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Internal Approach to External Sets and Universes

Part 2

External universes over the universe of bounded set theory

Abstract. In this article [†] we show how the universe of **BST**, *bounded set theory* (a modification of **IST** which is, briefly, a theory for the family of those sets in **IST** which are members of standard sets) can be enlarged by definable subclasses of sets (which are not necessarily sest in internal theories like **BST** or **IST**) so that Separation and Replacement are true in the enlargement for all formulas, including those in which the standardness predicate may occur.

Thus **BST** is strong enough to incorporate external sets in the internal universe in a way sufficient to develop topics in nonstandard analysis inaccessible in the framework of a purely internal approach, such as Loeb measures.

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Preface

This paper is second in the series of three articles devoted to set theoretic foundations of nonstandard mathematics. The first article [11], recently published in this Journal, was devoted to *bounded set theory* **BST**, a modification of internal set theory **IST** of Nelson [18]. It was shown in [11] that **BST** is a conservative extension of **ZFC**. Basically **BST** is a theory which describes those sets in the universe of **IST** which are members of standard sets.

To make the exposition in this article self-contained, we give a short review of **BST** including the main theorems of [11] in Section 1.

In this article the following question is considered: how "external sets" like ${}^\sigma\mathbb{N} = \{n \in \mathbb{N} : \operatorname{st} n\}$ can be consistently accomodated in **BST**. [1] For internal theories it is a natural idea to use st-$\in$-definable subclasses of internal sets as a substitute for external sets.

A definable subclass of a set is usually a set in standard set theories; in **ZFC** this is implied by the Separation axiom. Internal theories include Separation for $\in$-formulas only, so that, say, ${}^\sigma\mathbb{N}$ is *not* a set in **BST**.

DEFINITION. A *bounded definable class* is a subclass of a set.

To develop the idea fruitfully, we have to parametrize by sets *all* st-$\in$-definable subclasses using *one fixed* st-$\in$-formula. Fortunately **BST** provides such a parametrization, [2] see Theorem 2.2. (This is unknown for **IST**.)

We show in Section 2 that every bounded definable class has in **BST** the form $C_p = \bigcup_{a \in {}^\sigma A} \bigcap_{b \in {}^\sigma B} C_{ab}$, where $p = \langle A, B, \eta\rangle$, A and B are internal sets while $\eta = \langle C_{ab} : a \in A \ \&\ b \in B\rangle$ is an internal indexed family of internal sets. This allows to define the enlargement $\mathbb{E}$ of the internal universe $\mathbb{I}$ of **BST** by all bounded definable classes (of internal sets).

In the rigorous sense, $\mathbb{E}$ is simply $\mathbb{I}$ equipped by certain new relations $\overset{e}{=}$, ${}^e\!\in$, ${}^e\mathrm{st}$ (in particular, $x \overset{e}{=} y$ says that $C_x = C_y$), and an embedding which attaches a set ${}^e x$ to any x so that $x = C_{{}^e x}$. But it is completely

[1] We refer the reader to Diener and Stroyan [3], Henson and Keisler [4], Hurd and Loeb [7], Keisler, Kunen, Miller, and Leth [14], Lindstrøm [15], Luxemburg [17], Stroyan and Bayod [20] on matters of nonstandard analysis via nonstandard structures in the "standard" universe of **ZFC**.

[2] Reeken gave in [19] a detailed outline of this approach in the case when a parametrization is assumed to be given.

consistent to consider things as if $\mathbb{E}$ in fact enlarges the **BST** universe $\mathbb{I}$ so that $\mathbb{I}$ is the class of all sets which are elements of standard sets in $\mathbb{E}$.

We prove in Section 2 that $\mathbb{E}$ models **EEST**, a theory which describes exactly the universe of internal sets and external sets of internal elements. In particular this theory includes Comprehension for all st-$\in$-formulas and implies suitable forms of Saturation and Choice.

A defect of this approach is the absence of *sets* of external sets. Various codings may be used to fix this problem. The most advanced results can be obtained by the *cumulative coding* which utilizes the idea of construction of sets along a well-founded tree.

Suppose that some internal sets are placed at maximal points of a well-founded tree, and at every nonmaximal point we simply collect the sets obtained from the immediate successor points. The result appears at the root.

We run this construction in **EEST** using external well-founded trees. This leads to a much more advanced enlargement $\mathbb{H}$ of the internal universe $\mathbb{I}$ (Section 3) which includes $\mathbb{E}$ and many other external objects.

We finally prove in Section 4 that $\mathbb{H}$ models **HST**, an external theory containing all of **ZFC** minus Choice, Power Set, and Regularity (but with Separation and Replacement for all st-$\in$-formulas). In fact **HST** is close to the theory $\mathbf{NS_1}$ of Hrbaček [5].

This article ends with a brief discussion of how this idea can be used to develop external sets in the framework of the internal approach.

1. A review of BST

Bounded set theory **BST** is a theory in the st-$\in$-language [3] which contains all of **ZFC** (in the $\in$-language) together with the following axioms:

Bounded idealization **BI** :

$$\forall^{\text{stfin}} A \; \exists\, x \in X \; \forall\, a \in A \; \Phi(x,a) \;\longleftrightarrow\; \exists\, x \in X \; \forall^{\text{st}} a \; \Phi(x,a)$$

Standardization **S** : $\forall^{\text{st}} X \; \exists^{\text{st}} Y \; \forall^{\text{st}} x \; [\, x \in Y \;\longleftrightarrow\; x \in X \;\&\; \Phi(x) \,]$

Transfer **T** : $\exists\, x \; \Phi(x) \;\longrightarrow\; \exists^{\text{st}} x \; \Phi(x)$

Boundedness **B** : $\forall\, x \; \exists^{\text{st}} X \; (x \in X)$.

[3] The language containing membership $\in$, the unary predicate of standardness st, and the equality as the atomic predicates.

The formula Φ must be an $\in$-formula in **BI** and **T**, and Φ can contain only standard sets as parameters in **T**, but Φ can be any st-$\in$-formula in **S** and contain arbitrary parameters in **BI** and **S**. The quantifiers $\exists^{\text{st}}$ and $\forall^{\text{st}}$ have the obvious meaning: there exists standard, for all standard. $\forall^{\text{stfin}} A$ means: for all standard finite A. X is a standard set in **BI** (In the presence of **B** the set X may be any internal set in **BI**.

Thus **BI** is weaker than the Idealization **I** of internal set theory **IST** of Nelson [18] (**I** corresponds to the case of $X = \{\text{all sets}\}$ in **BI**), but the Boundedness axiom **B** is added.

The following is a brief exposition of the basic theorems of **BST**, proved in Kanovei and Reeken [11], which we refer to in this article.

The first theorem shows that **BI** could be replaced by another axiom, *Internal saturation* **IS** :

$$\forall^{\text{stfin}} A \subseteq A_0 \, \exists x \, \forall a \in A \, \Phi(x,a) \quad \longleftrightarrow \quad \exists x \, \forall^{\text{st}} a \in A_0 \, \Phi(x,a) \, ,$$

where A_0 is a standard set and Φ an $\in$-formula.

THEOREM 1.1. (Lemma 3.1 in [11].)
BI *is equivalent to* **IS** *in* **ZFC** *plus* **B** *plus* **T** .

THEOREM 1.2. (Reduction to Σ_2^{st} form. Theorem 1.5 in [11].)
Let $\Phi(x_1, ..., x_m)$ be an arbitrary st-$\in$*-formula. There exists an $\in$-formula $\varphi(x_1, ..., x_m, a, b)$ such that the following is a theorem of* **BST** :

$$\forall x_1 ... \forall x_m \, [\, \Phi(x_1, ..., x_m) \quad \longleftrightarrow \quad \exists^{\text{st}} a \, \forall^{\text{st}} b \; \varphi(x_1, ..., x_m, a, b) \,] \, .$$

(Σ_2^{st} denotes the class of all formulas $\exists^{\text{st}} a \, \forall^{\text{st}} b$ ($\in$-*formula*).) Thus every st-$\in$-formula is equivalent in **BST** to a Σ_2^{st} formula.

THEOREM 1.3. (Collection in **BST**. Theorem 1.8 in [11].)
Let $\Phi(x,y)$ be a st-$\in$*-formula having arbitrary sets as parameters. For any X there exists a standard set Y such that*

$$\forall x \in X \, [\, \exists y \, \Phi(x,y) \quad \longrightarrow \quad \exists y \in Y \; \Phi(x,y) \,] \, .$$

THEOREM 1.4. (Extension in **BST**. Theorem 1.9 in [11].)
Let $\Phi(x,y)$ be a st-$\in$*-formula containing arbitrary sets as parameters. Then for any standard X there exists a function f defined on X such that*

$$\forall^{\text{st}} x \in X \, [\, \exists y \, \Phi(x,y) \quad \longrightarrow \quad \Phi(x, f(x)) \,] \, .$$

THEOREM 1.5. (Theorem 1.10 in [11].)
The schema of dependent choice holds in **BST** .

Kanovei [10] proved Theorem 1.3. for **IST**. Theorems 1.2. and 1.4. are true in **IST** only for formulas of special type. (See Kanovei [8] on the matter of counterexamples.) It is not yet known whether Theorem 1.5. is true in **IST** .

2. Elementary external enlargement

This section shows how external sets can be accomodated in the internal theory **BST** in their most elementary form — as st-$\in$-definable subclasses of internal sets.

2.1. Parametrization and the enlargement

We begin with the definition of the parametrization

DEFINITION 2.1. [The parametrization]
Let $p = \langle A, B, \eta \rangle$, where A and B are standard sets, η an (internal) function defined on $A \times B$. We set $\mathcal{C}_p = \bigcup_{a \in {}^{\sigma}\!A} \bigcap_{b \in {}^{\sigma}\!B} \eta(a,b)$. If p is not of the form $\langle A, B, \eta \rangle$, we set $\mathcal{C}_p = \emptyset$.

THEOREM 2.2. [Parametrization Theorem] [**BST**]
Every bounded definable class C *is equal to some* $\mathcal{C}_p$.

PROOF. Let $C = \{x \in X : \Phi(x, q_0)\}$, where q_0 and X are arbitrary sets and Φ is a st-$\in$-formula. By the Boundedness axiom, one may assume that X is standard. By the Reduction Theorem 1.2. one may also assume that $\Phi(x,q)$ is a Σ_2^{st} formula $\exists^{\mathrm{st}} a\, \forall^{\mathrm{st}} b\, \varphi(x,a,b,q)$, where φ is an $\in$-formula. By Boundedness there exists a standard set Q containing q_0. Collection implies the existence of a pair of standard sets A, B such that

$$\forall x \in X\, [\, \Phi(x,q) \;\longleftrightarrow\; \exists^{\mathrm{st}} a \in A\, \forall^{\mathrm{st}} b \in B\; \varphi(x,a,b,q)\,] .$$

for all $q \in Q$. We put $\eta(a,b) = \{x \in X : \varphi(x,a,b,q_0)\}$ for $a \in A$, $b \in B$, and $p = \langle A, B, \eta \rangle$. Then $C = \{x \in X : \Phi(x, q_0)\} = \mathcal{C}_p$. ■

Thus the family of all definable bounded classes is parametrized as the collection of all classes $\mathcal{C}_p$. The principal idea is to enlarge $\mathbb{I}$, the **BST** universe, by adding all definable bounded classes $\mathcal{C}_p$, $p \in \mathbb{I}$, to obtain in this way a wider set universe in which every definable subclass of a set is a set itself, as it is the case in **ZFC** but not in internal theories like **BST** .

Let $\mathbb{E}$ denote the enlargement.

We have indeed to clarify how $\mathbb{E}$ can be legitimately defined within $\mathbb{I}$, the basic internal universe. Let $\epsilon(x,p)$ be the st-$\in$-formula of Definition 2.1. which expresses the relation $x \in \mathcal{C}_p$. We let, further,

$p \overset{e}{=} q$ be the formula $\mathcal{C}_p = \mathcal{C}_q$, i. e. $\forall x\, [\epsilon(x,p) \longleftrightarrow \epsilon(x,q)]$;

$p \overset{e}{\in} q$ be the formula $\mathcal{C}_p \in \mathcal{C}_q$, i. e.
$$\exists y\, [\epsilon(y,q)\ \&\ \forall x(\epsilon(x,p) \longleftrightarrow x \in y)];$$

${}^{e}\mathrm{st}\, p$ be the formula $\mathrm{st}\, \mathcal{C}_p$, i. e. $\exists^{\mathrm{st}} y\, \forall x\, [\epsilon(x,p) \longleftrightarrow x \in y]$.

This makes it possible to define $\mathbb{E} = \langle \mathbb{I}; \overset{e}{=}, \overset{e}{\in}, {}^{e}\mathrm{st} \rangle$, the *elementary external enlargement* of $\mathbb{I}$. To see that this is actually an enlargement, we put ${}^{e}x = \langle \{\emptyset\}, \{\emptyset\}, \eta_x \rangle$ for every x, where η_x is a function defined on $\{\langle \emptyset, \emptyset \rangle\}$ by $\eta_x(\emptyset, \emptyset) = x$. Then evidently $\mathcal{C}_{{}^{e}x} = x$, therefore

$$x = y \longleftrightarrow {}^{e}x \overset{e}{=} {}^{e}y; \quad x \in y \longleftrightarrow {}^{e}x \overset{e}{\in} {}^{e}y; \quad \mathrm{st}\, x \longleftrightarrow {}^{e}\mathrm{st}\, {}^{e}x \qquad (*)$$

for all x, y. Thus $\mathbb{E}$ really extends $\mathbb{I}$ via identification of x with ${}^{e}x$.

By definition, a closed st-$\in$-formula is true in $\mathbb{E}$ iff it is true (in $\mathbb{I}$) provided $=, \in, \mathrm{st}$ are interpreted as $\overset{e}{=}$, $\overset{e}{\in}$, ${}^{e}\mathrm{st}$ respectively. In other words, let ${}^{e}\Phi$ denote the relativization of Φ to $\mathbb{E}$: all subformulas $x \in y$, $x = y$, $\mathrm{st}\, x$ in Φ are changed to $x \overset{e}{\in} y$, $x \overset{e}{=} y$, ${}^{e}\mathrm{st}\, x$ respectively. We have by induction on the complexity of Φ :

PROPOSITION 2.3. *Let* $\Phi(x_1, ..., x_m)$ *be a* st-$\in$*-formula. Then, for all* $x_1, ..., x_n \in \mathbb{I}$,

$$[\, \Phi(x_1, ..., x_n) \text{ is true in } \mathbb{E}\,] \longleftrightarrow [\, {}^{e}\Phi(x_1, ..., x_n) \text{ is true in } \mathbb{I}\,].$$

EXAMPLE Let Φ be the st-$\in$-formula $x \in y$. Then ${}^{e}\Phi$ is $x \overset{e}{\in} y$, that is, $\mathcal{C}_x \in \mathcal{C}_y$. Let $x, y \in \mathbb{I}$. Then $x \in y$ in $\mathbb{E}$ iff $x \overset{e}{\in} y$ in $\mathbb{I}$.

2.2. Elementary external theory

The next step is to introduce a suitable theory in the st-$\in$-language which adequately describes $\mathbb{E}$. First of all we introduce a formula which formally distinguishes internal sets in $\mathbb{E}$. Let $\mathrm{int}\, x$ be the formula $\exists^{\mathrm{st}} X\, (x \in X)$. Let, for every formula Φ, Φ^{int} denote the relativization of Φ to int .

DEFINITION 2.4.
The *Elementary external set theory* **EEST** contains the axioms:

1. $\forall^{\text{st}} x \, (\text{int}\, x)$: all standard sets are internal;

   $\forall^{\text{int}} x \, \forall y \in x \, (\text{int}\, y)$: transitivity of the internal subuniverse;

   Standardization: $\forall X \, \exists^{\text{st}} Y \, \forall^{\text{st}} x \, (x \in Y \longleftrightarrow x \in X)$.
2. $\mathbf{BST}^{\text{int}}$: all axioms of **BST** relativized to the formula int ;
3. Extensionality and Separation for all st-$\in$-formulas;
4. The Parametrization axiom: $\forall X \, \exists^{\text{int}} p \, (X = \mathcal{C}_p)$.

The last axiom may be seen as a very artificial statement, but actually it postulates that *all sets are bounded definable classes from the point of view of the internal universe*, or, what is the same, that the universe of all (external) sets is the external enlargement of the internal universe $\mathbb{I}$ in the sense defined above. Usually one cannot express statements of this kind legitimately (unless, say, objects of a higher type enter in consideration); it is a very special property of the bounded set theory **BST** that an indirect formulation (via classes $\mathcal{C}_p$) becomes available.

THEOREM 2.5. *All axioms of* **EEST** *are true in* $\mathbb{E}$.

PROOF. It is asserted that, given an axiom Φ of **EEST**, ${}^{e}\Phi$ is a theorem of **BST**. Verification of axioms 1 and 2 in item 1 is quite elementary. $\mathbf{BST}^{\text{int}}$ holds in $\mathbb{E}$ because the internal subuniverse of $\mathbb{E}$ is a copy of the **BST** universe $\mathbb{I}$ via the embedding $x \longmapsto {}^{e}x$. Standardization can be reduced to the **BST** Standardization in $\mathbb{I}$ by Proposition 2.3. Extensionality holds by the definition of the basic relations ${}^{e}{=}$, ${}^{e}{\in}$, ${}^{e}\text{st}$. Separation in the form

$$\forall X \, \exists Y \, \forall x \, [x \in Y \longleftrightarrow x \in X \ \& \ \Phi(x)]$$

(where $\Phi(x)$ is an arbitrary st-$\in$-formula which may contain parameters from $\mathbb{E}$) is reduced to Theorem 2.2. again by Proposition 2.3. Finally Parametrization holds in $\mathbb{E}$ by the construction. ∎

COROLLARY 2.6. **EEST** *admits an interpretation in* **BST** *such that the class of all formally internal (satisfying* $\text{int}\, x$ *) sets in the sense of the interpretation coincides with (more precisely: is isomorphic to) the ground* **BST** *universe.*

PROOF. The ^{e}-transformation of formulas gives the interpretation. ∎

COROLLARY 2.7. **EEST** *is a conservative extension of* **BST** *in the following sense. Let* Φ *be a* st-$\in$*-sentence. Then* Φ *is a theorem of* **BST** *if and only if* Φ^{int} *is a theorem of* **EEST** .

It is a visible advantage of the Parametrization axiom, and a very essential property of **EEST** in general, that one can reduce the truth in the **EEST** universe $\mathbb{E}$ to the truth in the internal subuniverse $\mathbb{I}$.

PROPOSITION 2.8. [**EEST**] *Let* $\Phi(x_1, ..., x_n)$ *be a* st-$\in$*-formula. Then, for all internal* $p_1, ..., p_n$ *and* $x_1, ..., x_n$,

$$\text{(1)} \quad \Phi(\mathcal{C}_{p_1}, ..., \mathcal{C}_{p_n}) \quad \longleftrightarrow \quad [\, {}^{\text{e}}\Phi(p_1, ..., p_n) \text{ is true in } \mathbb{I} \,];$$

$$\text{(2)} \quad \Phi(x_1, ..., x_n) \quad \longleftrightarrow \quad [\, {}^{\text{e}}\Phi({}^{\text{e}}x_1, ..., {}^{\text{e}}x_n) \text{ is true in } \mathbb{I} \,].$$

(Quite similar to Proposition 2.3, and actually reflects the same property.)

PROOF. Induction by the complexity of Φ using the Parametrization axiom in the principal step for $\exists$. ■

2.3. Development of EEST

We argue in **EEST** in this subsection. Let $\mathbb{E}$ denote the ground **EEST** universe. We define $\mathbb{I} = \{x : \text{int}\, x\}$, $\mathbb{S} = \{x : \text{st}\, x\}$ (the subuniverses of internal and standard sets respectively); thus $\mathbb{S} \subseteq \mathbb{I} \subseteq \mathbb{E}$ and the axioms of **BST** hold in $\mathbb{I}$.

The following proposition gives an example of reasoning in **EEST** .

PROPOSITION 2.9. [**EEST**] *Every set* C *is a subset of a standard set.*

PROOF. Let, by Parametrization, $C = \mathcal{C}_p$, where p is internal. Then by definition either $C = \emptyset$ or $p = \langle A, B, \eta \rangle$, where A, B, η are internal, and in the latter case $C \subseteq Y = \text{ran}\, \eta$. By $\mathbf{BST}^{\text{int}}$ and transitivity of the internal subuniverse, Y is internal, therefore $Y \in s$ for a standard s. The set $S = \bigcup s$ is internal; indeed, S is equal to $\bigcup s$ *taken in* $\mathbb{I}$ by transitivity of $\mathbb{I}$. Hence S is standard by Transfer, and $Y \subseteq S$. ■

COROLLARY 2.10. [**EEST**] *For any* y, *if* $x \in y$ *then* x *is internal.*

PROOF. By Proposition 2.9, $y \subseteq S$ for a standard set S . ■

We present several theorems which are useful for the development of nonstandard mathematics in the theory **EEST** : Collection, Extension, Choice, Saturation. They reflect to some extent the familiar properties of external subsets of internal sets in nonstandard models.

THEOREM 2.11. [Collection] [**EEST**]
Let $\Phi(x,y)$ *be a* st-$\in$-*formula having arbitrary sets as parameters. For any (not necessarily internal) set* X *there exists a standard set* Y *such that*

$$\forall x \in X\, [\, \exists^{\text{int}} y\; \Phi(x,y) \;\longrightarrow\; \exists\, y \in Y\; \Phi(x,y)\,]\,.$$

PROOF. First of all we get rid of the "external side" of the problem. Using the Parametrization axiom, one may replace every external parameter that occurs in Φ by some C_p; thus we can assume from the beginning that Φ does not contain noninternal parameters. Let $\Psi(x,y)$ be the formula ${}^{\mathrm{e}}\Phi({}^{\mathrm{e}}x, {}^{\mathrm{e}}y)$. By Proposition 2.8, $\forall^{\text{int}} x\; \forall^{\text{int}} y\; [\, \Phi(x,y) \;\longleftrightarrow\; \mathbb{I} \models \Psi(x,y)\,]$. Using Proposition 2.9, we find a standard set X' such that $X \subseteq X'$, and then apply Theorem 1.3. (the **BST** Collection) in $\mathbb{I}$ to the formula Ψ . ■

To formulate Collection for the case when the variable y does not not necessarily denote an internal set, we are in need of some coding, since $\mathbb{E}$ does not contain collections of external sets. For $X \in \mathbb{E}$ we set $|X| = \{C_p : p \in X\}$. Thus, informally, one sees $|X|$ as a "set–size" collection of external sets. Formally, expressions like $Y \in |X|$ are shortcuts for legitimate st-$\in$-formulas, e.g. $\exists\, p \in X\; \forall\, y\; (y \in Y \;\longleftrightarrow\; y \in C_p)$ for the given example.

THEOREM 2.12. ["External" Collection] [**EEST**]
Let $\Phi(x,Y)$ *be a* st-$\in$-*formula having arbitrary sets as parameters. For any* X *there exists a standard set* $\mathcal{Y}$ *such that*

$$\forall x \in X\, [\, \exists\, Y\; \Phi(x,Y) \;\longrightarrow\; \exists\, Y \in |\mathcal{Y}|\; \Phi(x,Y)\,]\,.$$

PROOF. Apply the previous theorem to the formula $\Phi(x, C_y)$. ■

What is understood as *Extension* in the model theoretic setting of nonstandard analysis (see e. g. Lindstrøm [15]), can be formulated in the framework of the axiomatic approach as follows: if g is an external function defined on ${}^{\sigma}X = \{x \in X : \operatorname{st} x\}$ (a legitimate set in **EEST** by Separation), where X is standard, and taking internal values (this is automatic in **EEST** since all elements of sets are internal), then there exists an internal function f defined on X so that $g(x) = f(x)$ for all standard $x \in X$.

But sometimes f itself is obtained in the model theoretic setting by the axiom of Choice in the ambient **ZFC** universe. Since one has no direct analogy of this in the **EEST** setting, we formulate Extension so that it includes Choice together with the internality of the resulting function.

THEOREM 2.13. [Extension] [**EEST**]
Let $\Phi(x,y)$ *be a* st-$\in$-*formula having arbitrary sets as parameters. For any* X *there exists an internal function* f *such that* $X \subseteq \operatorname{dom} f$ *and*

$$\forall^{\mathrm{st}} x \in X\,[\,\exists^{\mathrm{int}} y\ \Phi(x,y) \longrightarrow \Phi(x,f(x))\,].$$

PROOF. First of all, by Collection, there exists a standard set Y such that

$$\forall^{\mathrm{st}} x \in X\,[\,\exists^{\mathrm{int}} y\ \Phi(x,y) \longrightarrow \exists\, y \in Y\ \Phi(x,y)\,].$$

Then $P = \{\langle x,y\rangle \in X \times Y : \Phi(x,y)\}$ is a set by Separation. Let $P = \mathcal{C}_p$, $p \in \mathbb{I}$, by the Parametrization axiom. It remains to apply Theorem 1.4. (the **BST** Extension) to $\langle x,y\rangle \in \mathcal{C}_p$ as the formula $\Phi(x,y)$. ■

The last theorem is *Choice*. To formulate this properly, we put $\widetilde{F}(x) = \mathcal{C}_{F(x)}$, whenever F is a function and $x \in \operatorname{dom} F$. To prove the result, simply apply Extension to the formula $\Phi(x,\mathcal{C}_y)$.

THEOREM 2.14. [Choice] [**EEST**]
Let $\Phi(x,Y)$ *be a* st-$\in$-*formula with arbitrary sets as parameters. For any* X *there exists an internal function* F *such that* $X \subseteq \operatorname{dom} F$ *and*

$$\forall^{\mathrm{st}} x \in X\,[\,\exists\, Y\ \Phi(x,Y) \longrightarrow \Phi(x,\widetilde{F}(x))\,].$$

The theory **EEST** is strong enough also to prove Dependent Choice and standard size Saturation. It is more suitable, indeed, to delay these theorems to Section 4 where they will be considered in a wider framework.

2.4. Discussion

In this subsection we shortly discuss how a mathematician willing to accept **BST** as his basic axiomatic theory (related to **ZFC** by the model enlargement and the reduction property as explained in part one [11]) may smoothly incorporate external sets into his argumentation without illegitimately transcending the limits of the internal theory.

Thesis. One can legitimately enlarge $\mathbb{I}$ by bounded definable classes.

First of all we want to understand the meaning of the thesis. It is convenient to develop this idea in parallel to the case of the enlargement of the real line $\mathbb{R}$ to the field of complex numbers. There are five principal points:

1. Every element of the enlargement has a *label* which identifies it with objects in the ground universe; thus a complex number $a+bi$ is labeled by the pair $\langle a, b \rangle$ of real numbers.
2. Every element of the ground universe has a label which determines it as an element of the enlarged universe. In particular, the pair $\langle a, 0 \rangle$ corresponds in this sense to a real number a .
3. The basic relations between objects in the enlarged universe are reflected as relations between *labels* in the ground universe. Thus the laws of addition, multiplication etc. in the complex field can be presented as certain operations over pairs of reals.
4. On the other hand, the basic relations in the ground universe retain its meaning in the enlarged universe; thus the addition etc. in $\mathbb{R}$ remain addition etc. in the field of complex numbers.
5. The enlargement satisfies certain principles or conditions which reflect, to some extent, the idea of what the enlargement should be. Say the field of compex numbers is algebraically closed.

$\mathbb{R}$ is thus enlarged to the complex field, but everything which happens in the enlargement can be completely traced back to $\mathbb{R}$.

It will not take a lot of efforts to see that the enlargement $\mathbb{E}$ of $\mathbb{I}$, the universe of **BST**, by bounded definable classes is of the same type, with the minor detail that in this case an element of the enlargement has many labels in the ground universe.

First, every element of $\mathbb{E}$ is C_p for some (not unique) $p \in \mathbb{I}$; theorem 2.2. tells that we do not miss any of the definable subclasses of sets. Second, every x has a certain label, ${}^{e}x$, which presents x as a member of $\mathbb{E}$. Third, the principal relations $\in$, $=$, st between classes C_p in $\mathbb{E}$ are expressed in $\mathbb{I}$ as relations between *labels* p, that is, relations ${}^{e}\!\in$, ${}^{e}\!=$, ^{e}st defined above. Fourth, as it is shown by $(*)$ in Subsection 2.1., the basic relations in $\mathbb{I}$ keep their meaning in $\mathbb{E}$. Finally, $\mathbb{E}$ satisfies Separation (among the other axioms of **EEST**), which is one of the most desired properties of a nonstandard set universe of "external" type.

The treatment of the external enlargement $\mathbb{E}$ in $\mathbb{I}$, the internal universe of **BST**, is therefore, in principle, analogous to the treatment of complex numbers as pairs of real numbers. In other words, assuming that $\mathbb{I}$ is enlarged to $\mathbb{E}$, the "universe" of all bounded definable classes, the **BST** mathematician does not face a problem with uncertainty or even illegality. As with complex numbers there is no need to translate everything back into the ground universe all the time. This is what we had in mind from the beginning.

It is not clear how this approach may be realized in **IST** instead of **BST**. So far we can neither prove a comparable parametrization theorem in **IST** nor demonstrate that **IST** does not admit a parametrization in general.

3. Cumulative external enlargements

The aim of this section is to extend the method of Section 2 to sets of external sets and more complicated objects of this type. One can use various codings, like $|X|$ or $\widetilde{F}$ in Section 2, to code special types of external objects, more complicated than external sets of internal elements. The most powerful of the coding systems is based on the procedure of construction of sets along a well-founded tree.

DEFINITION 3.1. [Informal] Let T be a well-founded tree and F be a function defined on $\operatorname{Max} T$, the collection of maximal points of T. The family of sets $F_T(t)$, $t \in T$, is defined by

1) if $t \in \operatorname{Max} T$ then $F_T(t) = F(t)$;
2) if $t \in T \setminus \operatorname{Max} T$ then $F_T(t) = \{F_T(t^\wedge a) : t^\wedge a \in T\}$.

We define finally $F[T] = F_T(\Lambda)$.

Let, for example, $T = \{\Lambda\}$ and $F(\Lambda) = x$. Then $F[T] = F_T(\Lambda) = x$.

Of course, if we run this in **BST** then nothing except the already existing (internal) sets can be obtained since **BST** includes **ZFC**. The picture changes in **EEST** : collections of external sets are not formally admitted, so we are creating new objects in this way.

It is the aim of this and the next section to demonstrate the possibilities of this method. Starting from a **EEST** universe, we obtain a standard size saturated external enlargement in which all of **ZFC** minus Choice, Power Set, and Regularity (but with Separation and Replacement for all st-$\in$-formulas) is satisfied — this is essentially the theory $\mathbf{NS}_1$ of Hrbaček [5]. [4]

[4] This way of introduction of external sets into an internal universe is an essential simplification of a sophisticated construction of [5] based on the infinitary language $\mathrm{L}_{\infty\infty}$.

We shall argue in **EEST** and use the **EEST** universe $\mathbb{E}$ as the basic universe. Elements of $\mathbb{E}$ will be called *external sets* or simply *sets*.

$$\mathbb{S} = \{x \in \mathbb{E} : \mathrm{st}\, x\} \quad \text{and} \quad \mathbb{I} = \{x \in \mathbb{E} : \mathrm{int}\, x\} = \{x \in \mathbb{E} : \exists^{\mathrm{st}} X\ (x \in X)\}$$

denote the subuniverses of resp. standard and internal sets in $\mathbb{E}$. Actually we could begin from the **BST** universe $\mathbb{I}$, but then we would have to proceed with trees which are definable classes, which would be inconvenient (although possible technically).

3.1. Wellfounded trees

Let Seq denote the class of all internal sequences of a standard finite length. For $t \in \mathrm{Seq}$ and every set a, $t^\wedge a$ denotes the sequence in Seq obtained by adjoining a as the rightmost additional term to t. The notation $a^\wedge t$ is to be understood correspondingly. Λ is the empty sequence. The formula $t' \subseteq t$ means that the sequence $t \in \mathrm{Seq}$ *extends* $t' \in \mathrm{Seq}$ (perhaps $t' = t$).

A *tree* is a nonempty set $T \subseteq \mathrm{Seq}$ such that, for any pair of sequences $t', t \in \mathrm{Seq}$ satisfying $t' \subseteq t$, $t \in T$ implies $t' \in T$. Thus every tree contains Λ. $\mathrm{Max}\, T$ denotes the set of all $\subseteq$*-maximal* $r \in T$.

A tree T is *well-founded* (*wf tree*, in brief) if and only if every nonempty external $T' \subseteq T$ contains a $\subseteq$-maximal element.

LEMMA 3.2. [wf Induction]
Let T be a wf tree and $X \subseteq T$ be an external set satisfying $\mathrm{Max}\, T \subseteq X$ and having the property that $t \in X$ whenever $t \in T$ is such that every $t^\wedge a \in T$ belongs to X. Then $X = T$.

PROOF. Consider an $\subseteq$-maximal element $t \in T \setminus X$. ∎

Since the external **EEST** universe $\mathbb{E}$ is not, generally speaking, closed under the usual pairing, we define an *artificial ordered pair* by

$$\lceil XY \rceil = (\{0\} \times X) \cup (\{1\} \times Y), \qquad \text{for all sets } X, Y.$$

DEFINITION 3.3. $\mathcal{H}$ is the collection of all pairs $\lceil TF \rceil$ such that T is an (in general, external) wf tree and F is an (in general, external) function [5] defined on $\mathrm{Max}\, T$. Elements of $\mathcal{H}$ are called *wf pairs*.

Notice, however, that the problem we face here is much easier than the one in [5]. Indeed we already have the internal and elementary external universes while Hrbaček had to define simultaneously both internal and external universes from a **ZFC** universe. Actually most of the problems in our setting are solved by the Parametrization theorem.

[5] Notice that in **EEST** external functions take only internal values.

Let $\lceil TF \rceil \in \mathcal{H}$. One cannot expect, of course, that $F_T(t)$ is well defined in $\mathbb{E}$, the **EEST** universe, for all $t \in T$ by Definition 3.1. However, one can determine the domain of direct definability of F_T.

LEMMA 3.4. *Let $\lceil TF \rceil \in \mathcal{H}$. There exists a (unique) set $I = I(T,F) \subseteq T$ on which F_T is defined in accordance to Definition 3.1, but cannot be defined anywhere in $T \setminus I$.*

PROOF. Let a *good function* be any external function f such that

1) the domain $D = \operatorname{dom} f$ satisfies $\operatorname{Max} T \subseteq D \subseteq T$ and is closed upwards in T, that is, if $t \in D$ and $t^\wedge a \in T$ then $t^\wedge a \in D$;

2) f coincides with F on $\operatorname{Max} T$; and

3) $f(t) = \{f(t^\wedge a) : t^\wedge a \in T\}$ for all $t \in D$.

Any two good functions f and g coincide on the common domain by Lemma 3.2. It follows from the **EEST** Comprehension and Theorem 2.11. (Collection) that the union φ of all good functions is a set, therefore a good function. We put $I = I(T,F) = \operatorname{dom} \varphi$ and $F_T(t) = \varphi(t)$ for $t \in I$. ■

The set $I = I(T,F)$ includes $\operatorname{Max} T$ and is closed upwards in T. Furthermore if $t \in T$ satisfies $t^\wedge a \in I$ whenever $t^\wedge a \in T$ then the necessary and sufficient condition for $t \in I$ is the internality of the set $X = \{F_T(t^\wedge a) : t^\wedge a \in T\}$; if this holds, one can define $F_T(t) = X$, thus expanding F_T on t. Take notice that $F[T] = F_T(\Lambda)$, therefore $F[T]$ is well defined iff $\Lambda \in I(T,F)$ (which is equivalent to $T = I(T,F)$).

We can expand F_T even a bit more. Let indeed $t \in T \setminus I(T,F)$ be such that all $t^\wedge a \in T$ belong to $I(T,F)$. We put in this case ${}^{\mathrm{e}}F_T(t) = \{F_T(t^\wedge a) : t^\wedge a \in T\}$, so that ${}^{\mathrm{e}}F_T(t)$ is an external (*not* internal) set.

EXAMPLE 3.5. Let x be internal. Let $T = \{\Lambda\}$ and $C^x(\Lambda) = x$. Then $\lceil TC^x \rceil \in \mathcal{H}$, $\Lambda \in I(T, C^x)$, and $C^x[T] = C^x{}_T(\Lambda) = x$.

To conclude, an occurrence of $F_T(t)$ in a formula, say, $F_T(t) = x$ makes sense only in the case when $t \in I(T,F)$ — then $F_T(t)$ denotes the internal set "computed" via Definition 3.1. In particular, the equality $F[T] = x$ preassumes that $\Lambda \in I(T,F)$ and is a shortcut for $F_T(\Lambda) = x$. But if $t \notin I(T,F)$ then $F_T(t)$ is only a symbol for what it should be provided all the operations intended by Definition 3.1. could have been performed.

3.2. Basic relations

However, one can figure out, using only the tools available in $\mathbf{E}$, whether it is implied by Definition 3.1. that, say, $F_T(t)$ should belong, or be equal, to $G_R(r)$ even in the case when the objects themselves are not well defined. This is realized by the binary relations $\overset{h}{=}$ and $^{h}\!\in$ and the unary relation $^{h}\mathrm{st}$ on $\mathcal{H}$ which denote the equality, membership and standardness as implied, first, by the relations in $\mathbf{I}$, and second, by Definition 3.1.

DEFINITION 3.6. Let $\lceil TF \rceil$ and $\lceil RG \rceil$ belong to $\mathcal{H}$. An (external, in general) set $\mathcal{E} \subseteq T \times R$ is a *correspondence set* (*c-set* in brief) for these wf pairs, if the following conditions are satisfied:

1. Assume that $\langle t, r \rangle \in \mathcal{E}$ and either $t \in I(T,F)$ or $r \in I(R,G)$. Then both $t \in I(T,F)$ and $r \in I(R,G)$, and $F_T(t) = G_R(r)$.
2. Let $t \notin I(T,F)$, $r \notin I(R,G)$, and $\langle t, r \rangle \in \mathcal{E}$. Then 1st, for every $r^\wedge b \in R$ there exists $t^\wedge a \in T$ such that $\langle t^\wedge a, r^\wedge b \rangle \in \mathcal{E}$, and 2nd, for every $t^\wedge a \in T$ there exists $r^\wedge b \in R$ such that $\langle t^\wedge a, r^\wedge b \rangle \in \mathcal{E}$.

We write $\lceil TF \rceil(t) \overset{h}{=} \lceil RG \rceil(r)$ if there exists a c-set $\mathcal{E}$ for $\lceil TF \rceil$ and $\lceil RG \rceil$ such that $\langle t, r \rangle \in \mathcal{E}$. If one or both of r, t is equal to Λ, it may be omitted: say $\lceil TF \rceil \overset{h}{=} \lceil RG \rceil$ means $\lceil TF \rceil(\Lambda) \overset{h}{=} \lceil RG \rceil(\Lambda)$.

Take notice that it is not assumed that a c-set is something like a *maximal* c-set (although the latter actually exists); it is not excluded that $t \in I(T,F)$, $r \in I(R,G)$, $G_R(r) = F_T(t)$, but $\langle r, t \rangle \notin \mathcal{E}$, where $\mathcal{E}$ is a c-set.

LEMMA 3.7. $\overset{h}{=}$ *is an equivalence relation.*

PROOF. Let $\lceil RG \rceil(r) \overset{h}{=} \lceil TF \rceil(t)$ and $\lceil TF \rceil(t) \overset{h}{=} \lceil UH \rceil(u)$; we prove that $\lceil RG \rceil(r) \overset{h}{=} \lceil UH \rceil(u)$. Let $\mathcal{A}$ and $\mathcal{B}$ be c-sets for $\lceil RG \rceil$, $\lceil TF \rceil$ and $\lceil TF \rceil$, $\lceil UH \rceil$ respectively such that $\langle r, t \rangle \in \mathcal{A}$ and $\langle t, u \rangle \in \mathcal{B}$. Thus we have to verify that $\mathcal{E} = \{\langle r', u' \rangle : \exists\, t' \,(\langle r', t' \rangle \in \mathcal{A} \;\&\; \langle t', u' \rangle \in \mathcal{B})\}$ is a c-set for $\lceil RG \rceil$, $\lceil UH \rceil$. (Obviously $\langle r, u \rangle \in \mathcal{E}$.)

To check condition 1 of Definition 3.6, we assume that $r' \in I(R,G)$ and $\langle r', u' \rangle \in \mathcal{E}$, so that $\langle r', t' \rangle \in \mathcal{A}$ and $\langle t', u' \rangle \in \mathcal{B}$ for some t'. Then $t' \in I(T,F)$ and $G_R(r') = F_T(t')$ since $\mathcal{A}$ is a c-set. Further, $u' \in I(U,H)$ and $F_T(t') = H_U(u')$ since $\mathcal{B}$ is a c-set. Thus $G_R(r') = H_U(u')$.

Condition 2. Assume that $r' \notin I(R,G)$ and $u' \notin I(U,H)$ but $\langle r', u' \rangle \in \mathcal{E}$ so that $\langle r', t' \rangle \in \mathcal{A}$ and $\langle t', u' \rangle \in \mathcal{B}$ for some $t' \in T$. Then $t' \notin I(T,F)$. Assume that $r'^\wedge a \in R$. Then, since $\mathcal{A}$ is a c-set, there exists b such that $t'^\wedge b \in T$ and $\langle r'^\wedge a, t'^\wedge b \rangle \in \mathcal{A}$. Similarly, there exists c such that we have $\langle t'^\wedge b, u'^\wedge c \rangle \in \mathcal{B}$. Then $\langle r'^\wedge a, u'^\wedge c \rangle \in \mathcal{E}$, as required. ∎

LEMMA 3.8. *Let* $\lceil TF \rceil$ *and* $\lceil RG \rceil$ *belong to* $\mathcal{H}$, $t \in T$, $r \in R$. *Then*

1. $\lceil TF \rceil(t) \stackrel{\mathrm{h}}{=} \lceil RG \rceil(r) \longleftrightarrow r \in I(R,G) \ \& \ F(t) = G_R(r)$, *whenever* $t \in \operatorname{Max} T$.
2. *If* $t \notin \operatorname{Max} T$ *and* $r \notin \operatorname{Max} R$ *then* $\lceil TF \rceil(t) \stackrel{\mathrm{h}}{=} \lceil RG \rceil(r) \longleftrightarrow (a) \ \& \ (b)$, *where*

$$\forall t^\wedge a \in T \ \exists r^\wedge b \in R \, [\, \lceil TF \rceil(t^\wedge a) \stackrel{\mathrm{h}}{=} \lceil RG \rceil(r^\wedge b) \,] \,, \qquad (a)$$

$$\forall r^\wedge b \in R \ \exists t^\wedge a \in T \, [\, \lceil TF \rceil(t^\wedge a) \stackrel{\mathrm{h}}{=} \lceil RG \rceil(r^\wedge b) \,] \,. \qquad (b)$$

PROOF. Assertion 1 and the direction $\longrightarrow$ in 2 are easy. We prove the opposite direction in 2. Let $A = \{a : t^\wedge a \in T\}$, $B = \{b : r^\wedge b \in R\}$,

$$E = \{\langle a, b \rangle \in A \times B : \lceil TF \rceil(t^\wedge a) \stackrel{\mathrm{h}}{=} \lceil RG \rceil(r^\wedge b)\} \,.$$

Then $\operatorname{dom} E = A$ and $\operatorname{ran} E = B$ by the assumption of (a) & (b). In other words for any pair $\langle a, b \rangle \in E$ there exists a c-set $\mathcal{E}'$ containing $\langle t^\wedge a, r^\wedge b \rangle$. One may assume in this case that $\mathcal{E}'$ satisfies the condition $\langle t', r' \rangle \in \mathcal{E}' \longrightarrow t^\wedge a \subseteq t' \ \& \ r^\wedge b \subseteq r'$ (otherwise $\mathcal{E}'$ can be properly restricted). Then the union $\mathcal{E}_{ab}$ of all of such c-sets $\mathcal{E}'$ is a c-set as well. Finally $\mathcal{E} = \{\langle t, r \rangle\} \cup \bigcup_{\langle a,b \rangle \in E} \mathcal{E}_{ab}$ is a c-set containing $\langle t, r \rangle$. ■

3.3. The cumulative enlargement

From now on, elements of $\mathcal{H}$ (wf pairs), when not presented as pairs, will be denoted sometimes by *boldface* letters: **a**, **A**, ... The next definition introduces the basic relations which transform $\mathcal{H}$ to an external universe.

DEFINITION 3.9. The relation $\lceil TF \rceil \stackrel{\mathrm{h}}{=} \lceil RG \rceil$ has been defined (Definition 3.6). We write $\lceil RG \rceil \stackrel{\mathrm{h}}{\in} \lceil TF \rceil$ in either of the two cases:

1. $T \neq \{\Lambda\}$ and there exists a 1-term sequence $\langle a \rangle \in T$ such that $\lceil RG \rceil \stackrel{\mathrm{h}}{=} \lceil TF \rceil(\langle a \rangle)$.
2. $T = \{\Lambda\}$, $\Lambda \in I(R, G)$ — then both $x = F[T] = F_T(\Lambda)$ and $y = G[R] = G_R(\Lambda)$ are internal sets, — and $y \in x$.

We finally write ${}^{\mathrm{h}}\mathrm{st} \lceil TF \rceil$ iff $\Lambda \in I(T, F)$ and $F[T]$ is standard.

COROLLARY 3.10. $\stackrel{\mathrm{h}}{=}$ *is an equivalence on* $\mathcal{H}$. *The relations* $\stackrel{\mathrm{h}}{\in}$ *and* ${}^{\mathrm{h}}\mathrm{st}$ *are* $\stackrel{\mathrm{h}}{=}$*-invariant.*

PROOF. Apply Lemma 3.7. ■

DEFINITION 3.11. We put $\mathbb{H} = \langle \mathcal{H}; {}^{\mathrm{h}}{=}, {}^{\mathrm{h}}{\in}, {}^{\mathrm{h}}\mathrm{st} \rangle$.
Let Φ be a st-$\in$-formula. ${}^{\mathrm{h}}\Phi$ denotes the st-$\in$-formula obtained the following way: 1) all quantifiers are relativized to $\mathcal{H}$, and

2) the relations $=, \in, \mathrm{st}$ are changed to ${}^{\mathrm{h}}{=}, {}^{\mathrm{h}}{\in}, {}^{\mathrm{h}}\mathrm{st}$.

Thus, for all $\mathbf{x}, \mathbf{y}, \ldots \in \mathcal{H}$, the formula $\Phi(\mathbf{x}, \mathbf{y}, \ldots)$ is true in $\mathbb{H}$ if and only if ${}^{\mathrm{h}}\Phi(\mathbf{x}, \mathbf{y}, \ldots)$ is true in $\mathbb{E}$.

To conclude, we have defined in **EEST** the structure $\mathbb{H}$ based on the class $\mathcal{H} \subseteq \mathbb{E}$ and equipped with relations ${}^{\mathrm{h}}{=}, {}^{\mathrm{h}}{\in}, {}^{\mathrm{h}}\mathrm{st}$, the first of which satisfies the logical properties of equality. The next proposition shows that the relations satisfy the requirements naturally implied by Definition 3.1.

DEFINITION 3.12. Assume that $\lceil TF \rceil \in \mathcal{H}$. We set $\operatorname{Min} T = \{a : \langle a \rangle \in T\}$. For any $a \in \operatorname{Min} T$, we put $T^a = \{t : a^\wedge t \in T\}$. For all $a \in \operatorname{Min} T$ and $t \in T^a$, we set $F^a(t) = F(a^\wedge t)$.

Obviously $\lceil T^a F^a \rceil \in \mathcal{H}$. Notice that $\lceil T^a F^a \rceil \,{}^{\mathrm{h}}{=}\, \lceil TF \rceil(\langle a \rangle)$ (via the c-set $\{\langle t, a^\wedge t \rangle : t \in T^a\}$), therefore $\lceil T^a F^a \rceil \,{}^{\mathrm{h}}{\in}\, \lceil TF \rceil$. It is important that wf pairs $\lceil T^a F^a \rceil$, $a \in \operatorname{Min} T$, actually exhaust all the ${}^{\mathrm{h}}{\in}$-members of $\lceil TF \rceil$.

LEMMA 3.13. *Let $\lceil TF \rceil$ and $\lceil RG \rceil$ belong to $\mathcal{H}$. If $T \neq \{\Lambda\}$ then*

$$\lceil RG \rceil \,{}^{\mathrm{h}}{\in}\, \lceil TF \rceil \longleftrightarrow \exists\, a \in \operatorname{Min} T\, (\lceil RG \rceil \,{}^{\mathrm{h}}{=}\, \lceil T^a F^a \rceil).$$

If $T = \{\Lambda\}$ then $\lceil RG \rceil \,{}^{\mathrm{h}}{\in}\, \lceil TF \rceil \longleftrightarrow \exists\, y \in x\, (G[R] = y)$, [6] *where $x = F(\Lambda)$.*

LEMMA 3.14. *Let $\lceil TF \rceil$ and $\lceil RG \rceil$ belong to $\mathcal{H}$. If neither T nor R is equal to $\{\Lambda\}$ then $\lceil TF \rceil \,{}^{\mathrm{h}}{=}\, \lceil RG \rceil \longleftrightarrow (a)\ \&\ (b)$, where*

$$\forall\, a \in \operatorname{Min} T\, \exists\, b \in \operatorname{Min} R\, (\lceil T^a F^a \rceil \,{}^{\mathrm{h}}{=}\, \lceil R^b G^b \rceil), \tag{a}$$

$$\forall\, b \in \operatorname{Min} R\, \exists\, a \in \operatorname{Min} T\, (\lceil T^a F^a \rceil \,{}^{\mathrm{h}}{=}\, \lceil R^b G^b \rceil). \tag{b}$$

If $T = R = \{\Lambda\}$ then $\lceil TF \rceil \,{}^{\mathrm{h}}{=}\, \lceil RG \rceil \longleftrightarrow F_T(\Lambda) = G_R(\Lambda)$.
If $T = \{\Lambda\}$, so that $x = F_T(\Lambda) \in \mathbb{I}$ is defined, but $R \neq \{\Lambda\}$, then we have $\lceil TF \rceil \,{}^{\mathrm{h}}{=}\, \lceil RG \rceil \longleftrightarrow (c)\ \&\ (d)$, where

$$\forall\, y \in x\, \exists\, b \in \operatorname{Min} R\, (G^b[R^b] = y)\,, \text{ [7]} \tag{c}$$

$$\forall\, b \in \operatorname{Min} R\, \exists\, y \in x\, (G^b[R^b] = y). \tag{d}$$

[6] We recall that $G[R] = y$ means that $\Lambda \in I(R, G)$ and $G_R(\Lambda) = y$.
[7] The preceding footnote explains the meaning of $G^b[R^b] = y$ as well.

PROOF. Apply Lemma 3.8. and Definition 3.9. ■

Now the interrelations between $\mathbb{H}$ and the subuniverses $\mathbb{I}$ (internal sets), $\mathbb{S}$ (standard sets) of the basic **EEST** universe $\mathbb{E}$ will be considered. If $\mathbf{x} = \lceil TF \rceil \in \mathcal{H}$ and $t \in T$ then we use $\mathbf{x}(t) \Rightarrow x$ and $\mathbf{x} \Rightarrow x$ as shorthand for $F_T(t) = x$ and $F[T] = x$ respectively (this presupposes $t \in I(T, F)$ and $\Lambda \in I(T, F)$ respectively, and $x \in \mathbb{I}$, see above).

To see that $\mathbb{H}$ is in fact an enlargement of $\mathbb{I}$, the copies of the universes $\mathbb{I}$ and $\mathbb{S}$ in $\mathbb{H}$ are introduced:

$$^{\mathrm{h}}\mathbb{I} = \{\mathbf{x} \in \mathcal{H} : \exists^{\mathrm{int}} x\ (\mathbf{x} \Rightarrow x)\} \qquad \text{and} \qquad ^{\mathrm{h}}\mathbb{S} = \{\mathbf{x} \in \mathcal{H} : \exists^{\mathrm{st}} x\ (\mathbf{x} \Rightarrow x)\}.$$

Fortunately every $x \in \mathbb{I}$ has a distinguished copy in $^{\mathrm{h}}\mathbb{I}$: $^{\mathrm{h}}x = \lceil \{\Lambda\} C^x \rceil$, where $C^x(\Lambda) = x$ (see Example 3.5). Obviously $^{\mathrm{h}}x \in \mathcal{H}$ and $^{\mathrm{h}}x \Rightarrow x$.

LEMMA 3.15. *Let* $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, $\mathbf{x} \Rightarrow x$, $\mathbf{y} \Rightarrow y$. *Then*

$$x = y \longleftrightarrow \mathbf{x} \overset{\mathrm{h}}{=} \mathbf{y}, \qquad x \in y \longleftrightarrow \mathbf{x} \overset{\mathrm{h}}{\in} \mathbf{y}, \qquad \textit{and} \qquad \mathrm{st}\, x \longleftrightarrow {}^{\mathrm{h}}\mathrm{st}\, \mathbf{x}.$$

If in addition $\mathbf{w} \in \mathcal{H}$, $\mathbf{w} \overset{\mathrm{h}}{\in} \mathbf{x}$, *then there is* $w \in x$ *such that* $\mathbf{w} \Rightarrow w$.

PROOF. Routine verification, based on lemmas 3.13. and 3.14. ■

Thus actually $\mathbb{H}$ is an enlargement of $\mathbb{I}$: $^{\mathrm{h}}\mathbb{I}$ can be made isomorphic to $\mathbb{I}$ by factorization via $\overset{\mathrm{h}}{=}$.

PROPOSITION 3.16. [Extensionality]
Let $\mathbf{x} = \lceil TF \rceil$ *and* $\mathbf{y} = \lceil RG \rceil$ *belong to* $\mathcal{H}$, *and* $\neg\ \lceil TF \rceil \overset{\mathrm{h}}{=} \lceil RG \rceil$. *There exists* $\mathbf{w} = \lceil UH \rceil \in \mathcal{H}$ *such that* $\neg\, [\mathbf{w} \overset{\mathrm{h}}{\in} \mathbf{x} \longleftrightarrow \mathbf{w} \overset{\mathrm{h}}{\in} \mathbf{y}]$.

PROOF. Assume that $T = R = \{\Lambda\}$. Then $x = F_T(\Lambda)$ and $y = G_R(\Lambda)$ are internal sets and $x \neq y$. By definition there exists $w \in \mathbb{I}$ such that $\neg\, [w \in x \longleftrightarrow w \in y]$. Then $\mathbf{w} = {}^{\mathrm{h}}w$ is as required by Lemma 3.15.

Assume that neither T nor R are equal to $\{\Lambda\}$. Then, by Lemma 3.14, there exists $a \in \operatorname{Min} T$ such that for all $b \in \operatorname{Min} R$ $\lceil T^a F^a \rceil \overset{\mathrm{h}}{=} \lceil R^b G^b \rceil$ does not hold. Then $\mathbf{w} = \lceil T^a F^a \rceil$ is as required: $\mathbf{w} \overset{\mathrm{h}}{\in} \mathbf{x}$ but $\neg\, \mathbf{w} \overset{\mathrm{h}}{\in} \mathbf{y}$ by Lemma 3.13.

In the mixed case, say when $T = \{\Lambda\}$, so that $x = F_T(\Lambda) \in \mathbb{I}$ is defined, but $R \neq \{\Lambda\}$, at least one of conditions (c), (d) of Lemma 3.14. fails. If (d) fails via some $b \in \operatorname{Min} R$ then $\mathbf{w} = \lceil R^b G^b \rceil$ is as required by Lemma 3.13. If (c) fails via some $y \in x$ then $\mathbf{w} = {}^{\mathrm{h}}y$ is as required. ■

4. Hrbaček set theory

This section is the continuation of the previous one. We prove that the universe $\mathbb{H}$ models Hrbaček's standard size saturated external theory (Replacement, but not Power Set) plus some rather useful additional postulates.

4.1. The theory

DEFINITION 4.1. A set X is a *set of standard size* if there exists a standard set S and a (generally, external) function $f : {}^\sigma S$ onto X.

DEFINITION 4.2. **HST** is the theory in the st-$\in$-language containing:

1. The same as Item 1 in **EEST** (see Definition 2.4), that is:

   $\forall^{\mathrm{st}} x\,(\operatorname{int} x)$: all standard sets are internal;

   (We recall that $\operatorname{int} x$ is the st-$\in$-formula $\exists^{\mathrm{st}} y\,(x \in y)$.)

   $\forall^{\mathrm{int}} x\, \forall y \in x\,(\operatorname{int} y)$: transitivity of the internal subuniverse;

   Generalized Standardization: $\forall X\, \exists^{\mathrm{st}} Y\, \forall^{\mathrm{st}} x\,(x \in Y \longleftrightarrow x \in X)$.
2. $\mathbf{BST}^{\mathrm{int}}$: all axioms of **BST** relativized to the formula int .
3. The **ZFC** Pair, Union, Extensionality, Infinity axioms, together with Separation, Collection, Replacement for all st-$\in$-formulas.
4. Extension: assume that S is a standard set and F a function defined on the set ${}^\sigma S = \{x \in S : \operatorname{st} x\}$, and $F(x)$ contains internal elements for all $x \in {}^\sigma S$, then there exists an *internal* function f defined on S and satisfying $\forall x \in {}^\sigma S\,[f(x) \in F(x)]$.
5. Saturation: if X is a set of standard size such that every $x \in X$ is internal and the intersection $\bigcap X'$ is nonempty for any finite nonempty $X' \subseteq X$, then $\bigcap X$ is nonempty.
6. Choice in the case when the domain X of the choice function is a set of standard size, and Dependent Choice.
7. Weak regularity: if a nonempty set X contains only noninternal elements then there exists $x \in X$ such that $x \cap X = \emptyset$.

Thus **HST** is a theory rather similar to the theory $\mathbf{NS}_1(\mathbf{ZFC})$ of Hrbaček [5] and, to some extent, to external theories of Kawaï [12]. To be precise, **HST** includes $\mathbf{NS}_1(\mathbf{ZFC})$, but it can be verified that the extra tools of **HST** (Weak regularity, standard size Choice and Dependent Choice) hold in Hrbaček's model for $\mathbf{NS}_1$ in [5].

We prove in this section that $\mathbb{H}$ models **HST**. Before the proof begins, let us take some space to present some theorems of **HST**, in particular, related to the notions of *finite set* and *cardinal*.

4.2. Natural numbers and cardinals in external universe

Let $\mathbf{HST}^-$ be the theory containing the axioms of items 1 through 3 of Definition 4.2. Let $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$ denote the universe of all sets in $\mathbf{HST}^-$ and its subuniverses of internal (satisfying the formula int in $\mathbb{H}$) and standard sets. The reasoning is mostly based on ideas of Hrbaček [5], [6] and Kawaï [12].

LEMMA 4.3. [$\mathbf{HST}^-$] *If* $X \subseteq \mathbb{I}$ *then there exists standard* S *such that* $X \subseteq S$.

PROOF. Indeed, by definition of an internal set, $\forall x \in X\ \exists^{\text{st}} y\ (x \in y)$. Therefore by Collection and Standardization there exists standard Y such that $X \subseteq S = \bigcup Y$. However S coincides with $\bigcup Y$ in the sense of $\mathbb{I}$ because $\mathbb{I}$ is transitive. Therefore S is standard by Transfer. ■

LEMMA 4.4. [$\mathbf{HST}^-$] *Every set* $X \subseteq \mathbb{S}$ *is a set of standard size.*

PROOF. By Lemma 4.3, $X \subseteq S$ for some standard S. Then $X \subseteq {}^\sigma S$. ■

It is a fact in external theories that some basic set theoretic notions defined in the external universe coincide with their standard rather than internal counterparts. This is true, in particular, with respect to finite sets and wellorderable cardinals.

Let $\text{nat}\, x$ be the $\in$-formula of being a natural number, which says that x is an element of the least set containing $0 = \emptyset$ and closed with respect to the operation $x \longmapsto x^+ = x \cup \{x\}$. We put

$$\begin{aligned} \mathbb{N} &= \{n \in \mathbb{I} : \text{nat}\, n \text{ is true in the universe } \mathbb{I} \text{ of all internal sets}\}, \\ \omega &= \{n \in \mathbb{H} : \text{nat}\, n \text{ is true in the universe } \mathbb{H} \text{ of all sets}\}. \end{aligned}$$

the collections of all internal and external natural numbers. As usual a set X is *finite* iff it can be put in a 1–1 correspondence with a set $N_n = \{1, 2, ..., n\}$, where $n \in \omega$.

PROPOSITION 4.5. [**HST**$^-$] (Hrbaček [5], [6])
$\omega = {}^\sigma\mathbb{N} = \{n \in \mathbb{N} : \mathrm{st}\, n\}$. A set $X \subseteq \mathbb{I}$ is finite in $\mathbb{H}$ if and only if $X \in \mathbb{I}$ and X contains a standard finite number of elements in $\mathbb{I}$.

This explains the notion of finite set in the formulation of the Saturation axiom of **HST**. It is worth to notice that ${}^\sigma N_n = N_n$ for $n \in \omega$; hence finite sets are sets of standard size, but not vice versa.

LEMMA 4.6. [**HST**$^-$] *A set is wellorderable iff it has standard size.*

PROOF. We prove that if S is standard then ${}^\sigma S$ is wellorderable. Let $<$ be a standard relation which wellorders S in $\mathbb{S}$. Then $<$ may not wellorder S in $\mathbb{H}$ but $<$ wellorders ${}^\sigma S$. Indeed consider a nonempty set $W \subseteq {}^\sigma S$. By Standardization, there exists a standard set $S' \subseteq S$ such that $W = {}^\sigma S'$. Then, in $\mathbb{S}$, S' contains the $<$-least element s. Then s is standard by Transfer, therefore $s \in W$, and s is the $<$-least element in W.

For the converse let X be wellordered by $<$. Let IOrd and SOrd denote the classes of all internal and standard ordinals respectively. That is, say, $\gamma \in \mathrm{IOrd}$ iff γ is an internal set satisfying the property of being an ordinal[8] in $\mathbb{I}$. By Standardization, for every $W \subseteq \mathrm{SOrd}$ there exists a standard set $S \subseteq \mathrm{IOrd}$ such that $W = {}^\sigma S = S \cap \mathrm{SOrd}$. The least internal ordinal greater than all ordinals in S is standard by Transfer. This ordinal, denoted by $\mathrm{Ssup}\, W$, is equal to the least standard ordinal greater than all ordinals in W. Note that the axioms of **HST**$^-$ are strong enough to carry out definitions by transfinite recursion, in particular there exists a 1–1 function $f : X \longrightarrow \mathrm{SOrd}$ such that $f(x) = \mathrm{Ssup}\,\{f(y) : y < x\}$ for all $x \in X$. Then $W = \{f(x) : x \in X\}$ is a subset of SOrd, hence a set of standard size by Lemma 4.4. But X is the image of W via f^{-1}. ■

Thus the Saturation axiom of **HST** implies that $\mathbb{I}$, the universe of all internal sets, is saturated with respect to all wellorderable cardinals. The next lemma shows that Saturation can be eliminated from the list of **HST** axioms; actually we have put it into the list for the sake of convenience only.

LEMMA 4.7. *Extension implies Saturation in* **HST**$^-$.

PROOF. Let X be a set of standard size such that every $x \in X$ is internal and the intersection $\bigcap X'$ is nonempty for every finite nonempty

[8] We let *ordinal* mean *a transitive set wellordered by the membership relation.*

subset $X' \subseteq X$. Prove that $\bigcap X$ is nonempty. Let S be a standard set and $F : {}^\sigma S$ onto X. By Extension, there exists an *internal* function f such that $f(x) = F(x)$ for all standard $x \in S$. Thus Saturation takes the form:

$$\forall^{\text{fin}} S' \subseteq {}^\sigma S \; \exists^{\text{int}} y \, \forall x \in S' \, (y \in f(x)) \;\longrightarrow\; \exists^{\text{int}} y \, \forall^{\text{st}} x \in S \, (y \in f(x)) \,. \quad \text{(H)}$$

Both the left–hand and the right–hand sides of the implication say something about internal and standard sets. Let us consider the auxiliary implication

$$\forall^{\text{stfin}} S' \subseteq S \; \exists \, y \, \forall x \in S' \, (y \in f(x)) \;\longrightarrow\; \exists \, y \, \forall^{\text{st}} x \in S \, (y \in f(x)) \,. \qquad \text{(I)}$$

The latter is true in $\mathbb{I}$ by Theorem 1.1. On the other hand, by what is said above, the left–hand side of (H) in $\mathbb{H}$ is equivalent to the left–hand side of (I) in $\mathbb{I}$, and the same for the right–hand sides. ■

It is not true that *ordinals* and *cardinals* in $\mathbb{S}$ are the same as in $\mathbb{H}$, but we shall see that they coincide, to some extent, with *wellorderable* counterparts in $\mathbb{H}$. Let HOrd denote the class of all ordinals in $\mathbb{H}$.

We define the *collapse function* $\zeta : \text{SOrd} \longrightarrow \mathbb{H}$ by

$$\zeta(\alpha) = \{\zeta(\beta) : \beta \in \text{SOrd} \;\&\; \beta < \alpha\}$$

for all $\alpha \in \text{SOrd}$. Since SOrd is wellordered in $\mathbb{H}$ by the membership (see the first part of the proof of Lemma 4.6), this is well defined in $\mathbf{HST}^-$.

PROPOSITION 4.8. [$\mathbf{HST}^-$] *ζ maps* SOrd *onto* HOrd *in a 1–1 order preserving way.*

PROOF. The nontrivial point is to prove that ζ cannot map SOrd on a proper initial part Ω of HOrd. Let, on the contrary, this be the case. Taking an arbitrary $\gamma \in \text{HOrd} \setminus \Omega$ and using Separation and Replacement, we see that Ω is a set and ζ^{-1} maps Ω onto SOrd, which immediately leads to contradiction with Standardization. ■

Let SCard denote the class of all $\kappa \in \text{SOrd}$ which are cardinals (initial ordinals) in $\mathbb{S}$. HCard is defined the same way, but in the sense of $\mathbb{H}$.

PROPOSITION 4.9. [$\mathbf{HST}^-$] *ζ maps* SCard *onto* HCard *in a 1–1 order preserving way.*

PROOF. The principal point is to prove that if $\lambda < \kappa$ are cardinals in SCard then in $\mathbb{H}$ there does not exist a function which maps $\zeta(\lambda)$ onto

$\zeta(\kappa)$. Let, on the contrary, f be such a function. Using ζ, we obtain a map $f' : {}^{\sigma}\lambda$ onto ${}^{\sigma}\kappa$. By Standardization, f' may be assumed standard. By Transfer, f' maps λ onto κ, which leads to contradiction. ■

PROPOSITION 4.10. [**HST**$^-$] *Let* X *be a set of standard size (or, which is the same by Lemma 4.6, a wellorderable set). Then* $\mathrm{card}\, X \in \mathrm{HCard}$.

PROOF. By definition, we obtain a 1–1 map X onto a set ${}^{\sigma}S$, where S is a standard set, then onto ${}^{\sigma}\kappa$, where $\kappa \in \mathrm{SCard}$ is the cardinality of S in $\mathbb{S}$, finally onto $\zeta(\kappa) \in \mathrm{HCard}$. ■

Thus we have got a complete description of wellorderable cardinalities in **HST**$^-$; all of them are members of HCard and nothing more. One more conclusion is that only relatively small sets (those of standard size) have wellorderable cardinalities in **HST**$^-$. We can easily prove that a set $x \in \mathbb{I}$ not of a standard finite number of elements (e. g. a nonstandard natural number) is not a set of standard size in $\mathbb{H}$; the (non–wellorderable) cardinality of such a set in $\mathbb{H}$ is greater than any $\kappa \in \mathrm{HCard}$.

4.3. The main theorem

THEOREM 4.11. [**EEST**] $\mathbb{H}$ *models* **HST** : *if* Φ *is an* **HST** *axiom then* ${}^{\mathbf{h}}\Phi$ *is a theorem of* **EEST**. *In addition, the class of all* $\mathbb{H}$*-internal elements of* $\mathbb{H}$ *is isomorphic to* $\mathbb{I}$, *the internal subuniverse.*

PROOF. The proof of the theorem consists of two parts; in part 1 we check some of the axioms in $\mathbb{H}$ directly and prove two important lemmas, in part 2 we obtain the rest of the axioms as formal consequences.

Part 1

We argue in **EEST**; thus let $\mathbb{E}$, $\mathbb{I}$, $\mathbb{S}$ denote resp. the universe of all sets in **EEST** and its subuniverses of internal (satisfying the formula int in $\mathbb{E}$) and standard sets. We first consider the requirements related to internal sets. The next lemma is an easy consequence of Lemma 3.15.

LEMMA 4.12. *1. Let* $\mathbf{x} \in \mathcal{H}$. *Then* $\mathrm{int}\,\mathbf{x}$ *is true in* $\mathbb{H}$ *iff* $\mathbf{x} \in {}^{\mathbf{h}}\mathbb{I}$.

2. ${}^{\mathbf{h}}\mathbb{I}$ *is* ${}^{\mathbf{h}}\!\in$*-transitive in* $\mathbb{H}$. ⊣

The required isomorphism between elements of $\mathcal{H}$ internal in the sense of $\mathbb{H}$ is defined by the correspondence $\mathbf{x} \Rightarrow x$ (for all $\mathbf{x} \in {}^{\mathbf{h}}\mathbb{I}$). The fact that this is actually an isomorphism (that is, it preserves equality, membership, and standardness) is implied by Lemma 3.15.

Axioms of item 4.2.1. We prove *internality of standard sets* in $\mathbb{H}$. Let $\mathbf{x} = \lceil TF \rceil \in \mathcal{H}$ satisfy ${}^{\mathrm{h}}\mathrm{st}\,\mathbf{x}$. By definition $\Lambda \in I(T, F)$ and $x = F_T(\Lambda)$ is standard. Then $X = \{x\}$ is standard by Transfer, therefore $\mathbf{X} = {}^{\mathrm{h}}X$ is standard in $\mathbb{H}$ again by definition, and $\mathbf{x} \;{}^{\mathrm{h}}\!\!\in \mathbf{X}$ by Lemma 3.15.

The *transitivity of the internal subuniverse* follows from Lemma 4.12.2.

We prove *Standardization*. Let $\mathbf{X} = \lceil TF \rceil \in \mathcal{H}$. Assume first that $T \neq \{\Lambda\}$. Let $A = \operatorname{Min} T$ (a legitimate set by the **EEST** Separation). Moreover, $D = \{x : \exists\, a \in A\; (\lceil TF \rceil(\langle a \rangle) \Rightarrow x)\}$ is a set by Theorem 2.11. (Collection), therefore by the **EEST** Standardization there exists a standard set S which contains the same standard elements as D does. Let $\mathbf{S} = {}^{\mathrm{h}}S$, so that $\mathbf{S} \Rightarrow S$ by Lemma 3.15. One can easily verify that $\forall^{\mathrm{st}} x\, (x \in \mathbf{S} \longleftrightarrow x \in \mathbf{X})$ is true in $\mathbb{H}$ using Lemma 3.15.

Assume that $T = \{\Lambda\}$. Then $\mathbf{X} \Rightarrow X$, where $X = F(\Lambda)$ is an internal set. By the **EEST** Standardization there exists a standard set S which contains the same standard elements as X. Let $\mathbf{S} = {}^{\mathrm{h}}S$, and so on. ⊣

BST in the internal subuniverse follows from 3.15. and 4.12: the quotient of ${}^{\mathrm{h}}\mathbb{I}$ via ${}^{\mathrm{h}}\!\!=$ is isomorphic to $\mathbb{I}$. **Infinity** is inherited from $\mathbb{I}$. **Extensionality** is implied by Proposition 3.16. ⊣

Pairs. Let $\mathbf{x} = \lceil UH \rceil \in \mathcal{H}$, $\mathbf{y} = \lceil RG \rceil \in \mathcal{H}$, and

$$T = \{\Lambda\} \cup \{1^\wedge u : u \in U\} \cup \{2^\wedge r : r \in R\},$$

$$F(1^\wedge u) = H(u) \text{ for } u \in \operatorname{Max} U, \quad \text{and} \quad F(2^\wedge r) = G(r) \text{ for } r \in \operatorname{Max} R,$$

so that $\mathbf{p} = \lceil TF \rceil \in \mathcal{H}$, $T^1 = U$, $F^1 = H$, $T^2 = R$, $F^2 = G$. (We recall that $T^a = \{t : a^\wedge t \in T\}$ and $F^a(t) = F(a^\wedge t)$ for $t \in T^a$.) Therefore $\mathbf{p} = \{\mathbf{x}, \mathbf{y}\}$ is true in $\mathbb{H}$ by Lemma 3.13. ⊣

For the remainder, let us denote this $\mathbf{p}$ by $\{\mathbf{x}, \mathbf{y}\}^{\mathrm{h}}$, and define

$$\langle \mathbf{x}, \mathbf{y} \rangle^{\mathrm{h}} = \{\{\mathbf{x}, \mathbf{x}\}^{\mathrm{h}}, \{\mathbf{x}, \mathbf{y}\}^{\mathrm{h}}\}^{\mathrm{h}},$$

so that it is true in $\mathbb{H}$ that $\langle \mathbf{x}, \mathbf{y} \rangle^{\mathrm{h}}$ is the ordered pair of $\mathbf{x}$, $\mathbf{y}$. The following two lemmas are quite essential to prove the rest of the theorem.

LEMMA 4.13. [S-images] *In* $\mathbb{H}$, *every set is an image of a standard set.*

PROOF. More accurately, we assert that the following holds in $\mathbb{H}$:

$$\forall X\; \exists^{\mathrm{st}} S\; \exists f\; [\, f \text{ is a function, } \operatorname{dom} f \subseteq S, \text{ and } X = \operatorname{ran} f\,].$$

Let $\mathbf{X} = \lceil TF \rceil \in \mathcal{H}$. Assume that $T \neq \{\Lambda\}$. (The case $T = \{\Lambda\}$ is quite easy: then $\mathbf{X}$ is formally internal in $\mathbb{H}$ by Lemma 4.12, therefore formally a subset of a standard set.) Let $A = \operatorname{Min} T$. We put $\mathbf{x}^a = \lceil T^a F^a \rceil$ for $a \in A$; thus $\mathbf{x}^a \in \mathcal{H}$. By Proposition 2.9 $A \subseteq S$ for a standard set S. Let $\mathbf{S} = {}^{\mathrm{h}}S$, $\mathbf{A} = {}^{\mathrm{h}}A$. We finally define $\mathbf{f} = \lceil RG \rceil \in \mathcal{H}$ so that $\operatorname{Min} R = A$ and $\mathbf{f}^a = \langle {}^{\mathrm{h}}a, \mathbf{x}^a \rangle^{\mathrm{h}}$ for all $a \in A$, so that it is true in $\mathbb{H}$ that $\mathbf{f}$ maps $\mathbf{A}$ onto $\mathbf{X}$ and $\mathbf{A}$ is a subset of the standard set $\mathbf{S}$. ⊣

LEMMA 4.14. *It is true in* $\mathbb{H}$ *that if* S *is standard and* $X \subseteq S$ *then* $X = \mathcal{C}_p$ *for an internal* p *.*

PROOF.[9] Let $\mathbf{S} \in \mathcal{H}$ be standard in the sense of $\mathbb{H}$, so that $\mathbf{S} \Rightarrow S$ for a standard set S by Lemma 3.15. Let $\mathbf{X} = \lceil TF \rceil \in \mathcal{H}$ and $\mathbf{X} \subseteq \mathbf{S}$ in $\mathbb{H}$. Then $X = \{x \in S : {}^{\mathrm{h}}x \;{}^{\mathrm{h}}\!\!\in \mathbf{X}\}$ is a st-$\in$-definable subclass of S, therefore a set by the **EEST** Separation. Hence $X = \mathcal{C}_p$ for some p by the **EEST** Parametrization axiom. Applying lemmas 4.12 and 3.15, one can easily verify that $\mathbf{X} = \mathcal{C}_{{}^{\mathrm{h}}p}$ is true in $\mathbb{H}$, as required. ⊣

Weak regularity.[10] Let $\mathbf{x} = \lceil TF \rceil \in \mathcal{H}$. Assume that $\mathbf{x}$ is nonempty and does not contain internal elements in $\mathbb{H}$. Then $A = \operatorname{Min} T$ is nonempty (otherwise $T = \{\Lambda\}$, therefore $x = F(\Lambda)$ is internal in $\mathbb{E}$, and Lemma 3.15. easily gives a contradiction with the choice of $\mathbf{x}$). Hence

$$T' = \{t \in T : \exists\, a \in \operatorname{Min} T\; [\, \lceil TF \rceil(t) \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a \rangle)\,]\}$$

is nonempty, too; actually $\operatorname{Min} T \subseteq T'$. As T is well-founded there exists $t \in T'$ such that none among the expansions $t^\wedge b$ belongs to T'. Let $a \in \operatorname{Min} T$ demonstrate that $t \in T'$. Then $\mathbf{y} = \lceil T^a F^a \rceil \in \mathcal{H}$ (see Definition 3.12.) and $\mathbf{y} \;{}^{\mathrm{h}}\!\!\in \mathbf{x}$ by Lemma 3.13. Notice that $\langle a \rangle \notin I(T, F)$ since otherwise $\Lambda \in I(T^a, F^a)$ and $\mathbf{y}$ would be internal in $\mathbb{H}$ by Lemma 4.12.

We assert that $\mathbf{x} \cap \mathbf{y} = \emptyset$ in $\mathbb{H}$. Let, on the contrary, $\mathbf{w} = \lceil UH \rceil \in \mathcal{H}$ satisfy $\mathbf{w} \;{}^{\mathrm{h}}\!\!\in \mathbf{x}$ and $\mathbf{w} \;{}^{\mathrm{h}}\!\!\in \mathbf{y}$. Then, in particular, there exists $a' \in \operatorname{Min} T$ such that $\mathbf{w} \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a' \rangle)$. On the other hand, there exists $b \in \operatorname{Min} T^a$ such that $\mathbf{w} \;{}^{\mathrm{h}}\!\!= \lceil T^a F^a \rceil(\langle b \rangle)$, which means that $\mathbf{w} \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a, b \rangle)$. We conclude that $\lceil TF \rceil(\langle a' \rangle) \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a, b \rangle)$.

Since $\lceil TF \rceil(t) \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a \rangle)$ and $\langle a \rangle \notin I(T, F)$, there exists some b' such that $t' = t^\wedge b' \in T$ and $\lceil TF \rceil(t') \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a, b \rangle)$. Then $\lceil TF \rceil(t') \;{}^{\mathrm{h}}\!\!= \lceil TF \rceil(\langle a' \rangle)$, therefore $t' \in T'$, leading to contradiction. ⊣

[9] Take notice that since the class of internal sets is transitive in $\mathbb{H}$, the notion of $\mathcal{C}_p$ retains its "internal" sense in $\mathbb{H}$: $\mathcal{C}_p = (\mathcal{C}_p$ in the sense of $\mathbb{I})$.

[10] The set of all nonstandard $\mathbb{I}$-natural numbers shows that the usual **ZFC** Regularity fails in **HST**.

Separation. Let $\mathbf{X} = \lceil TF \rceil \in \mathcal{H}$, and $\Phi(x)$ be a st-$\in$-formula which may contain elements of $\mathcal{H}$ as parameters. Assume first that $T \neq \{\Lambda\}$. We put

$$A = \operatorname{Min} T\,, \quad A^+ = \{a \in A : {}^{\mathrm{h}}\Phi(\mathbf{x}^a)\}\,, \quad T^+ = \{\Lambda\} \cup \bigcup_{a \in A^+} \{a^\wedge t : t \in T^a\}\,,$$

and finally $F^+ = F \restriction T^+$. By Lemma 3.13, $\mathbf{Y} = \lceil T^+F^+ \rceil \in \mathcal{H}$ is as required for Separation, that is, $\forall\, x\, [x \in \mathbf{Y} \longleftrightarrow x \in \mathbf{X}\ \&\ \Phi(x)]$ in $\mathbb{H}$. If $T = \{\Lambda\}$, so that $F(\Lambda) =$ some internal x, then we first define

$$T' = \{\Lambda\} \cup \{\langle y \rangle : y \in x\} \qquad F'(\langle y \rangle) = y \ \text{ for all } \ y \in x\,,$$

so that $\lceil TF \rceil \overset{\mathrm{h}}{=} \lceil T'F' \rceil$, and apply the former construction to $\lceil T'F' \rceil$. ⊣

Collection. Let $\Phi(x,y)$ be a st-$\in$-formula having elements of $\mathcal{H}$ as parameters. Separation and Lemma 4.13. reduce the problem to the following form. *Let S be a standard set and $\mathbf{S} = {}^{\mathrm{h}}S \in \mathcal{H}$. Find $\mathbf{Y} \in \mathcal{H}$ satisfying*

$$\forall\, x \in \mathbf{S}\, [\,\exists\, y\, \Phi(x,y) \longrightarrow \exists\, y \in \mathbf{Y}\, \Phi(x,y)\,]$$

in $\mathbb{H}$. By Theorem 2.12. there is a standard set Y such that, for every $x \in S' = \{x \in S : \exists\, \mathbf{y} \in \mathcal{H}\ {}^{\mathrm{h}}\Phi({}^{\mathrm{h}}x, \mathbf{y})\}$, there exists some $\mathbf{y} \in \mathcal{H} \cap |Y|$ satisfying ${}^{\mathrm{h}}\Phi({}^{\mathrm{h}}x, \mathbf{y})$. (We recall that $|Y| = \{\mathcal{C}_b : b \in Y\}$.) In other words,

$$\forall\, x \in S'\, \exists\, b \in Y\, [\,\mathcal{C}_b \in \mathcal{H}\ \&\ {}^{\mathrm{h}}\Phi({}^{\mathrm{h}}x, \mathcal{C}_b)\,]\,.$$

We define $Y' = \{b \in Y : \mathcal{C}_b \in \mathcal{H}\}$. Let $\mathcal{C}_b = \lceil R^b G^b \rceil$ for $b \in Y'$. It remains to put $R = \{\Lambda\} \cup \{b^\wedge r : r \in R^b\ \&\ b \in Y'\}$ and respectively $G(b^\wedge r) = G^b(r)$ for $r \in \operatorname{Max} R^b$; then $\mathbf{Y} = \lceil RG \rceil \in \mathcal{H}$ is as required. ⊣

Part 2.

The remaining **HST** axioms are proved as *formal consequences* of what has already been checked in $\mathbb{H}$, in other words, as theorems in the theory of $\mathbf{HST}^-$ yet minus Replacement and Union but plus lemmas 4.13. and 4.14. Thus we argue in the universe $\mathbb{H}$ of this theory and let $\mathbb{S}$ and $\mathbb{I}$ denote the classes of standard and internal (that is, those distinguished in $\mathbb{H}$ by formulas st and int) sets respectively.

Replacement is a consequence of Separation and Collection. ⊣

Union. We first prove the existence of *cartesian products* in $\mathbb{H}$. Let X_1, X_2 be arbitrary sets. By Lemma 4.13, there exist standard sets S_i and functions $f_i : S_i$ onto X_i, $i = 1, 2$. Then

$$X_1 \times X_2 = \{\langle f_1(s_1), f_2(s_2) \rangle : s_1 \in S_1\ \&\ s_2 \in S_2\}$$

is a set by the already proved axioms of Pairs, Collection, Separation, and **ZFC** in $\mathbb{S}$ (to prove the existence of $S_1 \times S_2$). We now prove the axiom of Union. Let X be an arbitrary set. By Collection, Standardization, and Lemma 4.13. there exist standard S and a set F of functions such that

$$\forall x \in X \, \exists s \in S \, \exists f \in F \, (f \text{ maps } s \text{ onto } x).$$

Let U denote the union $\bigcup S$ of S *taken in* $\mathbb{I}$, the internal subuniverse. Then U is standard, and, on the other hand, $U = \bigcup S$ also in $\mathbb{H}$ since $\mathbb{I}$ is transitive. Thus $\bigcup X \subseteq \{f(u) : \langle u, f\rangle \in U \times F \ \& \ u \in \operatorname{dom} f\}$ is a set by Replacement, Separation, and the already proved existence of $U \times F$. ⊣

Extension. Let S be a standard set, F a function defined on S so that $F(x)$ contains internal elements for all $x \in {}^{\sigma}S = \{s \in S : \operatorname{st} s\}$. Using Replacement and Standardization, we obtain a *standard* function G defined on S such that $F(x) \cap G(x)$ contains an internal element for all $x \in {}^{\sigma}S$. Then $R = \bigcup_{x \in S} G(x)$ is standard and $F(x) \cap R \cap \mathbb{I} \neq \emptyset$ for all $x \in {}^{\sigma}S$. By Lemma 4.14, there exists an internal p such that

$$C_p = \{\langle x, y\rangle \in S \times R : y \in F(x)\}.$$

Thus we have $\forall^{\text{st}} x \in S \, \exists y \, (\langle x, y\rangle \in C_p)$, which implies, by the **BST** Extension theorem (Theorem 1.4), the existence of an internal function f defined on S and satisfying $f(x) \in F(x)$ for all standard $x \in S$, as required. ⊣

Therefore we also have *Saturation* by Lemma 4.7. ⊣

Standard Size Choice. Let F be a function, $\operatorname{dom} F = X$ be a set of standard size, and $F(x) \neq \emptyset$ for all $x \in X$. We have to find a function f defined on X and satisfying $f(x) \in F(x)$ for all $x \in X$. By the definition of standard size one may assume that $X = {}^{\sigma}S$, where S is standard. By Union, $U = \bigcup_{x \in X} F(x)$ is a set, and one may assume, by Lemma 4.13, that U contains only internal elements. It remains to apply Extension. ⊣

Dependent Choice. Let X be a nonempty set, R a binary relation on X, and $\operatorname{dom} R = X$. It is asserted that there is an infinite sequence $\langle x_n : n \in \omega\rangle$ of elements of X such that $\langle x_n, x_{n+1}\rangle \in R$ for all $n \in \omega$. Indeed, Lemma 4.13. reduces the question to the case when the set X is standard. We may assume that $R = C_p$ for some internal p by Lemma 4.14. Applying in $\mathbb{I}$ Theorem 1.5. to the formula $\langle x, y\rangle \in C_p$, we obtain an internal function f defined on the set $\mathbb{N}$ of all $\mathbb{I}$-natural numbers and satisfying $\forall^{\text{st}} n \in \mathbb{N} \, [\langle f(n), f(n+1)\rangle \in R]$. It remains to set $x_n = f(n)$ for all standard $n \in \mathbb{N}$ and recall that $\omega = {}^{\sigma}\mathbb{N} = \{n \in \mathbb{N} : \operatorname{st} n\}$. ⊣

This ends the proof of Theorem 4.11. ■

COROLLARY 4.15. **HST** *admits an interpretation in* **BST** *such that the class of all formally internal (satisfying the formula* int *) sets in the sense of the interpretation is isomorphic to the ground* **BST** *universe.*

PROOF. Follows from Corollary 2.6. and the theorem. ■

4.4. Discussion

This is a continuation of the discussion in Section 2.4. In the same sense as was considered there, a mathematician working in **BST** can legitimately assume that the universe $\mathbb{I}$ of **BST** is an internal part of an external universe which satisfies the axioms of **HST**. The elements of this enlargement may be seen as the objects which can be obtained via Definition 3.1 applied to T and F which are bounded definable classes in the sense of $\mathbb{I}$.

Let us consider a couple of examples.

A monad. Let $\mathbb{R}$ denote the set of real numbers in $\mathbb{I}$, the universe of **BST**. A *monad* of a standard $x \in \mathbb{R}$ is the "set" $\mu_x = \{y \in \mathbb{R} : x \approx y\}$ (not a set in $\mathbb{I}$), where $x \approx y$ means $\forall^{\text{st}}\varepsilon > 0\,(|x - y| < \varepsilon)$. We put $T = \{\Lambda\} \cup \{\langle y\rangle : y \approx x\}$ and $F(\langle y\rangle) = y$ for $y \approx x$; so that $F[T] = \mu_x$. ⊣

The set of all monads. Every monad is a bounded definable class, so that this is still in the framework of Section 2. However the collection of all monads is not a bounded definable class (of internal sets), therefore this is the point where the "cumulative" construction seriously enters the reasoning. We put $F(\langle x, y\rangle) = y$ for $x \in {}^\sigma\mathbb{R}$ and $y \approx x$, and

$$T = \{\Lambda\} \cup \{\langle x\rangle : x \in {}^\sigma\mathbb{R}\} \cup \{\langle x, y\rangle : x \in {}^\sigma\mathbb{R} \;\&\; y \approx x\},$$

so that $F[T]$ represents the collection of all monads of standard reals (that is, $\mathbf{x} = [TF]$ is the set of all monads of standard $\mathbb{I}$-reals in $\mathbb{H}$). ⊣

One can develop in this manner most of nonstandard mathematics in **BST** using the external **HST** enlargement $\mathbb{H}$ of the **BST** universe $\mathbb{I}$. This is restricted by the capacities of the theory **HST** itself, mainly by the lack of the Power Set axiom in **HST**. In particular we shall face problems if we try to develop in **HST**, say, Loeb measures or hyperfinite descriptive set theory, that is, topics where we actually need *external power sets*.

One cannot hope to fix this by a minor modification of the model for **HST** we use, or **HST** itself, since it was demonstrated by Hrbaček [5] that Power Set is incompatible with Saturation plus Collection.

Fortunately there is a suitable solution. It happens that **HST** admits a system of subuniverses which satisfy Power Set and a weakened version of **HST** where Saturation is reduced to a previously fixed standard cardinal. Thus we can gain Power Set and save as much Saturation as necessary.

This will be the topic of the 3rd article in this series.

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