choice, provided one keeps Saturation for families of standard size.

Hrbaček considered two possibilities: (i) to retain Replacement, which is the theory NS_1, and (ii) to retain Power Set plus Choice, which is NS_2,

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equivalent in **IST** to Ψ^{st} for any $\in$-sentence Ψ. Obviously such a sentence Φ is independent in **IST** .

[7] We use the word *enlargement* to denote relations between universes. We use the word *extension* to denote relations between theories, and also in the notion of the *Extension principle*, see below.

[8] It is an interesting and important open problem to figure out whether **BST** can be interpreted in **ZFC** so that the class of all standard sets in the sense of the interpretation coincides (or: is isomorphic to) the basic **ZFC** universe. (The proof of Theorem 2.4. below shows that **BST** has such an interpretation in **ZFGC**, the extension of **ZFC** by the axiom of global choice.) This problem may have a relation to the following question suggested to one of the authors by V. A. Uspensky: prove that in the Solovay model for **ZFC** plus "all projective sets are measurable" there does not exist a real–ordinal–definable nonstandard enlargement of $\mathbb{R}$.

or **NZFC**. [9] The first one satisfies the Model enlargement property (this was actually established by Hrbaček) and possibly the Reduction property, with respect to the second theory the question is open. But even apart from Reduction, the lack of one of the fundamental set theoretic principles of modern mathematics makes it impossible to treat either **NS$_1$** or **NZFC** as the base for nonstandard mathematics chosen once and forever.

The third possibility might be the version of **NS$_1$** containing Saturation restricted by a cardinal κ, and including the Power Set axiom. Here κ can be either a fixed cardinal, large enough to capture all instances of Saturation one may practically need, or a constant added to the language together with the axiom saying that κ is a standard infinite cardinal. This could be a solution; indeed, external sets are accomodated, **ZF** minus Regularity holds, a suitable amount of Choice (for families of standard κ-size) can be added, the Reduction and Model enlargement properties can be verified, and a sufficient size of κ can be guaranteed by the axiom which postulates that κ is, say, greater than any cardinal definable by a Σ_{17} formula. More exactly, this could be a *practical* solution, but by no means a *theoretical system of foundations*; indeed, if κ is an explicitly defined cardinal one may ask why this cardinal, not another one, is accepted as a fundamental constant of the theory. There is no reasonable answer.

These problems forced mathematicians who were interested in the topic to introduce two more flexible ideas. Fletcher [7] suggested **SNST**, a theory which arranges the external universe E as the union of the built-in system of subuniverses E_κ, where κ is a standard cardinal, almost satisfying the κ-version of **NS$_1$** mentioned above, but with Saturation replaced by the weakened form, weak Saturation or Idealization, directly relativized to the internal universe. [10] This lack of the external form of Saturation is

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[9] A convenient theory to develop nonstandard mathematics, as it was shown in [10], but one would like not to miss Replacement ... . Another interesting property of **NZFC** should be mentioned. The proof of Theorem 4a in [9] shows that either the class of all standard ordinals in the **NZFC** universe contains a cofinal image of its initial segment under a map definable in the external universe, or a strong class theory can be interpreted, therefore the standard subuniverse is a model of **ZFC** + Cons **ZFC**, not merely **ZFC**. This may be an evidence that **NZFC** does not satisfy the Model enlargement property. *An alternative idea how this can be demonstrated was suggested by the referee.*

[10] Saturation can be obtained from Idealization with the help of another useful principle, *Extension*, which makes it possible to extend external functions to internal ones in certain cases. Kawaï's system includes Extension, so Saturation becomes a theorem. Fletcher's theory of [7] does not include Extension; perhaps, the latter holds in the proposed model, but this needs a separate study. On the other hand, Saturation implies Extension, see Hrbaček [10], but Idealization, in general, does not imply it: Kanovei [12] showed that failure of Extension is consistent with the completely idealized theory **IST** .

a principal defect which does not allow to consider the theory as properly external as, say, Hrbaček's and Kawaï's theories are: it demonstrates that all serious things going on actually happen in the internal subuniverse, the external "envelope" is nothing more than a camouflage, and some important examples of external reasoning, as, say, the proof of countable additivity of the Loeb measure, cannot be formalized as they stand.

The equipment of **SNST** includes also the system of internal universes I_κ, and the common standard universe S. Apart from the mentioned defect, and, perhaps, the Reduction property, which should be studied separately, the idea of a "stratified" principal set theoretic universe differs too much from the currently known traditions to be easily acceptable. [11]

Another approach was introduced by Ballard and Hrbaček [2]. They consider **ZFBC**, the Zermelo – Fraenkel – Boffa set theory, as the basic theory. (**ZFBC** replaces the Regularity axiom by a strong form of its negation and adds the axiom of global choice.) **ZFBC** is strong enough to model a stratified system of internal universes I_κ, with the common external universe E which coincides with the **ZFBC** universe, and a common standard universe which can be chosen in many ways, say, as the class of all wellfounded sets. However **ZFBC** is a bit too "exotic" a theory to be accepted easily by those practically working in nonstandard mathematics.

To conclude, several variants of external set theories were proposed. Perhaps, Kawaï's theory from [16] is the best equipped technically, but it certainly fails to satisfy the Reduction property and gives an unpleasant picture of enormously large sets. The other candidates miss either something essential from **ZFC** (we do not mean Regularity, of course), or some essential nonstandard tools (say Saturation in the general form, not reduced to internal Idealization), and besides this involve tools the philosophical acceptability of which for the everyday mathematical work is quite problematic.

Introduction

This section introduces the common content of the series of three articles.

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[11] When the work over the final version of this article had been almost completed, one of the authors received the manuscript of D. Ballard [1]. It contains, among many other important results in foundations of nonstandard mathematics, development of a theory which has a semblance of **SNST** but at a much more advanced level.

Part 1: this article

The system of foundations for nonstandard mathematics we propose is based on *bounded set theory* **BST**, a minor modification of Nelson's **IST**, and, at the same time, the "internal part" of Hrbaček's theories **NS₁** and **NZFC =** **NS₂**, explicitly formulated (as an internal theory) by Kanovei [12]. This theory (see the description of the axioms in Section 1.) is based on the four simple and philosophically acceptable ideas, approved by the current practice of internal theories (mostly **IST**). They are as follows:

1. The "universe of discourse" $\mathbb{I}$ is an elementary extension of the universe $\mathbb{S}$ of standard sets.

2. $\mathbb{I}$ satisfies Idealization (Saturation for $\in$-formulas) in the case when the "index set" is of the form $^\sigma A = A \cap \mathbb{S}$, where A is standard.

3. Every collection of the form $\{x \in {}^\sigma X : \Phi(x)\}$, where X is standard, is equal to $^\sigma Y = \{y \in Y : \operatorname{st} y\}$ for some standard Y .

4. Every set belongs to a standard set.

Thus **BST** is a close relative of **IST**, completely equivalent to **IST** in the known applications of the latter in the framework of "conventional" nonstandard analysis.

Section 1. This section presents the basic technical tools of **BST**. We show that **BST** provides an instrumentarium more advanced than **IST** does, in particular:

— the Collection, Extension, Dependent Choice theorems (only the first of them is known to be a theorem of **IST** as well, see Kanovei [15]);

— a theorem which allows to convert every formula to an equivalent Σ_2^{st} formula [12] (proved in **IST** by Nelson [23] only for a special type of formulas),

— the Reduction property introduced in the Preface, so that the truth of formulas with standard parameters in the **BST** universe can be reduced to the **ZFC** truth in the subuniverse of all standard sets.

These theorems will provide a base for the treatment of external sets.

Section 2. We study interrelations between **ZFC**, **BST**, and **IST**. The following are the main results:

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[12] That is, to the form $\exists^{st} \forall^{st}$ ($\in$-formula).

1. **BST** has an inner model in **IST**, the class of all *bounded* sets, i.e. elements of standard sets; this allows to prove that **BST**, similarly to **IST**, is a conservative extension of **ZFC** .

2. **BST** satisfies the Model enlargement property: any countable model of **ZFC** can be embedded as the class of all standard sets into a **BST** model.

3. The minimal transitive model of **ZFC** cannot be enlarged in the same manner to an **IST** model, therefore **IST** does *not* satisfy the Model enlargement property.

Part 2: external universes over the BST universe

The principal idea of how external sets will be incorporated in the internal world **I** of **BST** is very common for various branches of mathematics: it is the idea of *completion*. Indeed, since **I** is "incomplete" in the sense that some objects defined via Comprehension using formulas containing the standardness predicate st , as e.g. $^\sigma \mathbb{N} = \{n \in \mathbb{N} : \operatorname{st} n\}$, the collection of all standard natural numbers, are not legitimate sets in **BST**, we enlarge **I** by adding these collections, in a way similar to many other operations of this type in mathematics (e.g. the Dedekind completion, embedding of a field into an algebraically closed field, etc.).

This is actually the same as what one usually does when treating such objects as monads (not legitimate sets) in **IST** : "define" what is, say, a monad and consider them as "external sets". Certain provisions, however, must be taken. An enlargement of the "universe of discourse" by new objects is well defined in the "universe of discourse" if the newly added objects admit a common parametrization, that is, a common definition by a single formula in which only set parameters may vary.

E.g. in the case of monads one defines $M_x = \{y : y \approx x\}$ (not a set in internal theories) and freely considers sentences containing M_x as legitimate formulas being sure that every such a sentence can be treated, if necessary, as saying something about x rather than M_x .

To introduce *all* external objects rather than a certain part of them in this manner, a common parametrization of all definable subclasses of sets is indispensable.

Section 3. It is the principal point of our approach to external sets that such a parametrization can be given by $\mathcal{C}_p = \bigcup_{a \in \mathcal{A}} \bigcap_{b \in \mathcal{B}} C_{ab}$, where $p = \langle A, B, \eta \rangle$, A and B are internal sets while $\eta = \langle C_{ab} : a \in A \ \& \ b \in B \rangle$

is an internal indexed family of internal sets. We prove that in **BST** *every definable subclass of a set is equal to* C_p *for some (internal) set* p.

Notice that the collection of objects C_p is introduced by a single explicitly written formula, that is, in principle in the same manner as the family of all monads. Thus the parametrization theorem asserts that all st-$\in$-definable subclasses of sets are Σ_2^{st} in **BST**. It may be seen from this explanation that the parametrization system we use is very natural.

Then we introduce external sets, in their most primitive form of *external sets having only internal elements*, as

1) definable subclasses of (internal) sets — which makes them easily "visible" from the point of view of the "internal" observer;

2) (*and this is the same* !) objects of the form C_p — which makes it evident that the treatment is logically consistent.

This manner of incorporation of external sets into the internal universe is therefore a form of the completion procedure, very natural and systematically used in mathematics. On the other hand, it reflects the actually known practice of consideration of external objects in internal theories.

To conclude, we enlarge the **BST** universe I to the external universe $E = \{C_p : p \in I\}$ which contains external sets of internal elements and satisfies the Comprehension principle for all st-$\in$-formulas and certain useful forms of Extension and Saturation. [13] Therefore the **BST** mathematician can legitimately and freely avail himself of the methods provided by this type of external sets (including, in particular, quantifiers over external sets) although the external objects do not "physically" exist in the internal universe, seeing them as elements of the enlargement defined entirely in terms of the **BST** language.

Section 4. On the other hand, E will be the base for a more sophisticated construction of external sets, grounded on the idea of cumulation of sets along wellfounded trees. This is realized in the construction of the *cumulative external enlargement* H.

Section 5. We shall prove that H models **HST**, *Hrbaček set theory*, an external set theory which is approximately Hrbaček's theory NS_1 (the one which contains Saturation and Replacement but does not contain Power

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[13] *Extension* postulates that every function f defined on a set of the form $^\sigma S = \{x \in S : \operatorname{st} x\}$ and taking internal values can be extended to an internal function. *Saturation* in this case postulates that every standard size family of internal sets satisfying the finite intersection property has nonempty intersection.

Set) plus Extension, a standard size form of Choice, Dependent Choice, and a weakened form of Regularity. We also prove that in **HST** a set is of standard size if and only if it is wellorderable; thus all "standard size" results turn out to be "wellorderable" theorems. [14] Unfortunately the universe **H** does not model the Power Set axiom; in fact this axiom is incompatible with Saturation + Replacement.

Part 3: partially saturated universes

The idea to save the Power Set axiom at the cost of reduction of standard size properties to those having standard κ-size, [15] where κ is a standard infinite cardinal, can be realized in **HST** as a system of subuniverses which model the internal and external theories reduced in this way.

There are two principal steps in the construction of the subuniverses.

Internal step. We choose an internal subuniverse $I \subseteq \mathsf{I}$, an inner class in I which includes all standard sets and models a suitable κ-version of **BST**, for instance containing Idealization restricted somehow to sets of cardinality $\leq \kappa$ but enhanced by the assumption that every set belongs to a standard set of cardinality $\leq \kappa$.

There are at least two ways how I can be defined. First, we simply put $I = \mathsf{I}_\kappa$, the class of all elements of standard sets of cardinality $\leq \kappa$. It turns out that I_κ is an elementary submodel of I, the universe of all internal sets in **HST**, with respect to all $\in$-formulas; therefore if one is interested primarily in internal properties of internal sets, seeing external sets only as a research instrument, one can switch from I to I_κ without any harm.

The other version defines I as a κ-saturated ultrapower of the standard universe S; notice that the **BST** universe I is saturated enough to guarantee that the ultrapowers exist as inner classes in I. Of course the properties of the subuniverses of this type may depend on the choice of the ultrafilter.

External step. The class I is then expanded inside the **HST** universe **H** to a corresponding class H of external sets. This can be organized also in two different ways. First we can simulate inside I the above–mentioned cumulative construction of external sets. Second, for the internal universes

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[14] In particular Saturation postulates in **HST** that the internal subuniverse is saturated with respect to all wellorderable cardinals.

[15] A set of standard κ-size is an image of a set of the form ${}^\sigma S = \{s \in S : \mathrm{st}\, s\}$, where S is a standard set of cardinality $\leq \kappa$ in S.

of second type, we can run this construction outside (but starting from I), using the fact that in **H** every subset of I is a set of standard size.

External classes H of both types satisfy κ-forms of Extension, Saturation, and standard size Choice, and satisfy the Power Set axiom (false in **HST**). In addition, the universes H of second type satisfy the full Choice (in fact the statement that every set has standard size and is wellorderable). On the other hand, the universes of the first type have the following essential property: if $x \in I$ then there exists a standard cardinal κ such that x itself, all elements of x, all elements of elements etc. belong to I, which the universes of second type fail to satisfy: none of them contains even all internal natural numbers.

Thus we shall define in **HST** imprinted systems of "partially saturated" external subuniverses which model the corresponding κ-versions of **HST**, including the Power Set and, in the second version, full Choice, axioms.

Conclusion

Mathematicians working, informally speaking, in the internal universe **I** of **BST** have many possibilities how to carry out their investigations.

First, one can use purely internal methods, which work well in many cases.

Second, if an access to external sets is indispensable, but only to those of them which consist of internal elements, one can assume that the internal universe **I** is the internal part of **E**, a universe which provides, in particular, Comprehension for all st-$\in$-formulas (that is, any st-$\in$-definable subclass of a set is a set), and Saturation.

Third, if more complicated external sets are desirable, one can assume that the universe **I** is the internal part of a universe **H** of the powerful theory **HST**.

Fourth, if the Power Set axiom is requested, one can choose a standard cardinal κ such that the standard κ-size Saturation provides the amount of Saturation necessary for the particular research aim, and argue in an external subuniverse of **H** which models a κ-version of **HST** including Power Set, κ-size Saturation, and (full or standard size) Choice.

All the mentioned possibilities are completely legitimate in the logical sense. All of them are in accordance with the Reduction and Model enlargement properties (see the Preface). In all the cases the newly introduced external sets are definable and well visible from the point of view of **I**.   ■

This system of external subuniverses is somewhat similar to what is proposed by Fletcher, see the Preface. There is, however, an essential advantage. Indeed, Fletcher's theory **SNST** does not contain directly, and it is not seen how it may imply indirectly, the Extension principle, or, what is equivalent in this case, (external) Saturation, which is obligatory if one wants to develop certain important branches of nonstandard mathematics, say Loeb measures.

Thus external sets may be incorporated in **BST** as a mathematical instrumentarium, not as a primary element of the theory. External objects introduced by the interpretation are completely *describable* in **BST**, so that they can be considered as a tool which a mathematician working in **BST** is free to choose and apply, but actually being still based on the **BST** axioms. We would compare the treatment of external objects in **BST** via the mentioned interpretation with an advanced programming language, which is based on a low level language.

To conclude, the investigation is mainly devoted to logical details of nonstandard axiomatical systems rather than to demonstration how the proposed system of foundations practically works in this or another branch of nonstandard mathematics. We shall place in Part 2 and Part 3 some brief explanations as to how the external enlargements can be used by those who are interested mainly in practical applications but not in set theoretic analysis.

1.   Bounded set theory

Both **IST** and **BST** are theories in the st-$\in$-*language*, the language containing equality, the membership relation $\in$, and the standardness predicate st . Formulas of this language are called st-$\in$-*formulas* while formulas of the **ZFC** language are called $\in$-*formulas* or *internal* formulas. Abbreviations $\exists^{st} x$ and $\forall^{st} x$ have the obvious meaning.

1.1.   Internal theories: IST and BST

DEFINITION 1.1. $\mathbb{I}$ is the universe of all (internal) sets;

$\mathbb{S} = \{x : \operatorname{st} x\} = $ the class of all standard sets;

$^{\sigma}X = \{x \in X : \operatorname{st} x\} = X \cap \mathbb{S}$ for every set X .   ∎

It is very seldom that $^{\sigma}X$ is a legitimate set in internal theories. (Notice that the standardness predicate is not allowed to occur in the Comprehension

scheme.) In particular if X is standard then $^\sigma X$ is a set if and only if X is finite, and in this case $X = {}^\sigma X$. Thus occurrences like $^\sigma X$ in "internal" reasoning are nothing more than shortcuts for the corresponding (usually longer) legitimate expressions.

DEFINITION 1.2. [Nelson] **IST** [16] is a theory in the st-$\in$-language which includes all of **ZFC** (in the $\in$-language) together with the three principles:

Idealization **I** : $\forall^{\mathrm{stfin}}A \,\exists\, x \,\forall\, a \in A \,\, \Phi(x,a) \;\longleftrightarrow\; \exists\, x \,\forall^{\mathrm{st}}a \,\, \Phi(x,a)$

    for every $\in$-formula $\Phi(x,a)$ with arbitrary parameters in $\mathbb{I}$;

Standardization **S** : $\forall^{\mathrm{st}}X \,\exists^{\mathrm{st}}Y \,\forall^{\mathrm{st}}x \,[\, x \in Y \;\longleftrightarrow\; x \in X \,\&\, \Phi(x)\,]$

    for every st-$\in$-formula $\Phi(x)$ with arbitrary parameters in $\mathbb{I}$;

Transfer **T** : $\exists\, x \,\Phi(x) \;\longrightarrow\; \exists^{\mathrm{st}}x \,\Phi(x)$

    for every $\in$-formula $\Phi(x)$ with standard parameters.

$\forall^{\mathrm{stfin}}A$ means: for all standard finite A . ∎

Thus **IST = ZFC + I + S + T**.

The principal idea behind **BST**, the modified theory we consider, is to reduce the multitude of nonstandard sets. The following axiom is added:

Boundedness **B** : $\forall\, x \,\exists^{\mathrm{st}}X \,(x \in X)$.

This evidently contradicts Idealization, therefore the latter has to be weakened. The natural idea is to restrict the domain of one of the "active" variables x, a by a standard set. We obtain the following two weakened forms of Idealization, called *Bounded Idealization* and *Internal Saturation*:

BI : $\forall^{\mathrm{stfin}}A \,\exists\, x \in X \,\forall\, a \in A \,\, \Phi(x,a) \;\longleftrightarrow\; \exists\, x \in X \,\forall^{\mathrm{st}}a \,\, \Phi(x,a)$

    for every standard X and every $\in$-formula $\Phi(x,a)$,   and

IS : $\forall^{\mathrm{stfin}}A \subseteq A_0 \,\exists\, x \,\forall\, a \in A \,\, \Phi(x,a) \;\longleftrightarrow\; \exists\, x \,\forall^{\mathrm{st}}a \in A_0 \,\, \Phi(x,a)$

    for every standard A_0 and every $\in$-formula $\Phi(x,a)$.

As above, Φ may contain arbitrary parameters (syntactically – free variables other than x and a). Fortunately the two possibilities are equivalent to each other.

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[16] Papers and textbooks of van den Berg [3], F. Diener and Stroyan [5], F. Diener and Reeb [4], Kanovei [12, 14], Lutz and Gose [21], Nelson [23, 24], Reeken [25], Robert [26] give substantial information on **IST** and its applications.

LEMMA 1.3. **BI** *is equivalent to* **IS** *in* **ZFC** *plus Boundedness plus Transfer.*

PROOF. The case of many parameters in Φ can be easily reduced to the case of one parameter. Thus let Φ contain a single parameter p_0, so that Φ is $\Phi(x, a, p_0)$. We prove **BI** $\longrightarrow$ **IS**; thus **BI** and the left-hand side of **IS** are assumed, and we prove the right-hand side of **IS**. Let, by the Boundedness axiom, P be a standard set containing p_0, and $F = \mathcal{P}_{\mathrm{fin}}(A_0) = \{$all finite subsets of $A_0\}$; F is standard by Transfer. By the **ZFC** Collection there exists a set X such that

$$\forall p \in P \; \forall A \in F \left[\exists x \; \forall a \in A \; \Phi(x, a, p) \longrightarrow \exists x \in X \; \forall a \in A \; \Phi(x, a, p) \right].$$

One can choose a standard set X of this kind by Transfer. It remains to apply **BI** to the formula $a \in A_0 \longrightarrow \Phi(x, a, p_0)$ and the set X .

We prove the opposite direction: **IS** $\longrightarrow$ **BI**. Let X be standard and P as before. For any a, let $Z_a = \{\langle p, x \rangle \in P \times X : \Phi(x, a, p)\}$. As above, there exists a standard set A_0 such that $\forall a' \; \exists a \in A_0 \; (Z_a = Z_{a'})$. Using **IS**, one gets $x \in X$ such that $\Phi(x, a, p_0)$ for all standard $a \in A_0$. It remains to prove $\Phi(x, a', p_0)$ for all standard a' in general. Let a' be standard. By Transfer and the choice of A_0 we have $Z_a = Z_{a'}$ for some standard $a \in A_0$. Then

$$\Phi(x, a, p_0) \quad \longrightarrow \quad \langle p_0, x \rangle \in Z_a = Z_{a'} \quad \longrightarrow \quad \Phi(x, a', p_0),$$

as required.   ■

DEFINITION 1.4. *Bounded set theory* **BST** is the theory in the st-$\in$-language which includes all of **ZFC**, **S**, **T**, **B**, and **BI** (or, what is equivalent by the lemma, **IS**). [17]   ■

Thus **BST** provides a lesser amount of Idealization than **IST** does. On the other hand, **BST** contains the Boundedness axiom. One may consider theories with even weaker Idealization but stronger Boundedness. It is not immediately clear what we do gain by the introduction of the boundedness at the cost of idealization. However it will be demonstrated in parts 2 and 3 of this paper that actually the step from **IST** to **BST** provides the possibility to define external enlargements of the internal universe, while the further restrictions provide the Power Set axiom in the relevant external universes.

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[17] **BST** is the "internal part" of NS_1, an external theory of Hrbaček [9], i.e. the axioms of **BST** postulate the same for *all* sets what NS_1 implies for *internal* sets.

1.2.   Basic theorems of BST

This subsection presents several basic theorems [18] of **BST** .

A remarkable point in the **IST** development of the reduction to Σ_2^{st} form given by Nelson is that the involved standard variables (those bound by quantifiers $\exists^{st}$, $\forall^{st}$) have to be restricted by standard sets; otherwise one cannot change places of different types of quantifiers. It turns out that in **BST** no such restriction is necessary.

THEOREM 1.5. [Reduction to Σ_2^{st}]
Every formula is equivalent in **BST** *to a* Σ_2^{st} *formula with the same set of free variables. In other words, let* $\Phi(x_1, ..., x_m)$ *be a* st-$\in$-*formula. There exists an* $\in$-*formula* $\varphi(x_1, ..., x_m, a, b)$ *such that the following is a theorem of* **BST** :

$$\forall x_1 ... \forall x_m \left[\Phi(x_1, ..., x_m) \longleftrightarrow \exists^{st}a \, \forall^{st}b \, \varphi(x_1, ..., x_m, a, b) \right].$$

(Σ_2^{st} is the class of all formulas $\exists^{st}a \, \forall^{st}b \, \varphi$ where φ is an $\in$-formula.)

PROOF.   [19]   To simplify notation we write $\vec{x}$ instead of $x_1, ..., x_m$. The proof is carried out by induction on the number of logical symbols in Φ. After elimination of all occurrences of st z by $\exists^{st}w \, (z = w)$, it suffices to go through induction steps for $\neg$ and $\exists$.

The induction step for $\neg$. We search for a Σ_2^{st} formula equivalent to the formula $\forall^{st}a \, \exists^{st}b \, \varphi(\vec{x}, a, b)$ (where φ is an $\in$-formula), taken as $\Phi(\vec{x})$. Let $\mathrm{Ult}\, U$ be the formula: U *is an ultrafilter*. For any ultrafilter U, let $\mu(U) = \bigcap\{u \in U : \mathrm{st}\, u\}$ (the *monad* of U). It is asserted that

$$\Phi(\vec{x}) \longleftrightarrow \exists^{st}U \left[\mathrm{Ult}\, U \; \& \; \vec{x} \in \mu(U) \; \& \; \forall a \, \exists b \, \exists u \in U \, \forall \vec{y} \in u \; \varphi(\vec{y}, a, b) \right].$$

The rigth–hand side is evidently Σ_2^{st}, so the equivalence is enough to complete the step.

By the Boundedness and Standardization axioms, for every $\vec{x}$ there exists a standard ultrafilter U such that $\vec{x} \in \mu(U)$. It remains to verify

$$\Phi(\vec{x}) \longleftrightarrow \forall a \, \exists b \, \exists u \in U \, \forall \vec{y} \in u \; \varphi(\vec{y}, a, b)$$

---

[18]   Actually proved by Kanovei [12], with the exception of the Dependent Choices theorem. They are included partially for the sake of convenience, partially because a much simpler way to prove the Collection and Reduction to Σ_2^{st} theorems has been found.
[19]   This version of the proof is due to P. Andreev.

for every U of this type. By Transfer, this is equivalent to

$$\Phi(\vec{x}) \quad \longleftrightarrow \quad \forall^{\mathrm{st}} a\; \exists^{\mathrm{st}} b\; \exists^{\mathrm{st}} u \in U\; \forall \vec{y} \in u\; \varphi(\vec{y}, a, b),$$

that is, to $\varphi(\vec{x}, a, b) \longleftrightarrow \exists^{\mathrm{st}} u \in U\; \forall \vec{y} \in u\; \varphi(\vec{y}, a, b)$ for all standard a and b. Let a, b be standard and $u = u_{ab} = \{\vec{y} \in X : \varphi(\vec{y}, a, b)\}$, where $X = \bigcup U$. Both X and u are standard by Transfer. If $u \in U$ then both the left–hand and the right–hand parts of the last equivalence are true, otherwise both of them are false.

The step for $\exists$. Let φ be an $\in$-formula. We need a Σ_2^{st} formula which is equivalent to the formula $\exists w\; \exists^{\mathrm{st}} a\; \forall^{\mathrm{st}} b\; \varphi(\vec{x}, w, a, b)$ taken as $\Phi(\vec{x})$. Changing places $\exists w$ and $\exists^{\mathrm{st}} a$ and applying the Boundedness and Bounded Idealization axioms, we obtain

$$\Phi(\vec{x}) \quad \longleftrightarrow \quad \exists^{\mathrm{st}} a\; \exists^{\mathrm{st}} W\; \exists w \in W\; \forall^{\mathrm{st}} b\; \varphi(\vec{x}, w, a, b) \quad \longleftrightarrow$$

$$\longleftrightarrow \quad \exists^{\mathrm{st}} a\; \exists^{\mathrm{st}} W\; \forall^{\mathrm{stfin}} B\; \exists w \in W\; \forall b \in B\; \varphi(\vec{x}, w, a, b). \blacksquare$$

COROLLARY 1.6. [Reduction property] [**BST**]
Let $\Phi(x_1, ..., x_m)$ *be an arbitrary* st-$\in$-*formula. There exists an* $\in$-*formula* $\widehat{\Phi}(x_1, ..., x_m)$ *such that the following is a theorem of* **BST** :

$$\forall^{\mathrm{st}} x_1 ... \forall^{\mathrm{st}} x_m\; [\; \Phi(x_1, ..., x_m) \quad \longleftrightarrow \quad \widehat{\Phi}^{\mathrm{st}}(x_1, ..., x_m)\;].$$

($\widehat{\Phi}^{\mathrm{st}}$ denotes relativization of $\widehat{\Phi}$ to the standard universe: $\exists$ and $\forall$ are changed to $\exists^{\mathrm{st}}$ and $\forall^{\mathrm{st}}$.) Thus it is asserted that every st-$\in$-formula with standard parameters is provably equivalent in **BST** to an $\in$-formula with the same list of parameters. [20] In other words, the Reduction property discussed in the Preface holds for **BST** .

PROOF. Let φ be the formula given by Theorem 1.5. Then, by Transfer, the formula $\exists a\; \forall b\; \varphi(x_1, ..., x_m, a, b)$ can be taken as $\widehat{\Phi}$. $\blacksquare$

To demonstrate the power and significance of Theorem 1.5, we prove the Collection and Extension theorems. The first of them needs special form to be proved beforehand; besides the application to Collection in general form, the next lemma will be very useful in our parametrization theorems.

---

[20] This is true in **IST**, but with respect to formulas in which the standardness predicate occurs only as superscript in quantifiers $\exists^{\mathrm{st}} x \in X$, $\forall^{\mathrm{st}} x \in X$, where X is standard, see Nelson [23]. It is proved by Kanovei [13] that there exists a st-$\in$-sentence (not of the mentioned type) which is not provably equivalent in **IST** to an $\in$-formula.

LEMMA 1.7. [BST]

Let $\varphi(a, b, x)$ be an internal formula, X a standard set, $\kappa = \operatorname{card} X$. There exist standard sets A and B of cardinality $\leq 2^{2^{\kappa}}$ such that for all $x \in X$, $\exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b\, \varphi(a, b, x)$ $\longleftrightarrow$

$$\longleftrightarrow\ \exists^{\mathrm{st}} a \in A\ \forall^{\mathrm{st}} b\ \varphi(a, b, x)\ \longleftrightarrow\ \exists^{\mathrm{st}} a \in A\ \forall^{\mathrm{st}} b \in B\ \varphi(a, b, x) .$$

PROOF. We define, for all a and b,

$$
\begin{aligned}
X[a, b] &= & \{x \in X : \varphi(a, b, x)\} && \subseteq & X\,; \\
X[a] &= \{X[a, b] : b \text{ is an arbitrary set}\} && \subseteq & \mathcal{P}(X)\,; && \text{and} \\
X[\,] &= \{X[a] : a \text{ is an arbitrary set}\} && \subseteq & \mathcal{P}^2(X)
\end{aligned}
$$

Thus the set $X[\,]$ has cardinality at most $\lambda = 2^{2^{\kappa}}$ while every set $X[a]$ has cardinality at most 2^{κ}. Therefore, using the **ZFC** Collection and Choice, and then Transfer, one may choose standard sets A and B, both of cardinality $\leq 2^{2^{\kappa}}$ such that $\forall a'\, \exists\, a \in A\, (X[a] = X[a'])$, and for any $a \in A$:

$$\forall b'\, \exists\, b \in B\, (X[a, b] = X[a, b']) .$$

We assert that A and B are as required. Let (1), (2), (3) denote the parts of the equivalence of the lemma from left to right. Notice that (2) $\longrightarrow$ (1) & (3). On the other hand, both (1) and (3) imply

$$\exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b \in B\ \varphi(a, b, x)\,, \tag{4}$$

so that it remains to prove that (4) $\longrightarrow$ (2). So let a standard set a satisfy $\forall^{\mathrm{st}} b \in B\ \varphi(a, b, x)$. By the choice of A and Transfer, there exists a standard $a' \in A$ such that $X[a] = X[a']$. It is asserted that $\varphi(a', b', x)$ is true for every standard b'. Notice that $X[a', b']$ is a standard member of $X[a] = X[a']$, therefore by Transfer and the choice of B we have $X[a', b'] = X[a, b]$ for a suitable standard $b \in B$. Thus

$$\varphi(a', b', x)\ \longleftrightarrow\ x \in X[a', b'] = X[a, b]\ \longleftrightarrow\ \varphi(a, b, x)\,,$$

but the right–hand formula is true by the choice of a . ∎

THEOREM 1.8. [Collection] [BST]

Let $\Phi(x, y)$ be a st-$\in$-formula having arbitrary sets as parameters. For any X there exists a standard set Y such that

$$\forall x \in X\, [\,\exists\, y\, \Phi(x, y)\ \longrightarrow\ \exists\, y \in Y\ \Phi(x, y)\,] .$$

Collection is a theorem of **ZFC**; hence it is true in **BST** for all $\in$-formulas Φ. It is asserted, however, that the result holds for *all* formulas. [21]

PROOF. Let Φ be $\Phi(x,y,p)$, where p is an arbitrary set. By the Boundedness axiom $\exists y\,\Phi(x,y,p)$ implies $\exists^{st} z\,\exists y \in z\,\Phi(x,y,p)$. Let $\Psi(x,z,p)$ denote the following formula: $\exists y \in z\,\Phi(x,y,p)$. By the Reduction to Σ_2^{st} theorem, $\Psi(x,z,p)$ is equivalent to a Σ_2^{st} formula $\exists^{st} a\,\forall^{st} b\,\varphi(x,z,p,a,b)$. Covering the parameter p in Φ by a standard set P and applying Lemma 1.7, we obtain a standard set Z such that

$$\exists^{st} z\,\Psi(x,z,p) \;\longleftrightarrow\; \exists^{st} z \in Z\,\Psi(x,z,p) \quad - \quad \text{for all } x \in X.$$

The set $Y = \bigcup Z = \{y : \exists z \in Z\,(y \in z)\}$ is as required. ∎

THEOREM 1.9. [Extension] [**BST**]
Let $\Phi(x,y)$ be a st-$\in$-formula containing arbitrary sets as parameters. Then for any standard X there exists a function f defined on X such that

$$\forall^{st} x \in X\,[\,\exists y\,\Phi(x,y) \;\longrightarrow\; \Phi(x,f(x))\,].$$

PROOF. We may assume that $\forall^{st} x \in X\,\exists y\,\Phi(x,y)$ without any loss of generality (otherwise replace Φ by $\Phi(x,y) \bigvee \neg\exists y'\,\Phi(x,y')$). By Collection, there exists a standard set Y such that $\forall^{st} x \in X\,\exists y \in Y\,\Phi(x,y)$. We may now assume that Φ is a Σ_2^{st} formula $\exists^{st} a\,\forall^{st} b\,\varphi(x,y,a,b)$, where φ is an $\in$-formula. Covering parameters in Φ by standard sets and applying Collection again, we obtain a pair of standard sets A, B such that

$$\Phi(x,y) \;\longleftrightarrow\; \exists^{st} a \in A\,\forall^{st} b \in B\,\varphi(x,y,a,b) \quad - \quad \text{for all } x \in X \text{ and } y \in Y.$$

Thus $\forall^{st} x \in X\,\exists y \in Y\,\exists^{st} a \in A\,\forall^{st} b \in B\,\varphi(x,y,a,b)$.

Changing places the quantifiers on y and a and applying a theorem of Nelson [23] we get $\forall^{st} x \in X\,\exists y \in Y\,\forall^{st} b \in B\,\varphi(x,y,\tilde{a}(x),b)$ for a standard function $\tilde{a} : X \longmapsto A$. The next step is the idea of the Saturation theorem of Nelson [24]. We apply **BI** to the pair of quantifiers $\exists y\,\forall^{st} b$, use the fact that the quantifier $\forall^{st} x \in X$ is equivalent to the combination $\forall^{stfin} X' \subseteq X\,\forall x \in X'$, and get

$$\forall^{stfin} X' \subseteq X\,\forall^{stfin} B' \subseteq B\,\forall x \in X'\,\exists y \in Y\,\forall b \in B'\,\varphi(x,y,\tilde{a}(x),b),$$

therefore

$$\forall^{stfin} X' \subseteq X\,\forall^{stfin} B' \subseteq B\,\exists \hat{y} \in \hat{Y}\,\forall x \in X'\,\forall b \in B'\,\varphi(x,\hat{y}(x),\tilde{a}(x),b),$$

---

[21] It is rather surprising that Collection, unlike Reduction to Σ_2^{st}, holds in **IST** for all st-$\in$-formulas, see Kanovei [15].

where $\hat{Y}$ is the (standard) set of all functions $\hat{y}$ such that $\operatorname{dom} \hat{y}$ is a finite subset of X while $\operatorname{ran} \hat{y} \subseteq Y$.

Converting the pair of variables x, b into one variable and applying **BI** in the opposite direction we obtain $\exists \hat{y} \, \forall^{\mathrm{st}} x \in X \, \forall^{\mathrm{st}} b \in B \; \varphi(x, \hat{y}(x), \tilde{a}(x), b)$ finally, that is, $\exists \hat{y} \, \forall^{\mathrm{st}} x \in X \; \Phi(x, \hat{y}(x))$.   ∎

THEOREM 1.10. *The dependent choices scheme holds in* **BST** .

PROOF. We have to prove the following. Let X be an arbitrary set and $\Phi(x, y)$ an arbitrary st-$\in$-formula (having, perhaps, parameters which we do not indicate explicitly). Assume that $\forall x \in X \, \exists y \in X \, \Phi(x, y)$. It is asserted that there exists a function f such that

$$\forall^{\mathrm{st}} k \in \mathbb{N} \, [\, f(k) \text{ is defined and } \Phi(f(k), f(k+1)) \,] . \qquad (1)$$

First of all, by theorems 1.6. (Reduction to Σ_2^{st}) and 1.8. (Collection) Φ may be assumed to be a Σ_2^{st} formula $\exists^{\mathrm{st}} a \in A \, \forall^{\mathrm{st}} b \in B \, \varphi(a, b, x, y)$, where A and B are standard sets while φ is an $\in$-formula.

Let $\vec{a} = \langle a_1, ..., a_n \rangle$ be a finite sequence of elements of A. We put

$$X(\vec{a}) = \{ \langle x_1, ..., x_{n+1} \rangle \in X^{n+1} : \forall \, k \leq n \, \forall^{\mathrm{st}} b \in B \, \varphi(a_k, b, x_k, x_{k+1}) \} .$$

We say that $\vec{a}$ is *good* if $X(\vec{a}) \neq \emptyset$. The empty sequence Λ is evidently good: $n = 0$ and $X(\Lambda) = X$. If $\vec{a} = \langle a_1, ..., a_n \rangle$ is good then by the assumption there exists a *standard* $a_{n+1} \in A$ such that $\vec{a}^{\wedge} a_{n+1} = \langle a_1, ..., a_n, a_{n+1} \rangle$ is also good. By Standardization, there exists a standard set S whose elements are finite sequences of elements of A, and whose *standard* elements are all standard good sequences and nothing more. By what is said above and by Transfer, every sequence in S can be extended to a sequence in S by adding one more element. Therefore there exists an infinite sequence $\alpha = \langle a_n : 1 \leq n \in \mathbb{N} \rangle$ such that $\langle a_1, ..., a_n \rangle \in S$ for all n. By Transfer again, there exists a *standard* sequence α of this type. Then, for any standard n, a_n is standard and $\langle a_1, ..., a_n \rangle$ is standard and good. Thus,

$$\forall^{\mathrm{st}} n \, \exists \langle x_1, ..., x_{n+1} \rangle \, \forall \, k \leq n \, \forall^{\mathrm{st}} b \in B \, \varphi(a_k, b, x_k, x_{k+1}) .$$

To proceed with the proof, we first restrict the quantifier $\exists \langle x_1, ..., x_{n+1} \rangle$ by a standard set U using Collection. Thus

$$\forall^{\mathrm{st}} n \, \exists \langle x_1, ..., x_{n+1} \rangle \in U \, \forall \, k \leq n \, \forall^{\mathrm{st}} b \in B \, \varphi(a_k, b, x_k, x_{k+1}) .$$

By Bounded Idealization **BI**, we obtain

$$\forall^{\mathrm{st}} n \, \forall^{\mathrm{stfin}} B' \subseteq B \, \exists \langle x_1, ..., x_{n+1} \rangle \in U \, \forall \, k \leq n \, \forall \, b \in B' \, \varphi(a_k, b, x_k, x_{k+1}) .$$

Applying Bounded Idealization in the opposite direction, we get

$$\exists \langle x_1, ..., x_{n+1} \rangle \in U \ \forall^{st} k \ \forall^{st} b \in B \ \varphi(a_k, b, x_k, x_{k+1}) \, ;$$

in particular, n, the length of this sequence $\langle x_1, ..., x_{n+1} \rangle$ minus 1, is non-standard. We put $f(k) = x_k$ for $k \leq n$ and $f(k) = x_1$ (not essential) for $k > n$; (1) holds since a_k is standard provided k is standard. ∎

2. Models of BST and related topics

This section contains three main results. First we prove that **BST** has an inner model in **IST** : the class of all bounded sets. Then we demonstrate that every countable model of **ZFC** can be enlarged to a model of **BST**, therefore **BST** has the Model enlargement property discussed in the Preface, but the minimal transitive model of **ZFC** cannot be enlarged to a model of **IST** .

2.1. Inner interpretation in IST

We argue in **IST**. We say that a set x is *bounded* iff it is an element of a standard set. Thus all standard sets, all natural and real numbers are bounded in **IST**. (In **BST** all sets are bounded.) We put $\mathbb{B} = \{x : \exists^{st} X \ (x \in X)\}$ (the class of all bounded sets). Quantifiers $\exists^{bd}$ and $\forall^{bd}$ have the obvious meaning.

LEMMA 2.1. $\mathbb{B}$ *is a model of* **BST** *in* **IST**. *More exactly, let* A *be an axiom of* **BST**, *then* A^{bd} *is a theorem of* **IST** .

Here A^{bd} denotes the result of replacing every quantifier $\exists$ or $\forall$ in A by $\exists^{bd}$ or $\forall^{bd}$ respectively, that is, the relativization of A to $\mathbb{B}$.

PROOF. We check the statement $\exists^{bd} x \ \Phi^{bd}(x) \longrightarrow \exists^{st} x \ \Phi^{bd}(x)$ (that is, Transfer), where Φ is an $\in$-formula with standard parameters. This would be an immediate consequence of the **IST** Transfer if we could replace the formula Φ^{bd}, which certainly is not an $\in$-formula, by an equivalent $\in$-formula. So it is the following proposition that remains to be proved:

PROPOSITION 2.2. *Let* Φ *be an* $\in$-*formula having only bounded sets as parameters. Then* $\Phi \longleftrightarrow \Phi^{bd}$.

PROOF. The proof goes on by induction on the number of logical signs in Φ. The induction step for $\exists$ is the only one that needs a special consideration.

Thus let $\psi(x)$ be an $\in$-formula with bounded sets as parameters. We assert that $\exists z\, \psi(z) \longrightarrow \exists z \in \mathbf{B}\, \psi(z)$. Indeed, applying the **ZFC** Replacement and the **IST** Transfer, we get a standard set Z such that $\exists z\, \psi(z) \longrightarrow \exists z \in Z\, \psi(z)$. However all elements of a standard set are bounded.   ⊣

Thus Transfer in $\mathbf{B}$ has been checked. Therefore all **ZFC** axioms hold in $\mathbf{B}$. Standardization and the Boundedness axiom are evident. To verify Bounded Idealization, let X be standard and Φ be an $\in$-formula with bounded parameters; we have to prove

$$\forall^{\mathrm{stfin}}A\, \exists x \in X\, \forall a \in A\, \Phi(x,a) \longrightarrow \exists x \in X\, \forall^{\mathrm{st}}a\, \Phi(x,a)\,.$$

(The superscript bd is deleted from Φ by the proposition.) But this follows from the **IST** Idealization.   ∎

COROLLARY 2.3. **BST** *is an equiconsistent and conservative extension of* **ZFC**.

PROOF. Use a theorem of [23] which says that **IST** is a conservative and equiconsistent extension of **ZFC**.   ∎

2.2. BST enlargements of models of ZFC [22]

Let M be a model of **ZFC**, not necessary an $\in$-model. A model $^*M$ of **BST** or **IST** is called a *regular enlargement* of M if M is (isomorphic to) the class of all standard sets in $^*M$ (with the membership and equality inherited from $^*M$.)

THEOREM 2.4. [Model enlargement for **BST**]
Any countable model M *of* **ZFC** *has a regular* **BST** *enlargement.*

PROOF. [23] First of all we may assume that M is a model of **ZFGC**, the "global choice" version of **ZFC**, which contains $\mathbf{G}$ as an extra functional symbol in the language, and the axiom which says that $\mathbf{G}$ is a global

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[22] The authors are in debt to A. Enayat for the interest in this part of the paper and useful comments.

[23] A proof of this theorem modulo minor details was outlined in Hrbaček [9] as a part of the proof of a stronger enlargement result. We prefer to give here a somewhat different proof first to make the exposition self–contained and second to save some space in Part 3 where the construction of the proof is used for another theorem.

choice function for the universe, and it is assumed that **G** may occur also in the schemata of Separation and Replacement. (Indeed, by a theorem of Felgner [6] every countable model of **ZFC** can be enlarged, by forcing, to a model of **ZFGC** *containing the same sets*). Evidently **ZFGC** implies, by the Zermelo argument, the existence of a wellordering of the universe of all sets; let $\prec$ denote such a wellordering of M. In other words, M is a model of **ZFC** enriched by the assumption that $\prec$ may occur in the schemata, and it is true in M that $\prec$ wellorders the universe.

The principal idea is to apply the method of adequate ultrapowers used in the construction of **IST** models given in Nelson [23]. We introduce, however, some changes and simplifications.

Let C be an arbitrary set. We put $C^{\text{fin}} = \mathcal{P}_{\text{fin}}(C)$ (all finite subsets of C). A nonprincipal ultrafilter $U \subseteq \mathcal{P}(C^{\text{fin}}) = \{I : I \subseteq C^{\text{fin}}\}$ will be called *C-adequate* iff it contains all sets $I(C, i) = \{i' \in C^{\text{fin}} : i \subseteq i'\}$, where $i \in C^{\text{fin}}$. If in this case $D \subseteq C$ then we define $U \upharpoonright D = \{u \upharpoonright D : u \in U\}$ where $u \upharpoonright D = \{i \cap D : i \in u\}$ for any $i \subseteq C^{\text{fin}}$. Then $U \upharpoonright D$ will be a D-adequate ultrafilter.

There are two operations over ultrafilters which we shall use below.

Operation 1. Let U and U' be a C-adequate and a C'-adequate ultrafilters respectively. We suppose that $C \cap C' = \emptyset$. Then we let

$$U * U' = \{w \subseteq (C \cup C')^{\text{fin}} : U\, i\, U'\, i'\, (i \cup i' \in w)\}.\,^{24}$$

Then $U * U'$ is a $(C \cup C')$-adequate ultrafilter.   ⊣

Operation 2. Assume that U_α is a C_α-adequate ultrafilter for all $\alpha < \lambda$, λ a limit ordinal. Assume that $C_\alpha \subseteq C_\beta$ and $U_\alpha = U_\beta \upharpoonright C_\alpha$ for all pairs $\alpha < \beta$. We put $C = \bigcup_{\alpha < \lambda} C_\alpha$ and $U'_\lambda = \bigcup_{\alpha < \lambda} U_\alpha[\to C]$ where

$$U_\alpha[\to C] = \{u[\to C] : u \in U_\alpha\} \quad \text{and} \quad u[\to C] = \{j \in C : j \cap C_\alpha \in u\}.$$

Then U'_λ is an C-adequate *filter*, that is, there exists an C-adequate ultrafilter U_λ such that $U'_\lambda \subseteq U_\lambda$. We let $\lim_{\alpha < \lambda} U_\alpha$ denote the $\prec$-least of such U_λ in the case when a wellordering $\prec$ of the universe is fixed.   ⊣

Let $\mathcal{O} = \text{Ord}^M$. Using the relation $\prec$ which wellorders the universe in $\langle M; \prec \rangle$, and the two operations, we can easily define an $\mathcal{O}$-adequate ultrafilter U such that

1. The indexed family of ultrafilters $U_C = U \upharpoonright C$, where $C \in M$ is such that it is true in M that C is a set of ordinals, is definable in $\langle M; \prec \rangle$.

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[24] If U is an ultrafilter over a set I then $U\, i\, \phi(i)$ means: $\{i \in I : \phi(i)\} \in U$.

2. It is true in $\langle M; \prec \rangle$ that for any $C \subseteq \mathrm{Ord}$ and any cardinal κ there exists a set $D \subseteq \mathrm{Ord}$ of cardinality $\geq \kappa$ such that $D \cap C = \emptyset$ and $U_{C \cup D} = U_D * U_C$.

We argue in the model $\langle M; \prec \rangle$

We say that a formula $\varphi(i)$ is C-*reduced* if $\varphi(i) \longleftrightarrow \varphi(i \cap C)$ holds for all finite i. We say that φ is *reduced* if it is C-reduced for some set $C \subseteq \mathrm{Ord}$. Let $\varphi(i)$ be such a formula. The quantifier $\mathbf{U}$ is defined by:

$$\mathbf{U}\, i\, \varphi(i) \qquad \text{if and only if} \qquad U_C\, i\, \varphi(i).$$

where $C \subseteq \mathrm{Ord}$ is an arbitrary set such that φ is C-reduced. The independence on the choice of C is easily verifiable: indeed, every U_C is equal to $U \restriction C$ for one and the same U .

One can easily check the following properties of these quantifiers.

PROPOSITION 2.5. [Properties of the ultrafilter quantifiers]

1. *If* $c \in C$ *then* $U_C\, i\, (c \in i)$. $\qquad\qquad$ (By the adequacy of U_C .)

2. *If a formula* $\varphi(i)$ *is* C-*reduced, then it is* D-*reduced for all* $D \supseteq C$, *and in this case* $U_C\, i\, \varphi(i) \longleftrightarrow U_D\, i\, \varphi(i) \longleftrightarrow \mathbf{U}\, i\, \varphi(i)$.

3. *If* $\forall\, i \in C\, [\varphi(i) \longrightarrow \psi(i)]$, *then* $U_C\, i\, \varphi(i) \longrightarrow U_C\, i\, \psi(i)$.

4. $\mathbf{U}\, i\, \varphi(i)\ \&\ \mathbf{U}\, i\, \psi(i) \longleftrightarrow \mathbf{U}\, i\, [\varphi(i)\ \&\ \psi(i)]$ *for reduced* φ, ψ .

5. $\mathbf{U}\, i\, \neg\, \varphi(i) \longleftrightarrow \neg\, \mathbf{U}\, i\, \varphi(i)$, *whenever* φ *is reduced*. $\qquad\qquad \dashv$

For $C \subseteq \mathrm{Ord}$, we let $\mathcal{F}_C$ denote the class of all functions having C^{fin} as the domain and arbitrary values, and put $\mathcal{F} = \bigcup_{C \subseteq \mathrm{Ord}} \mathcal{F}_C$. For $f \in \mathcal{F}_C$ we put $C(f) = C$.

The set $\mathcal{F}$ is the base set (class, from the M th point of view) for the enlargement. To define the basic predicates $^*{=}$, $^*{\in}$, $^*\mathrm{st}$, acting on $\mathcal{F}$, we first put $f[i] = f(i \cap C)$ whenever $f \in \mathcal{F}_C$ but perhaps $i \not\subseteq C$. Then $f[i] = f(i)$ for $i \in C^{\mathrm{fin}}$. Now let $f, g \in \mathcal{F}$. We put

$$f\, ^*{=}\, g \qquad \text{if and only if} \qquad \mathbf{U}\, i\, (f[i] = g[i]);$$

$$f\, ^*{\in}\, g \qquad \text{if and only if} \qquad \mathbf{U}\, i\, (f[i] \in g[i]).$$

The formulas $f[i] = g[i]$ and $f[i] \in g[i]$ are $C(f) \cup C(g)$-reduced.

For $x \in M$, let $\underline{x} \in \mathcal{F}_{\{0\}}$ denote the function defined by $\underline{x}(0) = x$. We put $\mathbf{C} = \{\underline{x} : x$ an arbitrary set (in M)$\}$ and define, for $f \in \mathcal{F}$,

$$^*\mathrm{st}\, f \qquad \text{if and only if} \qquad \exists\, g \in \mathbf{C}\, (f\, ^*{=}\, g).$$

This is the standardness predicate. **The end of reasoning in** M. ⊣

Obviously $^*=$ is an equivalence on $\mathcal{F}$ and $^*\!\in$, *st are $^*=$-invariant. This allows to define $[f] = \{g \in \mathcal{F} : f \; ^*= g\}$, and then

$$[f] \; ^*\!\in [g] \qquad \text{if and only if} \qquad f \; ^*\!\in g \, ;$$

$$^*\text{st} \, [f] \qquad \text{if and only if} \qquad ^*\text{st} \, f \, .$$

Let finally $^*M = \{[f] : f \in \mathcal{F}\}$. We shall consider $^*M = \langle \, ^*M; =, \, ^*\!\in, \, ^*\text{st} \, \rangle$ as a st-$\in$-structure ($\in$ and st are interpreted as $^*\!\in$ and *st respectively). To treat $^*M$ as an enlargement of M, we put $^*x = [\underline{x}]$ for all $x \in M$; then $x = y \longleftrightarrow \; ^*x \; ^*= \; ^*y$, and the same for membership, so $x \longmapsto \; ^*x$ is in fact an embedding. The sets $^*x$ and only these sets satisfy *st in $^*M$.

Let Φ be an arbitrary $\in$-formula with parameters in $\mathcal{F}$. We define $C(\Phi) = \bigcup\{C(f) : f \text{ occurs in } \Phi\}$; then Φ is $C(\Phi)$-reduced. If i is finite then let $\Phi[i]$ denote the result of replacing each f occurring in Φ by $f[i]$, so that $\Phi[i]$ is an $\in$-formula having sets in M as parameters. Let $[\Phi]$ denote the result of changing each f occurring in Φ to $[f]$, so that $[\Phi]$ is an $\in$-formula with parameters from $^*M$.

LEMMA 2.6. [Łoś] *Let* Φ *be a closed* $\in$-*formula with parameters in* $\mathcal{F}$. *Then*

$$[\Phi] \quad \text{is true in } ^*M \qquad \longleftrightarrow \qquad \mathsf{U} \, i \; \Phi[i] \quad \text{is true in } M \, .$$

PROOF. Usual proof in M by induction on the logical complexity of Φ based on Proposition 2.5. To carry out the implication $\longleftarrow$ in the step for $\exists$ we define $C = C(\Phi)$ and replace $\mathsf{U} \, i$ by $U_C \, i$ in the right–hand side by Proposition 2.5.2. ⊣

COROLLARY 2.7. *Let* φ *be a closed* $\in$-*formula with parameters in* M, *and let* $^*\varphi$ *be obtained by replacing each parameter* $z \in M$ *by* $^*z$ *in* φ. *Then*

$$\varphi \quad \text{holds in } M \qquad \longleftrightarrow \qquad [^*\varphi] \quad \text{holds in } ^*M.$$

PROOF. Evidently $^*\varphi[i]$ coincides with φ for all i. ⊣

We assert that $^*M$ *is a regular* **BST** *enlargement of* M, as it is required by Theorem 2.4. Indeed, first of all M is (isomorphic to) the collection of all standard (satisfying *st) members of $^*M$ via the embedding $x \longmapsto \; ^*x$. Thus it remains to prove that $^*M$ is a model of **BST** .

Corollary 2.7 shows that all **ZFC** axioms hold in $^*M$ and easily implies Transfer in $^*M$. To check Standardization in $^*M$ it suffices to recall that

the predicates $\,{}^*\!\!=\,,\,\,{}^*\!\!\in\,,\,\,{}^*\!\mathrm{st}\,\,$ are expressed in $\,M\,$ by certain formulas of the $\{\in,\prec\}$-language.

We verify *Boundedness* in ${}^*\!M$. Let $f \in \mathcal{F}$; we have to find a set $X \in M$ such that $f \, {}^*\!\!\in {}^*\!X$. Take notice that f is an element of M, a **ZFC** model. Therefore there exists $X \in M$ such that every value of f belongs to X. Lemma 2.6 implies the required property of X.

We verify *Internal Saturation* **IS**. **We argue in** M. Let $\varphi(x,a)$ be an $\in$-formula containing elements of $\mathcal{F}$ as parameters, and $C = C(\Phi)$. We fix some $A_0 \in M$ and prove

$$\textbf{IS}: \quad \forall^{\mathrm{stfin}} A \subseteq {}^*\!A_0 \,\exists\, x \,\forall\, a \in A \,[\varphi(x,a)] \quad\longrightarrow\quad \exists\, x \,\forall^{\mathrm{st}} a \in {}^*\!A_0 \,[\varphi(x,a)]$$

in ${}^*\!M$. (The implication $\longleftarrow$ does not need a special consideration because it follows from Standardization that elements of finite standard sets are standard, see Nelson [23].)

Let $D \subseteq \mathrm{Ord}$ be any set of cardinality $card\,A_0$ in M disjoint from C and such that $U_{D \cup C} = U_D * U_C$. (Condition II above !) One may assume that in fact $A_0 = D$. (Let H be a map of D onto A_0. We define $\varphi'(x,\gamma)$ as $\varphi(x, H(\gamma))$.) Thus the statement we have to prove in ${}^*\!M$ takes the form:

$$\textbf{IS}: \quad \forall^{\mathrm{stfin}} A \subseteq {}^*\!D \,\exists\, x \,\forall\, \gamma \in A \,[\varphi(x,\gamma)] \quad\longrightarrow\quad \exists\, x \,\forall^{\mathrm{st}} \gamma \in {}^*\!D \,[\varphi(x,\gamma)].$$

One may convert the left-hand side by Lemma 2.6 to the form:

$$\forall^{\mathrm{fin}} A \subseteq D \,\,\mathbf{U}\,i\,\exists\, x \,\forall\, \gamma \in A \,\,\varphi(x,\gamma)\,[i]\,, \qquad \text{that is, to}$$

$$U_D\,i'\,\,U_C\,i\,\exists\, x \,\forall\, \gamma \in i'\,\,\varphi(x,\gamma)\,[i]\,, \qquad \text{then to}$$

$$U_{D \cup C}\,j\,\exists\, x \,\forall\, \gamma \in \mathcal{A}(j)\,\,\varphi(x,\gamma)\,[j]$$

by the choice of D, where $\mathcal{A} \in \mathcal{F}_{D \cup C}$ is defined on the domain $(D \cup C)^{\mathrm{fin}}$ by the equality $\mathcal{A}(j) = j \cap D$, and finally to $\mathbf{U}\,j\,\exists\, x\,(\forall\, \gamma \in \mathcal{A}\,\,\varphi(x,\gamma))\,[j]$. Then the sentence $\exists\, x \,\forall\, \gamma \in \mathcal{A}\,\,\varphi(x,\gamma)$ is true in ${}^*\!M$ by Lemma 2.6. Thus, to obtain the right–hand side of **IS**, it suffices to prove ${}^*\!\gamma \in \mathcal{A}$ in ${}^*\!M$ for all $\gamma \in D$. This is equivalent to the formula $U_{C \cup D}\,j\,(\gamma \in \mathcal{A})\,[j]$ by Lemma 2.6, then to $U_{C \cup D}\,j\,(\gamma \in j \cap D)$ by the definition of $\mathcal{A}$, and this is exactly Proposition 2.5.1 because $\gamma \in D$.

This ends the proof of Theorem 2.4.   ∎

2.3. A model of ZFC having no IST enlargements

THEOREM 2.8. *The minimal transitive model of* **ZFC** *does not have a regular* **IST** *enlargement.*

PROOF. [25] Let $M = \mathrm{L}_\vartheta$ be the minimal transitive **ZFC** model. (L_ϑ is the ϑ th level of the constructible hierarchy.) The proof is based on two ideas. The first of them is a known fact: [26] *every* $x \in M$ *is* $\in$-*definable in* M .

The other idea is the ability of **IST** to express the truth of $\in$-formulas in S, the class of all standard sets. [27] Let $\tau(n, T)$ be the st-$\in$-formula which says that n is a standard natural number, T is a set of closed $\in$-formulas (represented as finite sequences of symbols and, perhaps, sets used as parameters) such that the intersection of T with S satisfies the Tarski rules for the truth of in S restricted to Σ_n-formulas and below.

Let on the contrary $^*M$ be an **IST** enlargement of M in which M is the collection of all standard sets. For any standard n, $\exists T\, \tau(n, T)$ is true in $^*M$ — it suffices to utilize universal Σ_n-formula and use Transfer. Thus an $\in$-formula $\phi(x, y, ...)$ is true in M (where $x, y, ... \in \mathsf{S}$) iff

$$\exists\, n\, \exists\, T\, [\, \tau(n, T)\, \&\, \phi(x, y, ...) \in T\,]$$

is true in $^*M$. (We omit, generally speaking, the necessary procedure of "Gödelization" of formulas). Therefore, since in particular all M-countable ordinals are $\in$-definable in M, there exists a map ω which is st-$\in$-definable in $^*M$ and onto ω_1^M, the least uncountable ordinal in M. Applying Standardization, we obtain a map ω onto ω_1^M in M, contradiction. ∎

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[25] This theorem was proved in Kanovei [12], see also Kanovei [14]. We outline here the proof to keep the text self–contained.

[26] See e.g. P. Cohen, *Set Theory and the Continuum Hypothesis*, 1966.

[27] This is rather surprising because the existence of a truth definition witnesses usually that the theory that gives the definition is somewhat stronger than the underlying theory, but **ZFC** and **IST** are equiconsistent.

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