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Internal Approach to External Sets and Universes

Part 3
Partially saturated universes

Abstract. In this article ‡ we show how the universe of **HST**, *Hrbaček set theory* (a nonstandard set theory of "external" type, which includes, in particular, the **ZFC** Replacement and Separation schemata for all formulas in the language containing the membership and standardness predicates, and Saturation for "standard size" families of internal sets, but does not include the Power set axiom) admits a system of subuniverses which keep the Replacement, model Power set and Choice (in fact all of **ZFC**, with the exception of the Regularity axiom, which indeed is replaced by the Regularity over the internal subuniverse), and also keep as much of Saturation as it is necessary.

This gives sufficient tools to develop the most complicated topics in nonstandard analysis, such as Loeb measures.

Key words: nonstandard set theory, partially saturated subuniverses.

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Preface

This paper accomplishes the series of three articles on set theoretic foundations of nonstandard mathematics in this Journal.

The first article [12] introduced *bounded set theory* **BST**, a modification of Nelson's internal set theory **IST**, which, similarly to **IST**, is a theory in the st-$\in$-language [1], containing all of **ZFC** (Separation and Replacement formulated in the $\in$-language), together with the Transfer and Standardization axioms of **IST**, an Idealization somewhat weaker than the one of **IST**, and an axiom which postulates that every set is a member of a standard set (incompatible with **IST**).

It is an essential advantage of **BST** in comparison to **IST** that it implies several useful theorems impossible or unknown for **IST** in such a general form. In particular **BST** provides a uniform description of all *bounded definable classes* (i. e. st-$\in$-definable subclasses of sets — they are not necessarily sets in **BST** since Separation is available in **BST** only in the $\in$-language) — see Subsection 1.1. This description (or *parametrization*), introduced in [13], the second article in the series, allows to extend the universe I of **BST** to the universe $\mathsf{E} = \mathsf{E}(\mathsf{I})$ of all definable bounded classes in I, so that E models a reasonable nonstandard theory of "external" type (in particular, Separation for all st-$\in$-formulas holds in E). It was named **EEST**, *elementary external set theory* (see below).

Unfortunately non–internal sets cannot be elements of other sets in E; therefore E is e. g. not closed under pairing, which is a serious inconvenience. However E admits a further extension. We demonstrated in [13] that the known construction of "assembling" sets along well-founded trees leads to a wider universe $\mathsf{H} = \mathsf{H}(\mathsf{E})$ which models an "external" theory more advanced than **EEST** — we called it **HST**, *Hrbaček set theory*, since it is equal, modulo some details, to a theory introduced in [6]. Note that **HST** is a quite convenient nonstandard set theory; it contains Saturation, as well as all of **ZFC** (in particular, the Separation and Replacement schemata in the st-$\in$-language) with the exception of the Power set, Choice, and Regularity axioms.

In fact some amount of Choice (standard size Choice) and Regularity (Regularity over the internal subuniverse I) is provided in **HST**, but the Power set axiom straightforwardly contradicts Saturation plus Replacement. This can be considered as a serious defect of **HST**.

[1] The language which has the membership $\in$ and the standardness st as the atomic predicates.

The main goal of this paper is to show how to save the Power set axiom in this line of reasoning. We shall see that **HST** is strong enough to define, given an infinite standard cardinal κ, an inner universe $\mathbf{H}_\kappa$ (sections 2 and 3) which models a κ-version of **HST** (Saturation somehow restricted by κ) plus the Power set axiom, and another inner subuniverse $\mathbf{H}'_\kappa$ (Section 4) which models a slightly weaker κ-version of **HST** plus the Power set axiom and the full axiom of Choice.

It will be demonstrated at the end of the paper how these technical arrangements can be used for a practical development of nonstandard analysis.

1. Review of nonstandard set theories

To make the exposition more or less self–contained, we give a brief review of the nonstandard theories considered in the paper.

1.1. Bounded set theory

Bounded set theory **BST** is a theory in the st-$\in$-language which includes all of **ZFC** (in the $\in$-language) together with the following axioms:

Bounded Idealization **BI** :

$$\forall^{\text{stfin}} A\, \exists\, x \in X\, \forall\, a \in A\; \Phi(x,a) \;\longleftrightarrow\; \exists\, x \in X\, \forall^{\text{st}} a\; \Phi(x,a)\,;$$

Standardization **S** : $\forall^{\text{st}} X\, \exists^{\text{st}} Y\, \forall^{\text{st}} x\, [\, x \in Y \;\longleftrightarrow\; x \in X \;\&\; \Phi(x)\,]\,;$

Transfer **T** : $\exists\, x\, \Phi(x) \;\longrightarrow\; \exists^{\text{st}} x\, \Phi(x)\,;$

Boundedness **B** : $\forall\, x\, \exists^{\text{st}} X\, (x \in X)\,.$

The formula Φ must be an $\in$-formula in **BI** and **T**, and Φ may contain only standard sets as parameters in **T**, but Φ can be any st-$\in$-formula in **S** and contain arbitrary parameters in **BI** and **S**. The quantifiers $\exists^{\text{st}}$ and $\forall^{\text{st}}$ have obvious meaning: *there exists standard*, *for all standard*. $\forall^{\text{stfin}} A$ means: *for all standard finite* A . X is a standard set in **BI** .

Thus **BI** is weaker than the Idealization **I** of internal set theory **IST** of Nelson [21] (**I** results by replacing in **BI** the set X by the universe of all sets), but the Boundedness axiom **B** is added.

It occurs that **BI** is equivalent in $\mathbf{ZFC} + \mathbf{B} + \mathbf{T}$ to the following axiom of *Internal Saturation*:

$$\mathbf{IS} :\; \forall^{\text{stfin}} A \subseteq A_0\, \exists\, x\, \forall\, a \in A\; \Phi(x,a) \;\longleftrightarrow\; \exists\, x\, \forall^{\text{st}} a \in A_0\; \Phi(x,a)\,,$$

where A_0 is a standard set and Φ an $\in$-formula ([12], Lemma 1.3).

It is the key point in our development of external sets on the base of **BST** that, by Theorem 2.2 in [13], definable bounded classes (i. e. st-$\in$-definable subclasses of sets) have in **BST** the following regular form:

$\mathcal{C}_p = \bigcup_{a \in {}^\sigma\!A} \bigcap_{b \in {}^\sigma\!B} \eta(a,b)$, where $p = \langle A, B, \eta \rangle$, A and B are standard sets, η being a function defined on $A \times B$,

and ${}^\sigma S = \{s \in S : \mathrm{st}\, s\}$ for any set S. [2]

This result is an easy consequence of Theorem 1.5 in [12] (which asserts that every st-$\in$-formula is provably equivalent in **BST** to a $\Sigma_2^{\rm st}$ formula [3]), and the following lemma, which allows to restrict the two principal quantifiers in a $\Sigma_2^{\rm st}$ formula by standard sets.

LEMMA 1.1 (Lemma 1.7 in [12]) [**BST**]
Let $\varphi(a,b,x)$ be an $\in$-formula, X a standard set, $\kappa = \mathrm{card}\, X$. There exist standard sets A and B of cardinality $\le 2^{2^\kappa}$ such that for all $x \in X$, $\exists^{\rm st} a\, \forall^{\rm st} b\, \varphi(a,b,x) \longleftrightarrow$

$$\longleftrightarrow \exists^{\rm st} a \in A\, \forall^{\rm st} b\, \varphi(a,b,x) \longleftrightarrow \exists^{\rm st} a \in A\, \forall^{\rm st} b \in B\, \varphi(a,b,x)\,.$$

The proof of this lemma in [12] contained an incorrect argument [4]. We give here a corrected proof.

PROOF. We define, for all a and b,

$$\begin{array}{rcccc} X[a,b] & = & \{x \in X : \varphi(a,b,x)\} & \subseteq & X; \\ X[a] & = & \{X[a,b] : b \text{ is an arbitrary set}\} & \subseteq & \mathcal{P}(X); \quad \textit{and} \\ X[\,] & = & \{X[a] : a \text{ is an arbitrary set}\} & \subseteq & \mathcal{P}^2(X). \end{array}$$

Thus the set $X[\,]$ has cardinality at most $\lambda = 2^{2^\kappa}$ while every set $X[a]$ has cardinality at most 2^κ. Using the **ZFC** Collection and Choice, and then Transfer, we obtain standard sets A and B, of cardinality $\le 2^{2^\kappa}$ each, such that $\forall a'\, \exists a \in A\, (X[a] = X[a'])$, and $\forall b'\, \exists b \in B\, (X[a,b] = X[a,b'])$ for any $a \in A$. We assert that A and B are as required.

Let (1), (2), (3) denote the parts of the equivalence of the lemma from left to right. It is clear that (2) implies both (1) and (3).

[2] The definitions of $\mathcal{C}_p$ and ${}^\sigma S$ will be frequently used and very important in the remainder of the paper.

[3] We recall that $\Sigma_2^{\rm st}$ denotes the class of all formulas $\exists^{\rm st} a\, \forall^{\rm st} b$ ($\in$-*formula*).

[4] The wrong part was inserted by the authors into the final text of [12] after the paper had been refereed and accepted.

To prove (1) $\longrightarrow$ (2), let a standard set a satisfy $\forall^{\mathrm{st}} b\, \varphi(a,b,x)$. By the choice of A and Transfer, $X[a] = X[a']$ for a standard $a' \in A$. It is asserted that $\varphi(a',b',x)$ is true for every standard b'. We observe that $X[a',b']$ is a standard member of the set $X[a] = X[a']$, therefore by Transfer $X[a',b'] = X[a,b]$ for a suitable standard b. Then $\varphi(a,b,x)$ holds by the choice of a, so $x \in X[a,b] = X[a',b']$, and $\varphi(a',b',x)$, as required.

To prove (3) $\longrightarrow$ (2), let a standard $a \in A$ satisfy $\forall^{\mathrm{st}} b \in B\, \varphi(a,b,x)$. We assert that $\varphi(a,b',x)$ is true for every standard b'. Notice that $X[a,b']$ is a standard member of $X[a]$, therefore $X[a,b'] = X[a,b]$ for a standard $b \in B$ by Transfer and the choice of B. Then we have $\varphi(a,b,x)$ by the choice of a, so $x \in X[a,b] = X[a,b']$, and finally $\varphi(a,b',x)$. ■

Let κ be a standard cardinal, i. e. $\kappa \in \mathbb{S}$ and it is true in $\mathbb{S}$ (or in $\mathbb{I}$, which is equivalent by Transfer) that κ is a cardinal. The following weaker versions of some of the **BST** axioms will be of special interest:

$\mathbf{BI}_\kappa$ – Bounded Idealization **BI** in the case when $\operatorname{card} X \le \kappa$ in $\mathbb{S}$;

$\mathbf{IS}_\kappa$ – Internal Saturation **IS** in the case when $\operatorname{card} A_0 \le \kappa$ in $\mathbb{S}$;

$\mathbf{B}_\kappa$ – Boundedness: $\forall x\, \exists^{\mathrm{st}} X\, (x \in X \ \&\ \operatorname{card} X \le \kappa$ in $\mathbb{S})$.

Obviously $\mathbf{BI}_\kappa$ and $\mathbf{IS}_\kappa$ are weaker than resp. **BI** and **IS**, but $\mathbf{B}_\kappa$ is stronger than **B** .

PROPOSITION 1.2 *In the theory* $\mathbf{ZFC} + \mathbf{T} + \mathbf{B}_\kappa$, $\mathbf{BI}_\kappa$ *implies* $\mathbf{IS}_\kappa$ *while* $\mathbf{IS}_{2^\kappa}$ *implies* $\mathbf{BI}_\kappa$.

PROOF. The result can be obtained by a straightforward evaluation of cardinalities in the proof of Lemma 1.3 in [12] (the one which proves that the unrestricted forms of **IS** and **BI** are equivalent to each other). ■

1.2. Elementary external set theory

This theory was introduced in [13] to describe the "world" of all definable bounded classes over a universe of **BST** .

Let $\operatorname{int} x$ (" x is internal") be the st-$\in$-formula $\exists^{\mathrm{st}} y\, (x \in y)$ (saying: x belongs to a standard set). Thus the Boundedness axiom of **BST** postulates that all sets x satisfy $\operatorname{int} x$ ("are internal"). This is not true in **EEST**, although still only internal sets can be elements of other sets.

The *Elementary external set theory* **EEST** has the following list of axioms:

1. $\forall^{\text{st}} x\,(\text{int}\,x)$: all standard sets are internal;

   $\forall^{\text{int}} x\; \forall\, y \in x\;(\text{int}\,y)$: transitivity of the internal subuniverse;

   Standardization: $\forall\, X\; \exists^{\text{st}} Y\; \forall^{\text{st}} x\;(x \in Y \;\longleftrightarrow\; x \in X)$.

2. $\mathbf{BST}^{\text{int}}$: all axioms of **BST** relativized to the formula int ;

3. Extensionality and the **ZFC** Separation for all st-$\in$-formulas;

4. The Parametrization axiom: $\forall\, C\; \exists^{\text{int}} p\;(C = \mathcal{C}_p)$.

The last axiom may be seen as a quite artificial statement, but actually it postulates that *all sets are bounded definable classes from the point of view of the internal universe.* Usually one cannot express statements of this kind legitimately; it is a very special property of bounded set theory **BST** that an indirect formulation (via classes $\mathcal{C}_p$) is available.

1.3. Hrbaček set theory

Hrbaček set theory **HST** is also a theory in the st-$\in$-language, admitting non–internal sets, but in essential ways more powerful than **BST**. It relates to **EEST** in the same way as **ZFC** minus the Power set axiom relates to a second order Peano arithmetic. More exactly, **HST** includes:

1. and 2. The same as items 1 and 2 of **EEST** above.

3. The **ZFC** Pair, Union, Extensionality, Infinity axioms, together with Separation, Collection, Replacement for all st-$\in$-formulas.

4. *Extension*: assume that S is a standard set and F a is function defined on the set ${}^\sigma S = \{x \in S : \text{st}\,x\}$, and $F(x)$ contains internal elements for all $x \in {}^\sigma S$; then there exists an *internal* function f defined on S and satisfying $f(x) \in F(x)$ for every $x \in {}^\sigma S$.

5. *Saturation*: if X is a set of standard size such that every $x \in X$ is internal and the intersection $\bigcap X'$ is nonempty for any finite nonempty $X' \subseteq X$, then $\bigcap X$ is nonempty.

6. Choice in the case when the domain X of the choice function is a set of standard size (*standard size Choice*), and Dependent Choice.

7. *Weak regularity*: if a nonempty set X contains only noninternal elements then there exists $x \in X$ such that $x \cap X = \emptyset$.

We recall that, in "external" theories, *sets of standard size* are those of the form $\{f(x) : x \in {}^\sigma X\}$, where X is standard and f any function, but, in **HST**, "standard size" = "wellorderable", see [13].

2. Natural partially saturated internal universes

Hrbaček proved in [6] that the Power set axiom fails in (a prototype of) **HST**, because external subsets of an internal set which has more than a standard finite number of elements, are too numerous to be a "set–size" collection. Thus the only way to define, in **HST**, external subuniverses satisfying the Power set axiom is to reduce the multitude of external sets.

This section outlines one of the two available approaches how such a reduction can be achieved. It will be the key fact (Theorem 2.2 – item 4) that, given a standard set S and a standard cardinal κ, the family of all subsets of S of the form $C_p = \bigcup_{a \in {}^\sigma\! A} \bigcap_{b \in {}^\sigma\! B} \eta(a,b)$, where $p = \langle A, B, \eta \rangle$ belongs to a standard set of cardinality $\le \kappa$, is a "set–size" collection. Thus one has to find an external subuniverse which does not contain sets of internal sets other then those of the mentioned form. Following this idea, we introduce, in the next section, an external subuniverse (the universe $\mathbb{H}_\kappa$) satisfying a suitable κ-form of **HST** plus the Power set axiom at the cost of a κ-restriction imposed on the "standard size" parts of **HST** .

The first step is to define the relevant *internal* subuniverse $\mathbb{I}_\kappa \subseteq \mathbb{I}$, which then will be the internal part of $\mathbb{H}_\kappa$. This is the aim of this section.

We argue in **HST** *in this section*; thus let $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$ denote the ground **HST** universe and the classes of all internal and standard sets respectively. Let κ be a fixed standard infinite cardinal, that is, a standard set which is an infinite cardinal in the sense of $\mathbb{I}$ or $\mathbb{S}$, which is equivalent by Transfer.

Definition 2.1 $\mathbb{I}_\kappa = \{x : \exists^{\rm st} X \ (x \in X \ \& \ \mathrm{card}\, X \le \kappa \text{ in } \mathbb{S})\}$, the class of all internal sets of *order* κ, introduced in [9] [5] .

Take notice that $\mathbb{I}_\kappa$ is not a transitive subclass of $\mathbb{I}$. However Theorem 2.2 (item 1) implies that every nonempty $X \in \mathbb{I}_\kappa$ contains an element in $\mathbb{I}_\kappa$.

Theorem 2.2 [**HST**] $\mathbb{I}_\kappa$ *contains all standard sets. Furthermore,*

1. $\mathbb{I}_\kappa$ *is an elementary submodel of* $\mathbb{I}$ *with respect to all* $\in$*-formulas. Moreover, if a set* $x \in \mathbb{I}$ *is* st-$\in$*-definable in* $\mathbb{I}$ *using sets in* $\mathbb{I}_\kappa$ *as parameters then* $x \in \mathbb{I}_\kappa$.
2. $\mathbb{I}_\kappa$ *satisfies* $\mathbf{BST}_\kappa$, *the theory containing all axioms of* **ZFC** *(in the* $\in$*-language), Transfer, Standardization, and the* κ*-forms* $\mathbf{BI}_\kappa$, $\mathbf{IS}_\kappa$, $\mathbf{B}_\kappa$ *of Bounded Idealization, Internal Saturation, and Boundedness.*

[5] In the particular case $\kappa = \aleph_0$, these sets were introduced by Luxemburg [20] under the name of σ-quasistandard objects. The general definition was first given in a nonpublished version of Hrbaček [7]. — *Pointed out by the referee.*

3. *If* $f \in \mathbb{I}$ *is a function defined on a standard set* X, $\operatorname{card} X \leq \kappa$ *in* $\mathbb{S}$, *and* $f(x) \in \mathbb{I}_\kappa$ *for all standard* $x \in X$, *then there exists a function* $f' \in \mathbb{I}_\kappa$ *such that* $f(x) = f'(x)$ *for all standard* $x \in X$.
4. *For any standard* X *there exists standard* P *such that every set* $C \subseteq X$ *definable (as a class) in* $\mathbb{I}$ *by a* st-$\in$*-formula having only elements of* $\mathbb{I}_\kappa$ *as parameters, is equal to some* C_p, $p \in P \cap \mathbb{I}_\kappa$. [6]

PROOF. 1. If x is standard then $X = \{x\}$ is standard, too; thus $\mathbb{S} \subseteq \mathbb{I}_\kappa$. Let $\Phi(x, p_0)$ be an $\in$-formula having a set $p_0 \in \mathbb{I}_\kappa$ as a parameter. We prove that, in $\mathbb{I}$, $\exists\, x\, \Phi(x, p_0) \longrightarrow \exists\, x \in \mathbb{I}_\kappa\, \Phi(x, p_0)$. Let $p_0 \in P$, where P is standard, $\operatorname{card} P \leq \kappa$ in $\mathbb{S}$. By the **ZFC** Collection in $\mathbb{I}$, there exists a set $X \in \mathbb{I}$ of cardinality $\leq \kappa$ such that

$$\forall\, p \in P\, [\, \exists\, x\, \Phi(x, p) \longrightarrow \exists\, x \in X\, \Phi(x, p)\,]$$

in $\mathbb{I}$. By Transfer there is a *standard* X of this kind. We put $p = p_0$.

To prove the "moreover" assertion, let $x \in \mathbb{I}$ be the unique set satisfying $\Phi(x)$ in $\mathbb{I}$, where Φ is a st-$\in$-formula with parameters in $\mathbb{I}_\kappa$. One may assume, by Theorem 1.5 in [12], that Φ is a Σ_2^{st} formula, say $\exists^{\mathrm{st}} a\ \forall^{\mathrm{st}} b\ \varphi(x, a, b)$, where φ is an $\in$-formula. Following Nelson [21], we observe that there exists a standard a such that x is the unique set satisfying $\forall^{\mathrm{st}} b\, \varphi(x, a, b)$, that is, $\forall\, \xi\, [\forall^{\mathrm{st}} b\, \varphi(\xi, a, b) \longleftrightarrow \xi = x\,]$, in $\mathbb{I}$. Using **BI**, we find a standard finite set B such that, in $\mathbb{I}$,

$$\forall\, \xi\, [\, \forall\, b \in B\, \varphi(\xi, a, b) \longrightarrow \xi = x\,].$$

On the other hand, all elements of a standard finite set are standard; hence the implication can be replaced by the equivalence in the displayed formula. We conclude that x is definable in $\mathbb{I}$ by an $\in$-formula with parameters in $\mathbb{I}_\kappa$. This implies $x \in \mathbb{I}_\kappa$ by the already proved elementary equivalence.

2. Therefore $\mathbb{I}_\kappa$ is a model of **ZFC** satisfying Transfer. Standardization holds since $\mathbb{S} \subseteq \mathbb{I}_\kappa$. Bounded Idealization $\mathbf{BI}_\kappa$ holds in $\mathbb{I}_\kappa$ because **BI** holds in $\mathbb{I}$ and every standard set X of cardinality $\leq \kappa$ in $\mathbb{S}$ retains all elements in $\mathbb{I}_\kappa$. Boundedness $\mathbf{B}_\kappa$ holds by definition. Finally Internal Saturation $\mathbf{IS}_\kappa$ is a consequence of $\mathbf{BI}_\kappa$ by Proposition 1.2.

3. *We argue in* $\mathbb{I}$. Since $f(x) \in \mathbb{I}_\kappa$ for all $x \in {}^\sigma X = \{x \in X : \operatorname{st} x\}$, we obtain, using a known consequence of Standardization, a standard function F defined on X and such that $f(x) \in F(x)$ and $\operatorname{card} F(x) \leq \kappa$ for every

[6] Since the class $\mathbb{I}$ of all internal sets is transitive in **HST**, the definition of C_p (see Subsection 1.1.) retains its "internal" sense.

standard $x \in X$. By Transfer, $\operatorname{card} F(x) \le \kappa$ holds for all $x \in X$, therefore $R = \bigcup_{x \in X} F(x)$ is a standard set of cardinality $\le \kappa$. Furthermore $f(x) \in R$ for all standard $x \in X$. Let $X' \subseteq X$ be a finite set containing all standard elements of X and satisfying $f(x) \in R$ for all $x \in X'$. Then the restriction $g = f \restriction X'$ belongs to the standard set G of all functions mapping a finite subset of X to R. Therefore $g \in \mathbb{I}_\kappa$ since $\operatorname{card} G \le \kappa$. It remains to extend, in $\mathbb{I}_\kappa$, g to a function f' defined on X.

4. *We argue in* $\mathbb{I}$. Let $C = \{x \in X : \Phi(x, q_0)\}$, where $q_0 \in Q$, Q a standard set of cardinality $\le \kappa$. Let $\theta = \max\{\operatorname{card} X, \kappa\}$, $\lambda = 2^{2^\theta}$, and $P = \{\langle \lambda, \lambda, \eta \rangle : \eta$ maps $\lambda \times \lambda$ onto $\mathcal{P}(X)\}$. We prove that $C = \mathcal{C}_p$ for some $p \in P \cap \mathbb{I}_\kappa$.

One can assume, by Theorem 1.5 of [12], that $\Phi(x, q)$ has the form $\exists^{\mathrm{st}} a \, \forall^{\mathrm{st}} b \, \varphi(x, a, b, q)$, where φ is an $\in$-formula. By Lemma 1.1, there exist standard sets A, B of cardinality $\le \lambda$, satisfying

$$\Phi(x, q) \quad \longleftrightarrow \quad \exists^{\mathrm{st}} a \in A \, \forall^{\mathrm{st}} b \in B \, \varphi(x, a, b, q) \quad - \quad \text{for all } x \in X \text{ and } q \in Q.$$

Let f and g be standard maps from λ onto A and B respectively. We define $\eta(\alpha, \beta) = \{x \in X : \varphi(x, f(\alpha), g(\beta), q_0)\}$ for all $\alpha, \beta < \lambda$, and then $p = \langle \lambda, \lambda, \eta \rangle$, thus $\mathcal{C}_p = \{x \in X : \Phi(x, q_0)\} = C$ and $p \in P$.

We verify that $p \in \mathbb{I}_\kappa$. Since λ is standard, it suffices to prove that $\eta \in \mathbb{I}_\kappa$. To see this, we note that η belongs to the set $H = \{\eta_q : q \in Q\}$, where each η_q is a function defined on $\lambda \times \lambda$ by

$$\eta_q(\alpha, \beta) = \{x \in X : \varphi(x, f(\alpha), g(\beta), q)\}.$$

However H is a standard set (because X, λ, f, g are standard) of cardinality not greater than $\operatorname{card} Q$, that is, $\le \kappa$. ∎

We conclude this section with a useful additional property of $\mathbb{I}_\kappa$.

LEMMA 2.3 *Let $I \in \mathbb{I}_\kappa$. If $X = I \cap \mathbb{I}_\kappa$ is a set of standard size then $I \subseteq \mathbb{I}_\kappa$ and I is a set of a standard finite number of elements.*

PROOF. Let, in $\mathbb{I}_\kappa$, f be a function mapping an ordinal $\alpha = \{\gamma : \gamma < \alpha\}$ bijectively on I. This property of f is then also true in $\mathbb{I}$ by Theorem 2.2. Then α is a standard natural number. (Otherwise there exists a *nonstandard* natural number $n < \alpha$. Since every $k < n$ belongs to $\mathbb{I}_\kappa$, the set $R = \{f(k) : k < n\}$ is a subset of $I \cap \mathbb{I}_\kappa$ having exactly n elements. Thus $W = \{k : k < n\}$ is a set of standard size in $\mathbb{I}$, a contradiction with **BI**.)

So I has a standard finite number n of elements. It follows easily from Theorem 2.2 (item 1) that $f(k) \in \mathbb{I}_\kappa$ for all $k < n$, hence $I \subseteq \mathbb{I}_\kappa$. ∎

3. Natural partially saturated external universes

We argue in **HST** in this section. As above, $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$ denote the ground **HST** universe and the classes of all internal and standard sets respectively.

Let $\kappa \in \mathbb{S}$ be an infinite cardinal in the sense of $\mathbb{S}$ or $\mathbb{I}$.

The plan is to define an external partially saturated "envelope" $\mathbb{H}_\kappa$ over the class $\mathbb{I}_\kappa$, which models a corresponding κ-fragment of **HST**. This class will consist of those sets in the basic **HST** universe $\mathbb{H}$ which one obtains using a known cumulative construction of assembling sets along well-founded trees definable in $\mathbb{I}_\kappa$. Item 4 of Theorem 2.2 will imply the Power set axiom in $\mathbb{H}_\kappa$ because all subsets $Y \in \mathbb{H}_\kappa$ of a standard set X have the form C_p, $p \in P$, for a certain standard set P (depending on X).

3.1. Assembling sets along well-founded trees

Let Seq denote the class of all internal sequences, of arbitrary (internal) sets, of standard finite length. For $t \in$ Seq and every set a, $t^\wedge a$ is the sequence in Seq obtained by adjoining a as the rightmost additional term to t. The notation $a^\wedge t$ is to be understood correspondingly. Λ is the empty sequence. The formula $t' \subseteq t$ means that the sequence $t \in$ Seq *extends* $t' \in$ Seq (perhaps $t' = t$ in this case).

A *tree* is a nonempty (possibly external) set $T \subseteq$ Seq such that, whenever $t', t \in$ Seq satisfy $t' \subseteq t$, then $t \in T$ implies $t' \in T$. Thus every tree contains Λ. $\operatorname{Max} T$ is the set of all $\subseteq$-*maximal* in T elements $t \in T$.

A tree T is *well-founded* (*wf tree*, in brief) if and only if every nonempty (possibly external) set $T' \subseteq T$ contains a $\subseteq$-maximal element.

DEFINITION 3.1 Let a *wf pair* be any pair $\langle T, F\rangle$ such that T is a wf tree and F is a function defined on $\operatorname{Max} T$. In this case, the family of sets $F_T(t)$, $t \in T$, is defined as follows:

1) if $t \in \operatorname{Max} T$ then $F_T(t) = F(t)$;
2) if $t \in T \setminus \operatorname{Max} T$ then $F_T(t) = \{F_T(t^\wedge a) : t^\wedge a \in T\}$.

We finally set $F[T] = F_T(\Lambda)$.

Let, for example, $T = \{\Lambda\}$ and $F(\Lambda) = x$. Then $F[T] = F_T(\Lambda) = x$.

Since **HST** contains Replacement, Definition 3.1 works well directly; thus for every $\langle T, F\rangle \in \mathcal{H}$ the function F_T is uniquely defined on T and the final set $F[T] = F_T(\Lambda)$ is also well defined.

3.2. Class of elementary external sets

In particular we shall be interested to study the construction of Definition 3.1 from the point of view of the class $\mathbb{E} = \{\mathcal{C}_p : p \in \mathbb{I}\}$, of all *elementary external* sets. It is shown in [13] that $\mathbb{E}$ models **EEST**, elementary external set theory, described in Subsection 1.2. above.

We observe that $\mathbb{I} \subseteq \mathbb{E}$ and every set $X \in \mathbb{E}$ satisfies $X \subseteq \mathbb{I}$.

Let $\mathcal{H}$ denote the class of all wf pairs $\langle T, F\rangle$ such that $T,\ F \in \mathbb{E}$.

LEMMA 3.2 *Let* $T \in \mathbb{E}$ *be a wf tree in the sense of* $\mathbb{E}$. *Then* T *is a wf tree in the sense of* $\mathbb{H}$, *too. Hence the class* $\mathcal{H}$ *is* st-$\in$-*definable in* $\mathbb{E}$ *as a subclass of* $\mathbb{E}\times\mathbb{E}$.

PROOF. Since $\mathbb{E}$ models Separation, the wellfoundedness of T in $\mathbb{E}$ allows to define, in $\mathbb{E}$, a standard ordinal $\rho(t)$ for all $t \in T$ by the scheme: $\rho(t) = 0$ for $t \in \operatorname{Max} T$, and $\rho(t) = \operatorname{Ssup}\{\rho(t^\wedge a) : t^\wedge a \in T\}$ for $t \notin \operatorname{Max} T$, where, for a set O of standard ordinals, $\operatorname{Ssup} O$ denotes the least standard ordinal bigger than all ordinals in O. (The Collection and Standardization axioms of **HST** prove that $\rho(t)$ is defined correctly for $t \notin \operatorname{Max} T$.)

The existence of the function ρ proves the wellfoundedness of T in $\mathbb{H}$ as well because by the **HST** Standardization the class of standard ordinals is well-founded in $\mathbb{H}$. ∎

Since $\mathbb{E}$ contains only those sets which have internal elements exclusively, for a wf pair $\langle T, F\rangle \in \mathcal{H}$ the set $F[T]$ can be not a member of $\mathbb{E}$. However, one can determine, in $\mathbb{E}$, when $F[T] \in G[R]$, or $F[T] = G[R]$, for given wf pairs $\langle T, F\rangle$ and $\langle R, G\rangle$ in $\mathcal{H}$.

PROPOSITION 3.3 *There exist 4-ary* st-$\in$-*predicates* ${}^{\mathrm{h}}\!=$ *and* ${}^{\mathrm{h}}\!\in$ *and a binary* st-$\in$-*predicate* ${}^{\mathrm{h}}\mathrm{st}$ *such that the following holds for all wf pairs* $\langle T, F\rangle$ *and* $\langle R, G\rangle$ *in* $\mathcal{H}$:

$$
\begin{array}{llll}
F[T] = G[R] & \textit{iff} & \textit{it is true in } \mathbb{E} \textit{ that} & \langle T, F\rangle \ {}^{\mathrm{h}}\!= \langle R, G\rangle ; \\
F[T] \in G[R] & \textit{iff} & \textit{it is true in } \mathbb{E} \textit{ that} & \langle T, F\rangle \ {}^{\mathrm{h}}\!\in \langle R, G\rangle ; \\
\mathrm{st}\, F[T] & \textit{iff} & \textit{it is true in } \mathbb{E} \textit{ that} & {}^{\mathrm{h}}\mathrm{st}\,\langle T, F\rangle .
\end{array}
$$

PROOF. Let us first distinguish the case when $F_T(t)$ takes an *internal* value. We set $\mathtt{id}_F(T) = \{t \in T : F_T(t) \text{ is internal}\}$ whenever $\langle T, F\rangle \in \mathcal{H}$ (the *domain of internal definability*); for instance, $\operatorname{Max} T \subseteq \mathtt{id}_F(T)$.

LEMMA 3.4 *If* $\langle T, F\rangle \in \mathcal{H}$ *then the set* $\mathtt{id}_F(T)$ *and the restriction* $F_T \restriction \mathtt{id}_F(T)$ *belong to* E *.*

PROOF. As is the proof of Lemma 3.2, one conducts the definition of an internal set $f(t)$ for $t \in T$ by the scheme:

1) $f(t) = F(t)$ for $t \in \operatorname{Max} T$;
2) if $t \in T \setminus \operatorname{Max} T$ and the set $X_t = \{f(t^\wedge a) : t^\wedge a \in T\}$ is internal then we put $F(t) = X_t$;
3) if X_t is not internal then the values $f(t')$, $t' \subseteq t$, are not defined.

It follows that $\operatorname{dom} f = \mathtt{id}_F(T) \in \mathbb{E}$ and $f = F_T \restriction \mathtt{id}_F(T)$ belong to $\mathbb{E}$ by the **EEST** Separation, true in E . ■

To continue the proof of the proposition, let us associate, with each pair of wf pairs $\langle T, F\rangle$ and $\langle R, G\rangle$, a function $E = E_{TF,RG} \in \mathbb{E}$ mapping $T \times R$ into $2 = \{0, 1\}$. The values $E(t, r)$ ($t \in T$ and $r \in R$) are defined by the same type of induction in E, as follows.

(i) If $t \in \mathtt{id}_F(T)$ and $r \in \mathtt{id}_G(R)$ then $E(t, r) = 1$ iff $F_T(t) = G_R(r)$.

(ii) If $t \in \mathtt{id}_F(T)$ but $r \notin \mathtt{id}_G(R)$, or vice versa, then $E(t, r) = 0$.

(iii) If $t \notin \mathtt{id}_F(T)$ and $r \notin \mathtt{id}_G(R)$ then $E(t, r) = 1$ iff 1st, for any $t^\wedge a \in T$ there exists $r^\wedge b \in R$ such that $E(t^\wedge a, r^\wedge b) = 1$, and 2nd, for any $r^\wedge b \in R$ there exists $t^\wedge a \in T$ such that $E(t^\wedge a, r^\wedge b) = 1$.

Since $E = E_{TF,RG} \in \mathbb{E}$, the formula $E_{TF,RG}(\Lambda, \Lambda) = 1$ can be taken as $\langle T, F\rangle \;{}^{\mathrm{h}}{=}\; \langle R, G\rangle$.

Let, for any set x, C^x be the function defined on the singleton $\{\Lambda\}$ by $C^x(\Lambda) = x$; then $\langle\{\Lambda\}, C^x\rangle$ is a wf pair and $C^x[\{\Lambda\}] = x$. One takes the formula $\exists^{\mathrm{st}} x\, (\langle T, F\rangle \;{}^{\mathrm{h}}{=}\; \langle\{\Lambda\}, C^x\rangle)$ as ${}^{\mathrm{h}}\mathrm{st}\, \langle T, F\rangle$.

Finally let $\langle T, F\rangle \;{}^{\mathrm{h}}{\in}\; \langle R, G\rangle$ be the formula which says that either $R \neq \{\Lambda\}$ and $E_{TF,RG}(\Lambda, \langle b\rangle) = 1$ for some b such that the one-term sequence $\langle b\rangle$ belongs to R, or $R = \{\Lambda\}$, and there exists a set $x \in G(\Lambda)$ such that $\langle T, F\rangle \;{}^{\mathrm{h}}{=}\; \langle\{\Lambda\}, C^x\rangle$. ■

3.3. The partially saturated subuniverse

We consider the internal subuniverse I_κ of Section 2 as the base for our construction of an external subuniverse H_κ. The construction involves Definition 3.1. To guarantee that no external subsets of internal sets except those of the form $\mathcal{C}_p$, $p \in \mathbb{I}_\kappa$, appear, we have to keep the construction under the control of $\mathbb{I}_\kappa$. For example we can admit only those wf trees T and associated functions F which have the form $\mathcal{C}_p$, where $p \in \mathbb{I}_\kappa$.

We put $\mathbb{E}_\kappa = \{\mathcal{C}_p : p \in \mathbb{I}_\kappa\}$; so that $\mathbb{E}_\kappa \subseteq \mathbb{E}$.

PROPOSITION 3.5 *1. If a set $X \subseteq \mathbb{I}$ is definable in $\mathbb{E}$ as a class by a* st-$\in$-*formula having sets in $\mathbb{E}$ as parameters then $X \in \mathbb{E}$.*

2. If all parameters in the formula belong to $\mathbb{E}_\kappa$ then $X \in \mathbb{E}_\kappa$.

3. If in the latter case $X \in \mathbb{I}$ then $X \in \mathbb{I}_\kappa$; so $\mathbb{E}_\kappa \cap \mathbb{I} = \mathbb{I}_\kappa$ — therefore internal sets not in $\mathbb{I}_\kappa$ do not enter $\mathbb{E}_\kappa$ via external definitions.

PROOF. 1. Use the **EEST** Separation, true in $\mathbb{E}$.

2. By definition of $\mathbb{E}_\kappa$ we may assume that all parameters in the formula which defines X in $\mathbb{E}$ belong to $\mathbb{I}_\kappa$. Then, since the truth in $\mathbb{E}$, the **EEST** universe, can be expressed in $\mathbb{I}$, its internal part (see Proposition 2.8 in [13]), X is st-$\in$-definable in $\mathbb{I}$ as a class via the same parameters. Therefore $X \in \mathbb{E}_\kappa$ by Theorem 2.2 (item 4).

3. Apply Theorem 2.2 (item 1). ■

One might consider it quite natural to define a subuniverse $\mathbb{H}_\kappa \subseteq \mathbb{H}$ having $\mathbb{I}_\kappa$ as its internal part as the collection of all sets $F[T]$ where both F and T belong to $\mathbb{E}_\kappa$. However this does not work properly because the class obtained this way is not extensional. (Note that even $\mathbb{E}_\kappa$ is not extensional. Indeed, let $\kappa = \aleph_0$ and $X = \omega_1$ in $\mathbb{S}$. Then both X and $Y = X \cap \mathbb{I}_\kappa$ belong to $\mathbb{E}_\kappa$ and have the same elements in $\mathbb{E}_\kappa$, but $X \neq Y$.)

To fix this problem, we have to impose a suitable restriction on wf pairs. This is realized by the notion of *κ-illegal point.*

DEFINITION 3.6 1. Let $\langle T, F\rangle \in \mathcal{H}$. $t \in T$ is a *κ-illegal point* in T if there exists a set $I \in \mathbb{I}_\kappa$ such that $I \cap \mathbb{I}_\kappa = F_T(t) \neq I$.

2. $\mathcal{H}_\kappa$ is the collection of all wf pairs $\langle T, F\rangle \in \mathcal{H}$ such that both T and F belong to $\mathbb{E}_\kappa$, $T \subseteq \mathbb{I}_\kappa$, and T does not contain κ-illegal points.

3. $\mathbb{H}_\kappa = \{F[T] : \langle T, F\rangle \in \mathcal{H}_\kappa\}$.

However we shall see (Lemma 3.12 below) that the restriction is not harmful: sets which are left out are suitably replaced by internal sets.

We end this subsection with a useful lemma.

LEMMA 3.7 *Let $\langle T, F\rangle \in \mathcal{H}_\kappa$. Then $F_T(t) \in \mathbb{H}_\kappa$ for all $t \in T$. Furthermore if $F_T(t)$ is internal then $F_T(t) \in \mathbb{I}_\kappa$.*

(In particular $F(t) \in \mathbb{I}_\kappa$ for all $t \in \operatorname{Max} T$, provided the wf pair $\langle T, F\rangle$ belongs to $\mathcal{H}_\kappa$. In general functions in $\mathbb{E}_\kappa$ map elements of $\mathbb{I}_\kappa$ to $\mathbb{I}_\kappa$.)

Proof. We put $T^t = \{r : t*r \in T\}$, where $t*r$ is the concatenation, and $F^t(r) = F(t*r)$ for $t*r \in \operatorname{Max} T$; thus $\langle T^t, F^t\rangle \in \mathcal{H}$, and actually $\in \mathcal{H}_\kappa$ by Proposition 3.5. Moreover $F_T(t*r) = F^t{}_{T^t}(r)$ for all $r \in T^t$. In particular, $F_T(t) = F^t{}_{T^t}(\Lambda) \in \mathbb{H}_\kappa$. This proves the first statement. The second one is implied by Lemma 3.4 and Proposition 3.5. ■

Corollary 3.8 *Let* $X \in \mathbb{H}_\kappa$. *If* $X \not\subseteq \mathbb{H}_\kappa$ *then* $X \in \mathbb{I}_\kappa$.

3.4. The principal theorem

We recall that, in external nonstandard theories, a set *of standard size* is a functional image of a set of the form ${}^\sigma S = \{s \in S : \operatorname{st} s\}$, where S is a standard set. The following definition is a variant of this notion.

Definition 3.9 Let κ be a standard cardinal. X is a *set of standard κ-size* if there exist a standard set S *of cardinality* $\le \kappa$ *in* $\mathbb{S}$ and a function F defined on ${}^\sigma S$ such that $X = \{F(x) : x \in {}^\sigma S\}$.

Let us demonstrate that $\mathbb{H}_\kappa$ models a κ-version of **HST** in which the axioms involving standard size (see Subsection 1.3.) are weakened to standard κ-size, but also models the Power set axiom.

Theorem 3.10 [**HST**] $\mathbb{I} \cap \mathbb{H}_\kappa = \mathbb{I}_\kappa$, *so* $\mathbb{I}_\kappa$ *is the class of all formally internal* [7] *sets in* $\mathbb{H}_\kappa$. *In addition the following statements hold in* $\mathbb{H}_\kappa$:

1. *The axioms of Pair, Union, Extensionality, Infinity, together with Collection, Separation, Replacement for all* st-$\in$-*formulas.*
2. *Extension in the form of item 4 in Subsection 1.3. for standard sets* S *satisfying* $\operatorname{card} S \le \kappa$ *in* $\mathbb{S}$.
3. *Saturation and Choice for sets* X *of standard* κ-*size, and Dependent Choice.*
4. *Weak regularity.*
5. *The* **ZFC** *Power set axiom.*

Finally $\mathbb{H}_\kappa$ *satisfies the following closure property: if* $\mathcal{Z} \subseteq \mathbb{H}_\kappa$ *is a set of standard* κ-*size then* $\mathcal{Z} \in \mathbb{H}_\kappa$.

[7] Satisfying the formula $\operatorname{int} x$, that is, $\exists^{\mathrm{st}} X\ (x \in X)$.

PROOF. We start with **internal sets.** Let $x \in \mathbf{I}_\kappa$. We put $T = \{\Lambda\}$ and $F(\Lambda) = x$, so that evidently $\langle T, F\rangle \in \mathcal{H}_\kappa$ and $F[T] = x$. This proves $\mathbf{I}_\kappa \subseteq \mathbf{H}_\kappa$. The inclusion $\mathbf{H}_\kappa \cap \mathbf{I} \subseteq \mathbf{I}_\kappa$ is guaranteed by Lemma 3.7. $\dashv$

Extensionality. Suppose that $\langle T, F\rangle$ and $\langle R, G\rangle$ belong to $\mathcal{H}_\kappa$ and the sets $X = F[T]$ and $Y = G[R]$ satisfy $X \cap \mathbf{H}_\kappa = Y \cap \mathbf{H}_\kappa$. We assert that then $X = Y$.

If neither of T, R is equal to $\{\Lambda\}$ then $X \cup Y \subseteq \mathbf{H}_\kappa$ by Lemma 3.7.

If $T = R = \{\Lambda\}$ Then both X and Y belong to $\mathbf{I}_\kappa$ by Lemma 3.7, so $X \cap \mathbf{I}_\kappa = Y \cap \mathbf{I}_\kappa$ implies $X = Y$ by Theorem 2.2 (item 1).

Assume finally that e. g. $R = \{\Lambda\}$ but $T \neq \{\Lambda\}$. Then $Y = G(\Lambda) \in \mathbf{I}_\kappa$ as above. In particular, $Y \subseteq \mathbf{I}$, so $Y \cap \mathbf{H}_\kappa = Y \cap \mathbf{I} \cap \mathbf{H}_\kappa = Y \cap \mathbf{I}_\kappa$ (since $\mathbf{I} \cap \mathbf{H}_\kappa = \mathbf{I}_\kappa$). On the other hand, $X \subseteq \mathbf{H}_\kappa$ by Lemma 3.7 because $T \neq \{\Lambda\}$, so $X \cap \mathbf{H}_\kappa = X$. We conclude that $X = Y \cap \mathbf{I}_\kappa$. But Λ is not κ-illegal in T, so $X = Y$. $\dashv$

Weak regularity is inherited from $\mathbf{H}$, the universe of **HST**, because if $X \in \mathbf{H}_\kappa$ but $X \notin \mathbf{I}_\kappa$ then $X \subseteq \mathbf{H}_\kappa$ by Corollary 3.8. $\dashv$

Infinity is inherited from $\mathbb{S}$. $\dashv$

Extension. The κ-version of Extension ($\operatorname{card} S \leq \kappa$ in the standard universe) is reduced to the **HST** Extension by Theorem 2.2 (item 3). $\dashv$

Saturation. The κ-size Extension reduces the κ-size Saturation to the case when the given standard size family has the form $\{f(a) : a \in {}^\sigma\! A_0\}$, where A_0 is a standard set of cardinality $\operatorname{card} A_0 \leq \kappa$ in $\mathbb{S}$ and f a function in $\mathbf{I}_\kappa$. This is simply $\mathbf{IS}_\kappa$, the κ-case of Internal Saturation in $\mathbf{I}_\kappa$. [8] However $\mathbf{IS}_\kappa$ is true in $\mathbf{I}_\kappa$ by Theorem 2.2 (item 2). $\dashv$

The verification of the other axioms in $\mathbf{H}_\kappa$ proceeds by certain transformations of wf pairs. Let us prove two technical lemmas.

LEMMA 3.11 *In* $\mathbf{H}$, *every set* $X \subseteq \mathbf{I}$ *is covered by a standard set.*

PROOF. By the Boundedness axiom in $\mathbf{I}$, for each $x \in X$ there exists a standard set s such that $x \in s$. By the **HST** Collection in $\mathbf{H}$, we have a set S' such that every $x \in X$ belongs to a standard $s \in S'$. By the **HST** Standardization, there exists a standard set S having the same standard elements as S'. We put $Y = \bigcup S$; then Y is standard and $X \subseteq Y$. ■

[8] Indeed, it is known (see e. g. [13] or [6]) that *finite* (in the sense of the ordinary **ZFC** definition) sets in **HST** are those having a standard $\mathbb{S}$-finite number of elements. If such a set contains only internal elements then it is internal.

LEMMA 3.12 *Let $\tau \subseteq \mathcal{H}_\kappa$ be a set definable in E as a subclass of $\mathsf{E} \times \mathsf{E}$, using only sets in E_κ as parameters, and $\mathcal{Z} = \{F[T] : \langle T, F\rangle \in \tau\}$. There exists $Z \in \mathsf{H}_\kappa$ such that $Z \cap \mathsf{H}_\kappa = \mathcal{Z}$. Each of the following two conditions is sufficient for $\mathcal{Z}$ itself to belong to H_κ :*

1. *$\mathcal{Z}$ contains at least one noninternal element.*
2. *$\mathcal{Z}$ is a set of standard size.*

PROOF. The idea is clear: present τ as $\tau = \{\langle T_a, F_a\rangle : a \in A\}$, put $R = \{\Lambda\} \cup \{a^\wedge t : t \in T_a\}$ and define G appropriately so that $G[R] = \mathcal{Z}$. We have only to keep the construction within E_κ and avoid illegality.

By definition for any $\langle T, F\rangle \in \tau$ there exist $p, q \in \mathsf{I}_\kappa$ such that $T = \mathcal{C}_p$ and $F = \mathcal{C}_q$. By the **HST** Collection and Lemma 3.11 there is a standard set S such that p, q of this kind can be found in S for every pair $\langle T, F\rangle \in \tau$. The set

$$A = \{\langle p, q\rangle \in S^2 \cap \mathsf{I}_\kappa : \langle \mathcal{C}_p, \mathcal{C}_q\rangle \in \tau\} \subseteq \mathsf{E}$$

is then definable in E using only sets in E_κ as parameters, therefore we have $A \in \mathsf{E}_\kappa$ by Proposition 3.5. We define $\langle R, G\rangle \in \mathcal{H}$ as follows:

1) $R = \{\Lambda\} \cup \{a^\wedge t : a = \langle p, q\rangle \in A \ \& \ t \in \mathcal{C}_p\}$, and

2) $G(a^\wedge t) = \mathcal{C}_q(t)$ for all $a = \langle p, q\rangle \in A$ and $t \in \mathcal{C}_p$.

Evidently $G_R(\langle a\rangle) = \mathcal{C}_q[\mathcal{C}_p]$ whenever $a = \langle p, q\rangle \in A$, so that $G[R] = \mathcal{Z} = \{F[T] : \langle T, F\rangle \in \tau\}$. Moreover $R, G \in \mathsf{E}_\kappa$ again by Proposition 3.5. If Λ is not κ-illegal in R then immediately $\langle R, G\rangle \in \mathcal{H}_\kappa$, and $\mathcal{Z} = G[R]$. If Λ is κ-illegal then there exists $Z \in \mathsf{I}_\kappa$ such that $Z \cap \mathsf{H}_\kappa = G[R] = \mathcal{Z}$.

Condition 1. Apply Corollary 3.8 to Z .

Condition 2. It suffices to show that $Z = \mathcal{Z}$. Let on the contrary $\mathcal{Z} \neq Z$, therefore $Z \not\subseteq \mathsf{H}_\kappa$. Then $Z \in \mathsf{I}_\kappa$ by Corollary 3.8. Furthermore $Z \subseteq \mathsf{I}_\kappa$ by Lemma 2.3, so that still $\mathcal{Z} = Z$, contradiction. ■

It follows from Lemma 3.12 that ${}^\sigma S \in \mathsf{H}_\kappa$ for every standard set S . (We continue the proof of the theorem.)

Pairs – an immediate consequence of Lemma 3.12. ⊣

Separation. Let $X = F[T]$, $\langle T, F\rangle \in \mathcal{H}_\kappa$, and all parameters in a st-$\in$-formula $\Phi(x)$ belong to H_κ. We define a set $\tau \subseteq \mathcal{H}_\kappa$ as follows.

If $T \neq \{\Lambda\}$ then the set $\operatorname{Min} T = \{a : \langle a\rangle \in T\}$ is nonempty. For $a \in \operatorname{Min} T$, we put $T^a = \{t : a^\wedge t \in T\}$ and $F^a(t) = F(a^\wedge t)$ for all

$a^\wedge t \in \operatorname{Max} T$. Obviously $\langle T^a, F^a\rangle \in \mathcal{H}_\kappa$ for all a. Let τ be the set of all wf pairs $\langle T^a, F^a\rangle$, where $a \in \operatorname{Min} T$, such that $\Phi(F^a[T^a])$ holds in $\mathbf{H}_\kappa$.

Suppose that $T = \{\Lambda\}$, so that $X = F(\Lambda) \in \mathbf{I}_\kappa$. In this case, we define $X' = \{x \in X \cap \mathbf{I}_\kappa : \Phi(x) \text{ holds in } \mathbf{H}_\kappa\}$. We recall that for any internal x the wf pair ${}^{\mathrm{h}}x = \langle\{\Lambda\}, C^x\rangle \in \mathcal{H}$ is defined by $C^x(\Lambda) = x$, so that $C^x[\{\Lambda\}] = x$. Let $\tau = \{{}^{\mathrm{h}}x : x \in X'\}$.

In both cases $\tau \subseteq \mathcal{H}_\kappa$, and the set $Y = \{G[R] : \langle R, G\rangle \in \tau\}$ satisfies the equality $Y = \{y \in X : \Phi(y) \text{ holds in } \mathbf{H}_\kappa\}$. Thus the required result would follow from Lemma 3.12 as soon as we have proved that τ is definable in $\mathbf{E}$ as a subclass of $\mathbf{E} \times \mathbf{E}$, using only sets in $\mathbf{E}_\kappa$ as parameters.

We first note that $\mathcal{H}$ is definable in $\mathbf{E}$ by Lemma 3.2, therefore $\mathcal{H}_\kappa$ is definable in $\mathbf{E}$ using only κ as a parameter (but κ is standard). Furthermore definability in $\mathbf{H}_\kappa$ (by the formula Φ) is reducible to definability in $\mathbf{E}$ by Proposition 3.3. Thus τ is actually definable (that is, st-$\in$-definable) in $\mathbf{E}$ using as parameters only κ, $\langle T, F\rangle$, and several wf pairs $\langle U, H\rangle \in \mathcal{H}_\kappa$ such that $H[U]$ occurs in Φ, as required. ⊣

Union. Suppose that $X = F[T] \in \mathbf{H}_\kappa$, where $\langle T, F\rangle \in \mathcal{H}_\kappa$. We have to prove that the set $\mathbf{H}_\kappa \cap \bigcup X$ belongs to $\mathbf{H}_\kappa$.

For a two–element sequence $\langle a, b\rangle \in T$, we define $T^{ab} = \{t : a^\wedge b^\wedge t \in T\}$ and $F^{ab}(t) = F(a^\wedge b^\wedge t)$ for all $a^\wedge t \in \operatorname{Max} T$. Obviously $\langle T^{ab}, F^{ab}\rangle \in \mathcal{H}_\kappa$. The sets $F^{ab}[T^{ab}]$ give the first group of elements of the union $\bigcup X$ in the subuniverse $\mathbf{H}_\kappa$.

The other group consists of all sets $x \in F(\langle a\rangle) \cap \mathbf{I}_\kappa$, where a is such that the one–element sequence $\langle a\rangle$ belongs to $\operatorname{Max} T$. Each x of this type is presented in $\mathcal{H}_\kappa$ by the wf pair ${}^{\mathrm{h}}x = \langle\{\Lambda\}, C^x\rangle$.

One easily sees that the set τ of the wf pairs determined by the two groups is definable in $\mathbf{E}$ as a subclass of $\mathbf{E} \times \mathbf{E}$, using only sets in $\mathbf{E}_\kappa$ as parameters. It remains to apply Lemma 3.12. ⊣

Collection. By the **HST** Collection, it suffices to verify the following: if a set X satisfies $X \subseteq \mathbf{H}_\kappa$ then there exists $X' \in \mathbf{H}_\kappa$ such that $X \subseteq X'$. Using Collection again and definition of $\mathbf{H}_\kappa$, we conclude that there exists a set $P \subseteq \mathbf{I}$ such that

$$\forall Y \in X \, \exists p, q \in P \, (\langle \mathcal{C}_p, \mathcal{C}_q\rangle \in \mathcal{H}_\kappa \;\&\; Y = \mathcal{C}_q[\mathcal{C}_p]).$$

By Lemma 3.11, we have $P \subseteq S$ for a standard set S. We finally apply Lemma 3.12 to the set τ of all wf pairs $\langle \mathcal{C}_p, \mathcal{C}_q\rangle \in \mathcal{H}_\kappa$, where $p, q \in S$. ⊣

Replacement is a consequence of Collection and Separation. ⊣

The closure property. By the **HST** standard size Choice, there exists a set $\tau \subseteq \mathcal{H}_\kappa$ of standard κ-size s. t. $\mathcal{Z} = \{F[T] : \langle T, F\rangle \in \tau\}$. By Lemma 3.12 (condition 2) it suffices to prove that τ is definable in $\mathbb{E}$ by a st-$\in$-formula having only sets in $\mathbf{E}_\kappa$ as parameters.

By the standard size Choice again, there exists a set $\Pi \subseteq \mathbf{I}_\kappa$ of standard κ-size satisfying the equality $\tau = \{\langle \mathcal{C}_p, \mathcal{C}_q\rangle : \langle p, q\rangle \in \Pi\}$. Using Theorem 2.2 (item 3) and then Proposition 3.5, we see that $\Pi \in \mathbf{E}_\kappa$. Thus τ is st-$\in$-definable in $\mathbf{E}$ using only $\Pi \in \mathbf{E}_\kappa$ and κ as parameters. $\dashv$

Choice in the κ-size version and **Dependent Choice** follow from the same **HST** axioms and the closure property. $\dashv$

Power set. We first prove that an arbitrary *standard* S has the power set $\mathcal{P}(S)$ in $\mathbf{H}_\kappa$. Then the result will be expanded to the general case.

Thus let $S \in \mathbb{S}$. Theorem 2.2 (item 4) implies the existence of a standard set P such that all subsets of S in $\mathbf{E}_\kappa$ belong to the collection $|P|_\kappa = \{\mathcal{C}_p : p \in P \cap \mathbf{I}_\kappa\}$. We assert that every set $X \in \mathbf{H}_\kappa$, $X \subseteq S$, satisfies $X = \mathcal{C}_p$ in $\mathbf{H}_\kappa$ for some $p \in P \cap \mathbf{I}_\kappa$; this is enough to get the power set of S in $\mathbf{H}_\kappa$ by the already proved axioms of Separation and Collection.

Let $X = F[T] \in \mathbf{H}_\kappa$, $\langle T, F\rangle \in \mathcal{H}_\kappa$, $X \subseteq S$ in $\mathbf{H}_\kappa$. We observe that the set $C = \{x \in S \cap \mathbf{I}_\kappa : {}^{\mathrm{h}}x \,{}^{\mathrm{h}}\!\!\in \langle T, F\rangle\}$ belongs to $\mathbf{E}_\kappa$ by propositions 3.3 and 3.5. It follows that $C = \mathcal{C}_p$ for some $p \in P \cap \mathbf{I}_\kappa$.

On the other hand, one easily proves that $X \cap \mathbf{H}_\kappa = C$ by the definition of ${}^{\mathrm{h}}x$ and the choice of the formula ${}^{\mathrm{h}}\!\!\in$.

We now consider the general case. It suffices to prove that every set is a functional image of a standard set in $\mathbf{H}_\kappa$.

To prove this proposition, let $\langle T, F\rangle \in \mathcal{H}_\kappa$, $X = F[T] \in \mathbf{H}_\kappa$. We may assume that X contains noninternal elements in $\mathbf{H}_\kappa$ (otherwise apply Lemma 3.11). Then $T \neq \{\Lambda\}$. In this case, the set $\operatorname{Min} T = \{a : \langle a\rangle \in T\}$ is nonempty. Since $\operatorname{Min} T \subseteq \mathbf{I}$ by definition, Lemma 3.11 gives a standard set S such that $A = \operatorname{Min} T \subseteq S$. The function f defined on S by

$$f(a) = \begin{cases} F_T(\langle a\rangle) & \text{when } a \in \operatorname{Min} T \\ \text{a fixed element } x_0 \in X & \text{otherwise} \end{cases}$$

maps S onto X in $\mathbf{H}_\kappa$, as required. (f belongs to $\mathbf{H}'_\kappa$ by the already proved axioms, in particular Separation.) $\dashv$

This ends the proof of the theorem. ■

4. Standard size partial external universes

One can prove easily that although some amount of Choice is available in the universes $\mathbb{H}_\kappa$ by Theorem 3.10 (item 3), the general form of Choice does not hold in $\mathbb{H}_\kappa$; for example $\mathbb{N}$, the (standard) set of all natural numbers in the sense of $\mathbb{I}$ or $\mathbb{S}$, cannot be wellordered even in $\mathbb{H}$, the ground **HST** universe. (Take notice that $\mathbb{N} \subseteq \mathbb{I}_\kappa$ since κ is infinite, therefore $\mathbb{N}$ belongs to $\mathbb{H}_\kappa$ with all its elements.)

It is the aim of this section to show that one can provide the full Choice in an external subuniverse, keeping the Power set axiom and the other items of Theorem 3.10, in particular the standard κ-size Saturation, only at the cost of $\mathbf{BI}_\kappa$ in the internal subuniverse — but preserving Internal Saturation $\mathbf{IS}_\kappa$ which is weaker in the κ-restricted case.

A different system of external subuniverses $\mathbb{H}'_\kappa$ will be defined so that every set has standard size in $\mathbb{H}'_\kappa$. This is quite sufficient to get the Power set and Choice axioms in $\mathbb{H}'_\kappa$.

The construction is based on two principal ideas: first we define the ground internal subuniverse $\mathbb{I}'_\kappa$ as an ultrapower of the standard universe, but actually the ultrapower turns out to be an inner class in $\mathbb{H}$ because **HST** provides a sufficient amount of Saturation to obtain ultraproducts as inner classes; second, taking care that only standard size sets belong to $\mathbb{H}'_\kappa$, we use only trees T of "standard size" branching in the operation $F[T]$ to obtain the required external "envelope" of $\mathbb{I}'_\kappa$.

Let, as above, $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$ denote the ground **HST** universe and its subuniverses of internal and standard sets respectively, and κ be a standard infinite cardinal.

4.1. Standard size internal subuniverse

The construction of the new system of internal subuniverses is not so straightforward as the definition of $\mathbb{I}_\kappa$ in Section 2. The idea resembles the one used in the proof of Theorem 2.4 in [12] (every model of **ZFC** is the standard part of a model of **BST**). We shall proceed the same way as if a κ-saturated enlargement of a **ZFC** universe were being constructed.

We argue in $\mathbb{I}$ *in this subsection.*

Let us recall several definitions from [12], Section 2.

Let C be an arbitrary set. We put $C^{\text{fin}} = \mathcal{P}_{\text{fin}}(C)$. A nonprincipal ultrafilter $U \subseteq \mathcal{P}(C^{\text{fin}}) = \{I : I \subseteq C^{\text{fin}}\}$ is C-*adequate* iff it contains all sets $I(C, i_0) = \{i \in C^{\text{fin}} : i_0 \subseteq i\}$, where $i_0 \in C^{\text{fin}}$. If in this case $D \subseteq C$ then we define $U \restriction D = \{u \restriction D : u \in U\}$, where $u \restriction D = \{i \cap D : i \in u\}$ for any $u \subseteq C^{\text{fin}}$. Then $U \restriction D$ is a D-adequate ultrafilter.

There are two useful operations over ultrafilters.

Operation 1. Let U and U' be a C-adequate and a C'-adequate ultrafilter respectively. Suppose that $C \cap C' = \emptyset$. We put

$$U * U' = \{w \subseteq (C \cup C')^{\text{fin}} : U\, i\; U' i'\; (i \cup i' \in w)\}.$$

($U\, i\; \phi(i)$ means: $\{i \in I : \phi(i)\} \in U$, whenever U is an ultrafilter over a set I. Here $I = C^{\text{fin}}$ for the quantifier $U\, i$ and $I' = C'^{\,\text{fin}}$ for the quantifier $U' i'$.) Then $W = U * U'$ is a $(C \cup C')$-adequate ultrafilter, $W \restriction C = U$, $W \restriction C' = U'$, and $W\, j\; \phi(j)$ is equivalent to $U\, i\; U' i'\; \phi(i \cup i')$. $\dashv$

Operation 2. Assume that λ is a limit ordinal, U_α is a C_α-adequate ultrafilter for all $\alpha < \lambda$, $C_\alpha \subseteq C_\beta$ and $U_\alpha = U_\beta \restriction C_\alpha$ whenever $\alpha < \beta$. Let $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$. Then there exists a C_λ-adequate ultrafilter U_λ such that $U_\alpha = U_\lambda \restriction C_\alpha$ for all ordinals $\alpha < \lambda$. $\dashv$

Let $\Xi = \kappa^+$; thus $\Xi \in \mathbb{S}$ and it is true in $\mathbb{I}$ and $\mathbb{S}$ that Ξ is the least cardinal greater than κ. There exists a (unique) standard increasing sequence $\langle \Omega_\alpha : \alpha < \Xi \rangle$ of ordinals $\Omega_\alpha < \Xi$ such that each set $D_\alpha = \Omega_{\alpha+1} \setminus \Omega_\alpha$ has order type κ and $\Omega_\lambda = \lim_{\alpha<\lambda} \Omega_\alpha$ for limit ordinals λ.

Operations 1 and 2 allow to define a standard (by Transfer) Ξ-adequate ultrafilter $U \subseteq \mathcal{P}(\Xi^{\text{fin}})$ such that $U_{\alpha+1} = U^\alpha * U_\alpha$, where $U_\alpha = U \restriction \Omega_\alpha$ and $U^\alpha = U \restriction D_\alpha$ — for all α. Such an ultrafilter is fixed for the sequel. We put $I_\alpha = {\Omega_\alpha}^{\text{fin}} = \{i \subseteq \Omega_\alpha : i \text{ finite}\}$ and

$$\mathcal{F}_\alpha = \{f : f \text{ is a function defined on } I_\alpha \text{ with arbitrary values}\}$$

for all $\alpha < \Xi$, and $\mathcal{F} = \bigcup_{\alpha<\Xi} \mathcal{F}_\alpha$, $I = I_\Xi = \Xi^{\text{fin}}$. Take notice that if $f \in \mathcal{F}$ is standard then $f \in \mathcal{F}_\alpha$ for a *standard* $\alpha < \Xi$ by Transfer.

By Bounded Idealization **BI** there exists $\mathbf{i} \in I$ which belongs to every *standard* set $u \in U$. (So $\mathbf{i}$ belongs to the *monad* of U.) We fix $\mathbf{i}$ and put

$$\mathbb{J}_\alpha = \{f[\mathbf{i}] : f \in \mathcal{F}_\alpha \text{ is standard}\} \qquad \text{and} \qquad \mathbb{I}'_\kappa = \mathbb{J}_{<\Xi} = \bigcup_{\alpha<\Xi} \mathbb{J}_\alpha\,,$$

where $f[\mathbf{i}] = f(\mathbf{i} \cap \Omega_\alpha)$ for $f \in \mathcal{F}_\alpha$. Thus $\mathbb{I}'_\kappa$ is a st-$\in$-definable class in $\mathbb{I}$, the ground **BST** universe.

The following theorem shows that the universe $\mathbb{I}'_\kappa = \langle \mathbb{I}'_\kappa; =, \in, \text{st} \rangle$ has properties rather similar to those of the universe $\mathbb{I}_\kappa$ of Section 2, although it models a somewhat weaker theory than $\mathbb{I}_\kappa$ does. One more difference is that $\mathbb{I}'_\kappa$ is not uniquely defined; the definition depends on the choice of the ultrafilter U and the particular element $\mathbf{i}$.

THEOREM 4.1 *$\mathbb{I}'_\kappa$ contains all standard sets. Furthermore,*

1. *$\mathbb{I}'_\kappa$ is an elementary submodel of $\mathbb{I}$ with respect to all $\in$-formulas.*
2. *$\mathbb{I}_\kappa$ satisfies $\mathbf{BST}'_\kappa$, the theory containing all axioms of $\mathbf{ZFC}$ (in the $\in$-language), Transfer, Standardization, and the κ-forms $\mathbf{IS}_\kappa$ of Internal Saturation and $\mathbf{B}_\kappa$ of Boundedness.*[9]
3. *If $g \in \mathbb{I}$ is a function defined on a standard set X, $\operatorname{card} X \le \kappa$ in $\mathbb{S}$, and $g(x) \in \mathbb{I}'_\kappa$ for all standard $x \in X$, then there exists a function $g' \in \mathbb{I}'_\kappa$ such that $g(x) = g'(x)$ for all standard $x \in X$.*

PROOF. The inclusion $\mathbb{S} \subseteq \mathbb{I}'_\kappa$ is obvious.

1. We prove the property of being an elementary submodel. It suffices to verify that every $\mathbb{J}_\alpha$, $\alpha < \Xi$, is an elementary submodel of $\mathbb{I}$. The proof proceeds by induction on the complexity of the involved $\in$-formulas. The step for $\exists$ is the only one which needs a special consideration.

Thus let $\Phi(x, y)$ be an $\in$-formula (all free variables indicated), the ordinal $\alpha < \Xi$ and $f \in \mathcal{F}_\alpha$ be standard. Assume that $\exists\, x\, \Phi(x, f[\mathbf{i}])$ holds in $\mathbb{I}$. The goal is to find $x \in \mathbb{J}_\alpha$ satisfying $\Phi(x, f[\mathbf{i}])$. One may assume that $\exists\, x\, \Phi(x, y)$ holds in $\mathbb{I}$ for *all* y. (Otherwise replace Φ by the formula $\Phi(x,y) \bigvee [\, x = \emptyset \;\&\; \neg \exists\, x'\, \Phi(x', y)\,]$.) By Transfer, $\forall\, i \in I_\alpha\, \exists\, x\, \Phi(x, f(i))$ is true in $\mathbb{S}$, therefore there exists a standard function $g \in \mathcal{F}_\alpha$ such that $\Phi(g(i), f(i))$ is true in $\mathbb{I}$ for all $i \in I_\alpha$. Take $x = g[\mathbf{i}] = g(\mathbf{i} \cap \Omega_\alpha)$.

2. Thus we have all of $\mathbf{ZFC}$ and Transfer in $\mathbb{I}'_\kappa$. Standardization holds in $\mathbb{I}'_\kappa$ since $\mathbb{S} \subseteq \mathbb{I}'_\kappa$. We check $\mathbf{B}_\kappa$. Let $x \in \mathbb{I}'_\kappa$. There exist a standard ordinal $\alpha < \Xi$ and a standard function $f \in \mathcal{F}_\alpha$ such that $x = f[\mathbf{i}]$. Hence x belongs to $X = \{f(i) : i \in I_\alpha\}$, a standard set of cardinality $\le \kappa$.

The continuation of the proof involves a Łoś–like lemma.

LEMMA 4.2 *Let $\varphi(x_1, ..., x_n)$ be an $\in$-formula, $f_1, ..., f_n \in \mathcal{F}$ and $\alpha < \Xi$ all be standard, and every f_k belongs to some $\mathcal{F}_{\alpha'}$, $\alpha' = \alpha'(k) \le \alpha$. Then the following is true in $\mathbb{I}$:*

$$\varphi(f_1[\mathbf{i}], ..., f_n[\mathbf{i}]) \;\longleftrightarrow\; U\, i\; \varphi(f_1[i], ..., f_n[i]) \;\longleftrightarrow\; U_\alpha\, i\; \varphi(f_1[i], ..., f_n[i]).$$

(We recall that $U\, i\; \psi([i])$ and $U_\alpha\, i\; \psi(i)$ mean that $\{i \in I : \psi(i \cap \Omega_\alpha)\} \in U$ and $\{i \in I_\alpha : \psi(i)\} \in U_\alpha$ respectively.)

[9] This is weaker than $\mathbf{BST}_\kappa$ of item 2 of Theorem 2.2: $\mathbf{BI}_\kappa$, the κ-form of Bounded Idealization is absent. In principle we expect a weaker version of $\mathbf{BI}$ by Proposition 1.2, say $\mathbf{BI}_\lambda$ provided $2^\lambda \le \kappa$, but it is not clear whether $\mathbf{BI}_\kappa$ itself holds in $\mathbb{I}'_\kappa$.

Proof. Either the set $X = \{i \in I_\alpha : \varphi(f_1[i], ..., f_n[i])\}$ or the complement $X^c = I_\alpha \setminus X$ belong to U_α. If $X \in U_\alpha$ then both the left–hand and the right–hand side are true, while in the case $X^c \in U_\alpha$ both of them are false. (Indeed, $i = \mathbf{i} \cap \Omega_\alpha \in X^c$.) The expression in the middle is equivalent to the right–hand one because $U \restriction \Omega_\alpha = U_\alpha$ and all f_i belong to $\mathcal{F}_{\leq\alpha}$. ■

We prove that $\mathbf{IS}_\kappa$, Internal Saturation in the case when the standard set A_0 satisfies $\operatorname{card} A_0 \leq \kappa$ in $\mathbb{S}$, holds in $\mathbb{I}'_\kappa$. One may assume that A_0 is equal to $D_\alpha = \Omega_{\alpha+1} \setminus \Omega_\alpha$ for a standard $\alpha < \Xi$, and all parameters in the $\in$-formula $\Phi(x, a)$ belong to $\mathbb{J}_\alpha$. Let, to simplify the notation, $\Phi(x, a)$ contain a single parameter $p = f[\mathbf{i}]$, where $f \in \mathcal{F}_\alpha$ is standard. Thus Φ is $\Phi(x, a, p)$, and $\mathbf{IS}$ takes the form

$$\forall^{\text{stfin}} A \subseteq D_\alpha\, \exists\, x\, \forall\, \delta \in A\; \Phi(x, \delta, p) \;\longleftrightarrow\; \exists\, x\, \forall^{\text{st}} \delta \in D_\alpha\; \Phi(x, \delta, p)\,.$$

It suffices to prove the direction $\longrightarrow$. Assume that the left–hand side is true in $\mathbb{I}'_\kappa$. Then it holds in $\mathbb{I}$ as well by the elementary submodel property. (All standard sets belong to $\mathbb{I}'_\kappa$.) Therefore by Lemma 4.2 we have

$$\forall^{\text{stfin}} A \subseteq D_\alpha\; U_\alpha\, i\, \exists\, x\, \forall\, \delta \in A\; \Phi(x, \delta, f(i))$$

in $\mathbb{I}$. Then $U^\alpha\, i'\, U_\alpha\, i\, \exists\, x\, \forall\, \delta \in i'\; \Phi(x, \delta, f(i))$, so that, by the choice of U,

$$U_{\alpha+1}\, j\, \exists\, x\, \forall\, \delta \in \mathcal{A}(j)\; \Phi(x, \delta, f[j])\,,$$

where $\mathcal{A} \in \mathcal{F}_{\alpha+1}$ is defined by $\mathcal{A}(j) = j \cap D_\alpha$. Let $A = \mathcal{A}[\mathbf{i}] = \mathbf{i} \cap D_\alpha$; obviously $A \in \mathbb{I}'_\kappa$. Again by Lemma 4.2, we obtain $\exists\, x\, \forall\, \delta \in A\; \Phi(x, \delta, p)$ in $\mathbb{I}$. Since $\mathbb{I}'_\kappa$ is an elementary submodel, such a set x exists in $\mathbb{I}'_\kappa$.

It remains to verify that $\mathbf{i}$ contains all standard elements of D_α. Let $\delta \in D_\alpha$ be standard. We observe that $U_{\alpha+1}\, i\; (\delta \in i)$ because $U_{\alpha+1}$ is $\Omega_{\alpha+1}$-adequate. Therefore $\delta \in \mathbf{i}$ by Lemma 4.2.

3. We prove item 3 of the theorem. Let S be a standard set of cardinality κ in $\mathbb{S}$ and g an internal function defined on S and satisfying $g(x) \in \mathbb{I}'_\kappa$ for all standard $x \in S$. We have to find a function $g' \in \mathbb{I}'_\kappa$ which coincides with g on the standard elements of S .

Since Ξ has cofinality $> \kappa$, we have a standard ordinal $\alpha < \Xi$ such that for any standard $x \in S$ there exists a standard function $f \in \mathcal{F}_\alpha$ such that $g(x) = f[\mathbf{i}]$. Using Standardization, we get a standard map $H : S \longrightarrow \mathcal{F}_\alpha$ such that for every standard $x \in S$ the function $h_x = H(x) \in \mathcal{F}_\alpha$ satisfies $g(x) = h_x[\mathbf{i}]$. Let, for $i \in I_\alpha$, f_i be a function defined on S by $f_i(x) = h_x(i)$. Then $F(i) = f_i$ is a standard function in $\mathcal{F}_\alpha$. Let $g' = F[\mathbf{i}]$; thus $g' \in \mathbb{I}'_\kappa$. Then $g'(x) = F[\mathbf{i}](x) = h_x[\mathbf{i}] = g(x)$ for all standard $x \in S$. ■

4.2. Standard size partially saturated external universe

The class $\mathbb{I}'_\kappa$ will be the internal base for an external subuniverse of $\mathbb{H}$. This is close to the approach of Section 3. However there is an essential difference: we use the "standard size" version of Definition 3.1 to assemble sets rather than the $\mathbb{E}_\kappa$-definable version applied in the previous section. Fortunately this makes the construction essentially easier.

We argue in the **HST** universe $\mathbb{H}$ in this subsection.

To keep extensionality, a definition similar to Definition 3.6 is necessary.

DEFINITION 4.3 1. Let $\langle T, F\rangle \in \mathcal{H}$. $t \in T$ is a *κ-illegitimate point* in T if there exists $I \in \mathbb{I}'_\kappa$ such that $I \cap \mathbb{I}'_\kappa = F_T(t) \neq I$.

2. $\mathcal{H}'_\kappa$ is the collection of all $\langle T, F\rangle \in \mathcal{H}$ such that both T and F are subsets of $\mathbb{I}'_\kappa$, and T does not contain κ-illegitimate points.

3. $\mathbb{H}'_\kappa = \{F[T] : \langle T, F\rangle \in \mathcal{H}'_\kappa\}$.

The treatment of $\mathbb{H}'_\kappa$ is quite similar to the development of $\mathbb{H}_\kappa$ in Section 3. One detail differs: we have to guarantee that the trees obtained by this or another transformation keep standard size rather than membership in $\mathbb{E}_\kappa$, which facilitates the reasoning a great deal.

The following lemma stands behind the technical treatment of $\mathbb{H}'_\kappa$. Proving the lemma, we shall also see that the sets of standard size have power sets, also of standard size, in **HST**, which will later reduce the verification of the Power set axiom in $\mathbb{H}'_\kappa$ to a triviality. [10]

LEMMA 4.4 *Let $X \subseteq \mathbb{H}'_\kappa$. Then X is a set of standard size.*

PROOF. *Fact 1.* We prove that at least each set $X \subseteq \mathbb{I}'_\kappa$ has standard size. Indeed, by definition every $x \in X$ has the form $x = f[\mathrm{i}]$ for a standard $f \in \mathcal{F}$. By the **HST** Collection in $\mathbb{H}$ and Lemma 3.11 there exists a standard set $F \subseteq \mathcal{F}$ such that every $x \in X$ is equal to $f[\mathrm{i}]$ for a standard function $f \in F$. Thus X is an image of the set ${}^\sigma F = \{f \in F : \mathrm{st}\, f\}$.

Fact 2. We prove (in **HST**) that if Y is a set of standard size then the (external) power set $\mathcal{P}(Y)$ exists and is also a set of standard size. It suffices to consider the case when $Y = {}^\sigma S$ and S is standard. Let P be the "standard" power set of S, that is, the power set taken in the standard universe $\mathbb{S}$. We put $P' = \{{}^\sigma U : U \in {}^\sigma P\}$. Thus P' is a set of standard

[10] This way of reasoning was suggested by the referee. Our original approach was more cumbersome.

size. Furthermore, by the **HST** Standardization P' contains all subsets of ${}^\sigma S$ and only those sets, that is, $P' = \mathcal{P}({}^\sigma S)$ in $\mathbb{H}$, as required.

We prove the lemma. Thus suppose that $X \subseteq \mathbb{H}'_\kappa$. By definition and the **HST** Collection, there exists a standard set S such that every $x \in X$ has the form $F[T]$ for a wf pair $\langle T, F\rangle$ satisfying $T \cup F \subseteq S' = S \cap \mathbb{I}'_\kappa$. We observe that S' is a set of standard size by Fact 1 and $\mathcal{P}(S')$ exists and is a set of standard size by Fact 2. ∎

LEMMA 4.5 *Let $\langle T, F\rangle \in \mathcal{H}'_\kappa$ and $t \in T$. Then $F_T(t) \in \mathbb{H}'_\kappa$. If in addition $F_T(t)$ is internal then $F_T(t) \in \mathbb{I}'_\kappa$.*

PROOF. The first assertion is quite obvious. We verify the additional part. If $t \in \operatorname{Max} T$ then $F_T(t) = F(t) \in \mathbb{I}'_\kappa$ by definition. The wellfoundedness assumption allows to use induction. Thus it suffices to prove the following: every internal set $X \subseteq \mathbb{I}'_\kappa$ belongs to $\mathbb{I}'_\kappa$.

By Lemma 4.4, X is a set of standard size. But, in **HST** an internal set of standard size is necassarily a set of standard finite number of elements by Saturation. Finally every subset of $\mathbb{I}'_\kappa$ having a standard finite number of elements belongs to $\mathbb{I}'_\kappa$ by Theorem 4.1 (item 3). ∎

The following theorem shows that the system of subuniverses $\mathbb{H}'_\kappa$ has a certain advantage with respect to the universes $\mathbb{H}_\kappa$ of Section 3: full Choice is achieved. But we should not forget a certain loss, too: first, Bounded Idealization $\mathbf{BI}_\kappa$ in the internal subuniverse $\mathbb{I}'_\kappa$ is missing, second the subuniverses are no longer unique.

THEOREM 4.6 [**HST**] $\mathbb{I}'_\kappa = \mathbb{I} \cap \mathbb{H}'_\kappa$. *Hence $\mathbb{I}'_\kappa$ is the class of all formally internal sets in $\mathbb{H}'_\kappa$. In addition, the following axioms hold in $\mathbb{H}'_\kappa$:*

1. *The axioms of Pair, Union, Extensionality, Infinity, together with Collection, Separation, Replacement for* st-$\in$*-formulas.*
2. *Extension in the form of item 4 in Subsection 1.3. for sets S satisfying $\operatorname{card} S \le \kappa$ in $\mathbb{S}$.*
3. *Saturation for sets X of standard κ-size, and the full Choice: every set has standard size and is well-orderable.*
4. *Weak Regularity.*
5. *The* **ZFC** *Power set axiom.*

Finally $\mathbb{H}'_\kappa$ satisfies the following closure property: if $X \subseteq \mathbb{H}'_\kappa$ then either $X \in \mathbb{H}'_\kappa$ or there exists $Y \in \mathbb{I}'_\kappa$ such that $X = Y \cap \mathbb{H}'_\kappa = Y \cap \mathbb{I}'_\kappa$.

PROOF. Lemma 4.5 proves the equality $\mathbb{I} \cap \mathbb{H}'_\kappa = \mathbb{I}'_\kappa$. $\dashv$

The closure property. Since X has standard size by Lemma 4.4, one gets, using the **HST** standard size Choice, a standard set A and sequences $\langle T^a : a \in {}^\sigma\!A\rangle$ and $\langle F^a : a \in {}^\sigma\!A\rangle$ such that $X = \{F^a[T^a] : a \in {}^\sigma\!A\}$ and $\langle T^a, F^a\rangle \in \mathcal{H}'_\kappa$ for all standard $a \in A$. Let $\langle T, F\rangle \in \mathcal{H}$ be defined by

$$T = \{\Lambda\} \cup \bigcup\{a^\wedge t : a \in {}^\sigma\!A \ \&\ t \in T^a\}$$

and $F(a^\wedge t) = F^a(t)$ for $t \in \operatorname{Max} T^a$. We observe that $X = F[T]$, and the wf pair $\langle T, F\rangle$ meets all the requirements of the definition of $\langle T, F\rangle \in \mathcal{H}'_\kappa$ except, perhaps, that Λ can be κ-illegitimate. If Λ is *not* κ-illegitimate then $\langle T, F\rangle \in \mathcal{H}'_\kappa$ and $X = F[T] \in \mathbb{H}'_\kappa$ as required. Suppose that Λ *is* κ-illegitimate. Then by definition there exists $Y \in \mathbb{I}'_\kappa$, hence $\in \mathbb{H}'_\kappa$, such that $X = Y \cap \mathbb{I}'_\kappa$, as required. $\dashv$

Extensionality. We argue as in the proof of Theorem 3.10. Assume that $\langle T, F\rangle$ and $\langle R, G\rangle$ belong to $\mathcal{H}'_\kappa$ and the sets $X = F[T]$ and $Y = G[R]$ satisfy $X \cap \mathbb{H}'_\kappa = Y \cap \mathbb{H}'_\kappa$; we prove $X = Y$. The case when neither of the trees T, R is equal to $\{\Lambda\}$ is easy: then both X and Y are subsets of $\mathbb{H}'_\kappa$. Let $T = \{\Lambda\}$, so that $X = F(\Lambda) \in \mathbb{I}'_\kappa$. If $R = \{\Lambda\}$, too, then $Y \in \mathbb{I}'_\kappa$ as well and the equality $X = Y$ follows from the fact that $\mathbb{I}'_\kappa$ is an elementary submodel of $\mathbb{I}$ by Theorem 4.1. Thus assume that $R \neq \{\Lambda\}$. Then $Y = X$ because Λ is not a κ-illegitimate point of R. $\dashv$

Extension and Saturation. The standard κ-size form of Extension follows from the **HST** Extension and item 3 of Theorem 4.1. Using this, one easily obtains the κ-size form of Saturation in $\mathbb{H}'_\kappa$ from the Internal Saturation $\mathbf{IS}_\kappa$ (true in $\mathbb{I}'_\kappa$ by Theorem 4.1). $\dashv$

Routine axioms. We reduce the axioms of Pair, Infinity, Weak Regularity, Union, Separation, Collection, and Replacement in $\mathbb{H}'_\kappa$ to the ground **HST** universe using the closure property. $\dashv$

Standard size and well-ordering. It follows from Lemma 4.4 and the closure property that every set has standard size in $\mathbb{H}'_\kappa$. This suffices to conclude that every set is well-orderable in $\mathbb{H}'_\kappa$, and therefore $\mathbb{H}'_\kappa$ satisfies the full Choice. (Basically $\mathbb{H}'_\kappa$ still only satisfies the standard size Choice — but all sets have standard size in $\mathbb{H}'_\kappa$!) $\dashv$

Power set. We simply use Fact 2 in the proof of Lemma 4.4 plus the closure property. $\dashv$

This ends the proof of the theorem. ∎

REMARK. We observe that the Separation, Collection, and Replacement schemata hold in $\mathbf{H}'_\kappa$ also in a form which does not presuppose that the core formula is relativized to $\mathbf{H}'_\kappa$. For instance, it follows from the closure property and Lemma 4.4 that for any $X \in \mathbf{H}'_\kappa$ and any st-$\in$-formula $\Phi(x)$ which may contain arbitrary sets in $\mathbf{H}$ (not only in $\mathbf{H}'_\kappa$) as parameters, the set $Y = \{x \in X \cap \mathbf{H}'_\kappa : \Phi(x) \text{ is true in } \mathbf{H}\}$ belongs to $\mathbf{H}'_\kappa$.

Thus, modulo the fact that the classes $\mathbf{I}'_\kappa$ and $\mathbf{H}'_\kappa$ are not uniquely defined, one can conclude that the system of subuniverses $\mathbf{I}'_\kappa$ and $\mathbf{H}'_\kappa$ models the stratified theory **SNST** of Fletcher [4] in the upgraded form which includes κ-size Saturation in the external subuniverses $\mathbf{H}'_\kappa$ rather than simply κ-size Internal Saturation $\mathbf{IS}_\kappa$ in the internal subuniverses $\mathbf{I}'_\kappa$.

5. Discussion

This section is written to explain, in brief, how the methods of the paper can be used to model various forms of nonstandard reasoning on the base of bounded set theory **BST**. It is not our intension here to give a practically useful introduction in this matter – this would need more space and another style of writing than the frameworks of this paper allow. We rather suggest an "intoduction to an introduction", which intends to show how the technique developed in the paper allows to realize the known patterns of reasoning in nonstandard set universes with external sets given by Hrbaček [6, 7] and Kawaï [14], but still on the base of **BST**, a nonstandard set theory of "internal" type similar to **IST**.

Let us take as an example some topics related to descriptive set theory on hyperfinite sets: the Loeb measure and Borel sets. These topics do not seem to be easily carried out in a theory which involves only internal sets, like **BST** in its straightforward setting or **IST**.

Let us first review how these topics are usually handled via nonstandard structures in the **ZFC** universe. [11] One considers subsets of a set S of the form $S = \{1, 2, 3, ..., H\}$, where H is a hyperfinite number in a fixed nonstandard structure. Some of them are called *internal* – those presented in the internal part of the superstructure.

If one is interested in Borel sets, one puts $\mathbf{\Pi}_0 = \mathbf{\Sigma}_0 =$ all internal subsets of S. Then one runs the known definition:

[11] Diener and Reeb [3], Henson and Keisler [5], Hurd and Loeb [8], Keisler [15], Keisler, Kunen, Miller, and Leth [16], Lindstrøm [17], Lutz and Goze [18], Luxemburg [19], Stroyan and Bayod [24] are suggested as the basic references on the matter of "superstructural" treatment of nonstandard analysis.

$$\begin{array}{lcl} \mathbf{\Sigma}_\alpha & = & \text{countable unions of sets in } \bigcup_{\gamma<\alpha}\mathbf{\Pi}_\gamma \\ \mathbf{\Pi}_\alpha & = & \text{complements of sets in } \mathbf{\Sigma}_\alpha \end{array}$$

for all countable ordinals α. If one is interested in Loeb measures, one proceeds the following way. Each internal set $X \subseteq S$ has a certain (hyperfinite or finite) number of elements $\#X \le H$. The hyperrational fraction

$$\mu X = \frac{\#X}{H}$$

is called the *counting measure* of X. There exists a unique standard real number $x = {}^\circ\mu X$, called the *standard part* of μX (and more often denoted by $\mathrm{st}\,\mu X$, but this notation is here occupied by the standardness predicate), such that $x \approx \mu X$. ($a \approx b$ means that the difference $a-b$ is infinitesimal.) If a set $Z \subseteq S$ (not necessarily internal) satisfies

$$\sup\{{}^\circ\mu X : X \subseteq Z \text{ is internal}\} = \inf\{{}^\circ\mu X : Z \subseteq X \subseteq S,\ X \text{ is internal}\}$$

then Z is called *Loeb measurable*, and the value determined by the displayed formula is called the *Loeb measure* of X, in symbols $L\mu(X)$.

To run these constructions on the base of **BST**, we first of all consider the **HST** enlargement $\mathbf{H}$ of the ground **BST** universe $\mathbf{I}$, where $\mathbf{I}$ is the subuniverse of all internal sets, as in [13]. Take notice that essentially $\mathbf{H}$ is (isomorphic to) a st-$\in$-definable structure in $\mathbf{I}$; therefore all sorts of activity related to $\mathbf{H}$ can be embedded down to $\mathbf{I}$ — in other words one does not leave the framework of **BST**.

Then we have to choose, in $\mathbf{I}$, an infinite standard cardinal κ, which is the desired amount of saturation. Let us put, for instance, $\kappa = 2^{2^{\aleph_0}}$, the cardinality of the *hypercontinuum*.

At this point the construction splits into two directions.

One may choose the *natural* system of subuniverses, described in sections 2 and 3. Then the choice of κ also guarantees that every internal set X of subsets of $\mathbb{N}$ belongs to $\mathbf{I}_\kappa$, just because $X \in \mathcal{P}(\mathcal{P}(\mathbb{N}))$, which is a standard set of cardinality κ. All internal real numbers and all internal sets of real numbers also belong to $\mathbf{I}_\kappa$. (If we need to guarantee more internal sets not to be missed, or more Saturation, we can increase κ as desired.)

Then one begins to argue in $\mathbf{H}_\kappa$, a subuniverse of $\mathbf{H}$ defined in Section 3. This class satisfies the axioms indicated in Theorem 3.10, in particular, the Power set axiom. Therefore, the external power set $P = \mathcal{P}_{\mathrm{ext}}(\mathbb{N})$ exists in $\mathbf{H}_\kappa$. ($\mathcal{P}_{\mathrm{ext}}$ is the usual power set defined in $\mathbf{H}_\kappa$; we adjoin the subscript ext to make a distinction from the internal power $\mathcal{P}(\mathbb{N})$ defined in $\mathbf{I}$.)

From the point of view of $\mathbb{I}$, P contains all internal subsets of $\mathbb{N}$, and a suitable part of st-$\in$-definable subclasses of $\mathbb{N}$, namely, those which can be defined in $\mathbb{I}$ by st-$\in$-formulas having only sets in $\mathbb{I}_\kappa = \mathbb{I} \cap \mathbb{H}_\kappa$ as parameters (but no restriction on the *type* of formulas involved is imposed).

Alternatively, one may choose the *standard size* system of subuniverses, described in Section 4. Then one begins to argue in $\mathbb{H}'_\kappa$, a subuniverse satisfying Theorem 4.6, in particular, the Power set and Choice axioms. But in this case, not all internal natural numbers belong to $\mathbb{I}'_\kappa = \mathbb{H}' \cap \mathbb{I}$.

In both the natural and the standard size cases, one has the standard κ-size Saturation in the chosen subuniverse, either $\mathbb{H}_\kappa$ or $\mathbb{H}'_\kappa$. Take notice that, due to a $1-1$ correspondence between the standard cardinals and the cardinals in the sense of $\mathbb{H}$ described in [13], the saturation property is in fact Saturation for the families (of internal sets) having cardinality less than the κ th infinite **ZFC**-cardinal in $\mathbb{H}$. In addition, one has all of **ZFC** in the subuniverse with the exception of Choice for the natural subuniverse (but Choice for standard size families is secured) and Regularity (but regularity over the internal subuniverse is secured) — which is quite sufficient to develop things like Loeb measures or Borel sets.

Let us see how one proceeds with this matter practically.

Arguing in the chosen subuniverse, one picks up a nonstandard (= hyperfinite) natural number H and then freely runs the definitions of the Borel hierarchy, starting from internal subsets of the set $S = \{1, 2, ..., H\}$ and the Loeb measure, because the tools which theorems 3.10 and 4.6 prove to be available in the subuniverses $\mathbb{H}_\kappa$ are strong enough to conduct these constructions exactly as they are carried out in the usual setting.

Perhaps one detail, related to the Standardization axiom, is worth to be especially indicated. One may ask why the $\mathtt{sup}$ and $\mathtt{inf}$ in the definition of the Loeb measure do exist. Thus the problem is how to prove in **HST** that every nonempty set X of standard real numbers, bounded by a standard real from above, has a standard supremum.

First of all, let us agree that the equality $x' = \sup X$ (where x is a standard real while X a nonempty set of standard reals) will be understood in **HST** so that $x \le x'$ for all $x \in X$ and no smaller *standard* real $x'' < x'$ satisfies the same property with respect to X .

To prove the assertion, let, by Standardization, Y be a standard set of reals such that $Y \cap \mathbb{S} = X \cap \mathbb{S}$ — that is, simply $X = {}^\sigma Y = Y \cap \mathbb{S}$ in this case. The standard set Y has the supremum $y = \sup Y \in \mathbb{S}$ in $\mathbb{S}$ since $\mathbb{S}$ is a **ZFC** universe. One easily proves that $y = \sup X$ in the external universe because $X = {}^\sigma Y$.

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