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Extending Standard Models of ZFC to Models of Nonstandard set Theories

Abstract. We study those models of **ZFC** which are embeddable, as the class of all standard sets, in a model of internal set theory **IST** or models of some other nonstandard set theories.

Key words: nonstandard set theories, standard sets, models of **ZFC** .

Introduction

In the early 60s Abraham Robinson demonstrated that nonstandard models of natural and real numbers could be used to interpret the basic notions of analysis in the spirit of mathematics of the 17-th and 18-th century, *i.e.*, including infinitesimal and infinitely large quantities.

Nonstandard analysis, the field of mathematics which has been initiated by Robinson's idea, develops in two different versions.

The *model theoretic* version, following the original approach, interprets "nonstandard" notions via nonstandard models in the **ZFC** universe.

On the other hand, the *axiomatic* version more radically postulates that the whole universe of sets (including all mathematical objects) is arranged in a "nonstandard" way, so that it contains both the objects of conventional, "standard" mathematics, called standard, and objects of different nature, called nonstandard. The latter type includes infinitesimal and infinitely large numbers, among other rather unusual objects.

Each of the two versions has its collective of adherents who use it as a working tool to develop nonstandard mathematics.

Many of those who follow the axiomatic version use *internal set theory* **IST** of Nelson [9] as the basic set theory. This is a theory in the $\mathtt{st}$-$\in$-language (that is, the language $\mathcal{L}_{\in,\mathtt{st}}$ containing the membership $\in$ and the unary predicate of standardness $\mathtt{st}$ as the only atomic predicates) which includes all axioms of **ZFC** in the $\in$-language together with three principles that govern the interactions between *standard* (*i.e.* those sets x which satisfy $\mathtt{st}\,x$) and *nonstandard* objects in the set universe.

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Some other nonstandard axiomatic set theories, introduced by Hrbaček [2, 3] and Kawaï [8], arrange the nonstandard "universe of discourse" in somewhat different way (see below) but all of them have the predicate $\mathtt{st}$ in the language and do not go essentially beyond **ZFC** in the sense that

⟨I⟩ they prove those and only those $\in$-sentences to be true in the class $\mathbb{S} = \{x : \mathtt{st}\, x\}$ of all standard sets, which are theorems of **ZFC**.

This property of "conservativity" may be considered as a reason to view **IST** and similar theories as a syntactical tool of proving **ZFC** theorems often in a more convenient way than "standard" mathematics allows.

However, working with a nonstandard set theory, one should be interested to know whether its axioms reflect some sort of mathematical reality.

- Let us say that a transitive model $\mathfrak{M} = \langle \mathfrak{M}; \in \rangle \models \mathbf{ZFC}$ *extends to a model of* a nonstandard set theory $\mathbf{T}$ iff there is a model $\mathfrak{N} \models \mathbf{T}$ such that the class of all standard sets of $\mathfrak{N}$ is isomorphic to $\mathfrak{M}$.

In view of the above, one could expect that the relations between **ZFC** and *e.g.* **IST** are similar to those between the real line and the complex plane, so that each model of **ZFC** extends to a model of **IST**. However this is not the case: we demonstrated in [5] that minimal $\in$-models of **ZFC** do not extend to a model of **IST**. This observation leads to the question:

- Let $\mathbf{T}$ be a nonstandard set theory, *e.g.*, **IST**. *Which "standard" models* (*i.e.*, transitive $\in$-models) *of* **ZFC** *extend to models of* $\mathbf{T}$ *?*

The aim of this paper is to answer this question for several nonstandard set theories. We shall see that, although ⟨I⟩ is their common property, there are striking differences in what they require from a standard model of **ZFC** to be extendible. This observation could perhaps lead to new insights into the philosophy of nonstandard mathematics.

1. Main results

Unfortunately, a complete description of "standard" models of **ZFC**, extendible to **IST**, has not yet been obtained. Our first main theorem solves this problem for $\mathbf{IST}^+$, an extension of **IST**.

DEFINITION 1. $\mathbf{IST}^+$ is the extension of **IST** by the following axiom:

Locally Standard Choice: There is a function $\mathbf{C}$ such that $\mathbb{S} \subseteq \mathtt{dom}\,\mathbf{C}$ and, for any non-empty standard x, $\mathbf{C}(x) \in x$ and $\mathbf{C}(x)$ is standard. □

This does not seem to be an essential strengthening, at least $\mathbf{IST}^+$ still satisfies (I) and follows from **ISTGC**, a "global choice" version of **IST**. See more comprehensive discussion in the beginning of Section 2.

Recall that **ZFGC** (with Global Choice) is the theory in the language $\mathcal{L}_{\in,<}$, containing all of the **ZFC** axioms (with the schemata of Separation and Replacement in $\mathcal{L}_{\in,<}$), together with the axiom saying that $<$ wellorders the universe in such a way that each initial segment is a set.

Suppose that $\mathfrak{M} = \langle \mathfrak{M}; ... \rangle$ is a transitive model of a theory including **ZF**. A family of sets $T_1, ..., T_n \subseteq \mathfrak{M}$ will be called *innocuous* for $\mathfrak{M}$ iff the structure $\langle \mathfrak{M}; ..., T_1, ..., T_n \rangle$ models Separation in the language which extends the language of the signature of $\mathfrak{M}$ by T_1 , ..., T_n as extra predicates.

Note that every $\mathcal{L}_{\in,<}$-formula having sets in $\mathfrak{M}$ as parameters can be naturally considered as an element of $\mathfrak{M}$. Let $\mathtt{Truth}^{\mathfrak{M}}_{\in,\prec}$ denote the set of all closed $\mathcal{L}_{\in,<}$-formulas, which are true in $\langle \mathfrak{M}; \in, \prec \rangle$, so that $\mathtt{Truth}^{\mathfrak{M}}_{\in,\prec} \subseteq \mathfrak{M}$.

$\mathtt{Truth}^{\mathfrak{M}}_{\in}$ will have similar meaning (the set of all $\in$-formulas ...).

THEOREM 2. *For any transitive model $\mathfrak{M} \models$ **ZFC**, the following conditions are equivalent:*[1]

(1) *$\mathfrak{M}$ extends to a model of $\mathbf{IST}^+$;*

(2) *$\mathfrak{M}$ extends to a model of **ISTGC** ;*

(3) *there is a well-ordering $\prec$ of $\mathfrak{M}$, such that the structure $\langle \mathfrak{M}; \in, \prec \rangle$ models **ZFGC** and $\mathtt{Truth}^{\mathfrak{M}}_{\in,\prec}$ is innocuous for $\langle \mathfrak{M}; \in, \prec \rangle$.*

The proof of $(3) \Longrightarrow (2)$ (Subsection 2.2) is a modification of the original construction of an **IST** model by Nelson [9]. Direction $(1) \Longrightarrow (3)$ (Subsection 2.1) is more interesting: the role of the truth relation is somewhat surprising. (However see Lemma 5.)

The other theories we deal with in this paper, **NST** of Hrbaček [2, 3] and **KST** of Kawaï [8], differ from **IST** or $\mathbf{IST}^+$ in the manner how they arrange the "universe of discourse". While **IST** sees it as an elementary extension of the standard subuniverse $\mathbb{S}$ in the $\in$-language, **NST** and **KST** view things so that the universe contains an intermediate transitive

[1] $\mathbf{IST}^+$-extendable transitive models of **ZFC** admit another characterization. Let $\langle \mathfrak{M}; \in, \prec \rangle$ be a model of **ZFGC**. Say that a collection $\mathfrak{X}$ of subsets of $\mathfrak{M}$ is *innocuous* for $\langle \mathfrak{M}; \in, \prec \rangle$ if we have $y \in \mathfrak{M}$ whenever $y \subseteq x \in \mathfrak{M}$ and y is definable in the 2nd order structure $\langle \langle \mathfrak{M}; \in, \prec \rangle; \mathfrak{X} \rangle$. Then, a transitive model $\mathfrak{M} \models$ **ZFC** extends to a model of $\mathbf{IST}^+$, iff there is a well-ordering $\prec$ of $\mathfrak{M}$ such that $\langle \mathfrak{M}; \in, \prec \rangle$ models **ZFGC** and the family $\mathfrak{X}$ of all sets $X \subseteq \mathfrak{M}$, definable in $\langle \mathfrak{M}; \in, \prec \rangle$, is innocuous for $\langle \mathfrak{M}; \in, \prec \rangle$.

The equivalence of this characterization and the one given by the theorem can in principle be verified directly without a reference to the $\mathbf{IST}^+$-extendibility.

class $\mathbb{I}$ of *internal* sets, which behaves approximately like an **IST** universe, while the whole universe is a sort of well-founded superstructure over $\mathbb{I}$, in particular it satisfies Separation in the language containing $\mathtt{st}$, hence allows to freely operate with *external sets* which causes trouble for **IST** users.

On the other hand **NST** and **KST** differ in some important details. For instance, **NST** does not have Replacement (but has Power Set) in the "universe of discourse", provides a weaker amount of Saturation comparably with the **IST** Idealization, and a rather strong form of Standardization which implies that, given a set X, the intersection $X \cap \mathbb{S}$ is covered by a standard set (a "boundedness" property, incompatible with **IST**). Internal sets in **NST** are elements of standard sets and only them.

THEOREM 3. *A transitive model* $\mathfrak{M} \models$ **ZFC** *extends to a model of* **NST** *iff there exist a transitive model* $\mathfrak{N}$ *of* **ZC** (Zermelo with Choice) *and an ordinal* $\kappa \in \mathfrak{N}$ *such that* κ *is a cardinal in* $\mathfrak{N}$ *while* $\mathfrak{M} = \mathfrak{N} \cap V_\kappa$.[2]

Unlike **NST**, Kawaï's theory **KST** arranges the "universe of discourse" so that $\mathbb{S} \subseteq \mathbb{I}$ are sets (hence the abovementioned "boundedness" property of $\mathbb{S}$ in **NST** fails) while Replacement in the $\mathtt{st}$-$\in$-language holds.

THEOREM 4. *A transitive model* $\mathfrak{M} \models$ **ZFC** *extends to a model of* **KST** *iff there are a model* $\mathfrak{N} = \langle \mathfrak{N} ; \epsilon \rangle$ [3] *of* **ZFC** *and a* $\mathfrak{N}$*-cardinal* $\kappa \in \mathfrak{N}$ *such that* $\langle \mathfrak{M} ; \in \rangle$ *is isomorphic to* $\langle (V_\kappa)^{\mathfrak{N}} ; \epsilon \rangle$.

We would be interested to prove Theorem 4 with the additional requirement that $\mathfrak{N}$ is a transitive set and ϵ is $\in \restriction \mathfrak{N}$. (See a short discussion in Subsection 5.4)

2. Internal set theory

Internal set theory **IST** of Nelson [9] is a theory in the $\mathtt{st}$-$\in$-language, containing all **ZFC** axioms (in the $\in$-language) and the following principles:

IST Transfer $\quad \Phi^{\mathrm{st}} \Longleftrightarrow \Phi$
 – for any $\in$-formula Φ with standard parameters;

Idealization: $\quad \forall^{\mathrm{stfin}} A \, \exists\, x \, \forall\, a \in A \; \Phi(a,x) \Longleftrightarrow \exists\, x \, \forall^{\mathrm{st}} a \; \Phi(a,x)$
 – for any $\in$-formula $\Phi(x)$ with arbitrary parameters;

IST Standardization $\quad \forall^{\mathrm{st}} X \, \exists^{\mathrm{st}} Y \, \forall^{\mathrm{st}} x \, (x \in Y \Longleftrightarrow x \in X \;\&\; \Phi(x))$
 – for any $\mathtt{st}$-$\in$-formula $\Phi(x)$ with arbitrary parameters.

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[2] Recall that V_κ is the κ-th level of the cumulative set hierarchy.

[3] In this paper, $\in$ always denotes the "true" membership, while ϵ will be used to denote membership relations in not necessarily standard models.

The quantifiers $\exists^{\text{st}} x$ and $\forall^{\text{st}} x$ have the obvious meaning (there exists a standard set x ...). $\forall^{\text{stfin}} A$ means: for any standard finite set A.

Φ^{st} is the relativization of an $\in$-formula Φ to $\mathbb{S} = \{x : \mathtt{st}\, x\}$.

ISTGC is a theory in the language $\mathcal{L}_{\in,<,\mathtt{st}}$, containing all of **ZFGC** (in the language $\mathcal{L}_{\in,<}$, see Section 1), together with the schemata of **IST** Transfer, Idealization, and **IST** Standardization, in each of which the global order predicate $<$ is allowed to occur in the formulas involved.

It is known that **IST** proves the existence of a set D such that $\mathbb{S} \subseteq D$, even more, D can be chosen as a formally finite set. (See Nelson [9].) By the Choice for finite sets, this yields a choice function $\mathbf{C}$ defined on D, hence on $\mathbb{S}$. In other words, $\mathbf{C}(x) \in x$ for any standard $x \neq \emptyset$. The special content of Locally Standard Choice is therefore to guarantee that $\mathbf{C}$ can be chosen so that $\mathbf{C}(x)$ is standard for any standard x.

We doubt (but do not know for sure) that Locally Standard Choice is provable in **IST**. However, it is an easy theorem of **ISTGC**. Indeed, take a set D such that $\mathbb{S} \subseteq D$, and, for any $x \neq \emptyset$ in D, define $\mathbf{C}(x)$ to be the $<$-least element in x. Then $\mathbf{C}(x)$ is standard whenever x is standard, by Transfer, where now the predicate $<$ can occur.

To end this discussion, let us cite the following lemma of Kanovei [4].

LEMMA 5. *There is a* $\mathtt{st}$-$\in$-*formula* $\tau(x)$ *such that, for any* $\in$-*formula* $\varphi(x_1, ..., x_n)$, *it is a theorem of* **IST** *that*

$$\forall^{\text{st}} x_1 \ldots \forall^{\text{st}} x_n \, (\varphi^{\text{st}}(x_1, ..., x_n) \Longleftrightarrow \tau(\ulcorner \varphi(x_1, ..., x_n) \urcorner)) \,. \qquad \square$$

Here $\ulcorner \psi \urcorner$ is the formula ψ considered as a finite sequence of (coded) symbols of the $\in$-language and sets which occur in ψ as parameters.

2.1. Getting the order

This subsection proves implication $(1) \Longrightarrow (3)$ in Theorem 2.

Consider a transitive model $\mathfrak{M} \models \mathbf{ZFC}$ which is the standard part $\mathbb{S} = \{x \in \mathbb{I} : \mathtt{st}\, x\}$ of a model $\mathbb{I} = \langle \mathbb{I}; \epsilon, \mathtt{st} \rangle$ of $\mathbf{IST}^+$, so $\in \restriction \mathfrak{M} = \epsilon \restriction \mathfrak{M}$. Our aim is to find a well-ordering $\prec$ of $\mathfrak{M}$ such that $\langle \mathfrak{M}; \in, \prec \rangle$ models **ZFGC** and $\mathtt{Truth}^{\mathfrak{M}}_{\in,\prec}$ is innocuous for $\langle \mathfrak{M}; \in, \prec \rangle$.

Let $\mathbf{C} \in \mathbb{I}$ be such that the following is true in $\mathbb{I}$: " $\mathbf{C}$ is a function which guararantees the axiom of Locally Standard Choice." Let $C = \mathbf{C} \restriction \mathfrak{M}$, so that C is a choice function for the standard $\in$-model $\mathfrak{M} = \langle \mathfrak{M}; \in \rangle$.

LEMMA 6. *The pair of sets* C *and* $\mathbf{T} = \mathtt{Truth}^{\mathfrak{M}}_{\in}$ *is innocuous for* $\langle \mathfrak{M}; \in \rangle$.

PROOF. By Standardization, it suffices to prove that $\mathbf{T}$ is $\mathtt{st}$-$\in$-definable in $\mathbb{I}$. But this follows from Lemma 5. $\blacksquare$

We define the required order $\prec$ as the limit of an increasing sequence of orders, the construction of which is based on a forcing-like idea.

We argue in the model $\langle \mathfrak{M}; \in, C, \mathbf{T} \rangle$.

Let Σ be the class of all structures of the form $\sigma = \langle X; < \rangle$, where $X \in \mathfrak{M}$ is transitive and has the form $X = \mathfrak{M}_\alpha = V_\alpha \cap \mathfrak{M}$ for some ordinal $\alpha \in \mathfrak{M}$, and $< \in \mathfrak{M}$ is a well-ordering of X. We say that $\sigma' = \langle X'; <' \rangle$ *extends* $\sigma = \langle X; < \rangle$, if $X \subseteq X'$ and $<'$ is an end-extension of $<$.

Define a relation $\sigma \;\mathtt{forc}\; \Phi(x_1, ..., x_n)$, where $\sigma = \langle X; < \rangle \in \Sigma$ while Φ is a $\mathcal{L}_{\in,<}$-formula and $x_1, ..., x_n \in X$, by induction on the complexity of Φ .

1. If Φ is an elementary formula of $\mathcal{L}_{\in,<}$, *i.e.* $x < y$, $x = y$, or $x \in y$, then $\sigma \;\mathtt{forc}\; \Phi$ iff Φ is true in σ (viewed as $\sigma = \langle X; \in, < \rangle$).
2. $\sigma \;\mathtt{forc}\; (\Phi \;\&\; \Psi)$ iff $\sigma \;\mathtt{forc}\; \Phi$ and $\sigma \;\mathtt{forc}\; \Psi$.
3. $\sigma \;\mathtt{forc}\; (\neg \Phi)$ iff no $\sigma' \in \Sigma$ extending σ satisfies $\sigma' \;\mathtt{forc}\; \Phi$.
4. $\sigma \;\mathtt{forc}\; \exists x \, \Phi(x)$ iff there is $x \in X$ such that $\sigma \;\mathtt{forc}\; \Phi(x)$.

For a $\mathcal{L}_{\in,<}$-formula Φ, a structure $\sigma = \langle X; < \rangle \in \Sigma$ is Φ-*complete* iff, for any subformula $\Psi(x_1, ..., x_n)$ of Φ and all $x_1, ..., x_n \in X$, we have $\sigma \;\mathtt{forc}\; \Psi(x_1, ..., x_n)$ or $\sigma \;\mathtt{forc}\; \neg \Psi(x_1, ..., x_n)$, and is *totally complete* if it is Φ-complete for any formula Φ of $\mathcal{L}_{\in,<}$. The following can be proved by a standard argument using induction on the complexity of Φ .

PROPOSITION 7. *If* Φ *is a closed* $\mathcal{L}_{\in,<}$*-formula with parameters in* X, *and* $\sigma = \langle X; < \rangle \in \Sigma$ *is* Φ*-complete, then* $\sigma \;\mathtt{forc}\; \Phi$ *iff* $\sigma \models \Phi$. □

We define an increasing sequence of structures $\sigma_\gamma = \langle X_\gamma; <_\gamma \rangle \in \Sigma$. The construction depends on the extendibility to totally complete structures. (Note that, given a formula Φ, every structure $\sigma \in \Sigma$ extends to a Φ-complete $\sigma' \in \Sigma$ because $\langle \mathfrak{M}; \in \rangle$ models **ZFC**. But, to prove that there is a totally complete extension, one would need that the set $\mathbf{T} = \mathtt{Truth}^{\mathfrak{M}}_{\in}$ does not destroy Replacement in $\mathfrak{M}$, which is not assumed.)

Case 1: each $\sigma \in \Sigma$ can be extended to a totally complete $\sigma' \in \Sigma$.

Define $\sigma_\gamma = \langle X_\gamma; <_\gamma \rangle \in \Sigma$ by induction on $\gamma \in \mathtt{Ord}$ (in $\mathfrak{M}$) so that $X_\gamma = \bigcup_{\delta<\gamma} X_\delta$ and $<_\gamma = \bigcup_{\delta<\gamma} <_\delta$ for all limit ordinals γ, and $\sigma_{\gamma+1} = C(W(\sigma_\alpha))$, where $W(\sigma_\alpha)$ is the set of all totally complete proper extensions of σ_γ in Σ of the least possible $\in$-rank, while $C = \mathbf{C} \restriction \mathfrak{M}$, see above.

Let λ be the largest ordinal such that σ_γ is defined (and belongs to Σ, hence to $\mathfrak{M}$) for all $\gamma < \lambda$; clearly $\lambda \leq$ "the least ordinal not in $\mathfrak{M}$ ". $\dashv$

Case 2: otherwise.

Fix a recursive enumeration $\{\Phi_n : n \in \omega\}$ of all formulas of $\mathcal{L}_{\in,<}$. A structure $\sigma \in \Sigma$ will be called n-*complete* if it is Φ_k-complete for any

$k \le n$. We set $\lambda = \omega$ in this case, pick a structure $\sigma_0 \in \Sigma$ not extendable to a totally complete structure, and define a sequence of structures $\sigma_n = \langle X_n ; <_n \rangle \in \Sigma$ such that $\sigma_{n+1} = C(W(\sigma_n))$ for any $n \in \omega$. $\dashv$

In each of the two cases $\langle \sigma_\gamma : \gamma < \lambda \rangle$ is a sequence of elements of $\mathfrak{M}$, definable in $\langle \mathfrak{M} ; \in, \mathbf{T}, C \rangle$, where, we recall, $\mathbf{T} = \mathtt{Truth}_\in^{\mathfrak{M}}$.

LEMMA 8. *λ is a limit ordinal and* $\bigcup_{\gamma < \lambda} X_\gamma = \mathfrak{M}$.

PROOF. Recall that $\lambda = \omega$ in Case 2. If $\lambda = \gamma + 1$ in Case 1 then, by the assumption of Case 1, we would be able to define σ_λ. Hence λ is a limit ordinal and the relation $\prec = \bigcup_{\gamma<\lambda} <_\gamma$ is a well-ordering of X.

Assume that $X = \bigcup_{\gamma<\lambda} X_\gamma \ne \mathfrak{M}$. Then $X \in \mathfrak{M}$ as each X_γ is $\mathfrak{M}_\alpha = V_\alpha \cap \mathfrak{M}$ for some α. Now the order $\prec$ belongs to $\mathfrak{M}$ by Lemma 6. It follows that $\sigma = \langle X ; \prec \rangle \in \Sigma$. Moreover σ is totally complete. (As the limit of an increasing sequence of totally complete structures in Case 1, and by similar reasons in Case 2.) This immediately contradicts the choice of σ_0 in Case 2, while, in Case 1, adds an extra term to the sequence, which contradicts the choice of λ. ■

We conclude that $\prec = \bigcup_{\gamma<\lambda} <_\gamma$ is a well-ordering of $\mathfrak{M}$.

LEMMA 9. $\langle \mathfrak{M} ; \in, \prec \rangle$ *is a model of* **ZFGC**. *The set* $\mathbf{T}_\prec = \mathtt{Truth}_{\in,\prec}^{\mathfrak{M}}$ *is innocuous for* $\langle \mathfrak{M} ; \in, \prec \rangle$.

PROOF. To see that $\langle \mathfrak{M} ; \in, \prec \rangle$ satisfies Collection, suppose that p, $X \in \mathfrak{M}$ and $\Phi(x, y, p)$ is a $\mathcal{L}_{\in,<}$-formula. We have to find $Y \in \mathfrak{M}$ such that the following holds in $\mathfrak{M}$: $\forall x \in X\, [\exists y\, \Phi(x, y, p) \Longrightarrow \exists y \in Y\, \Phi(x, y, p)]$.

In both Case 1 and Case 2, there is $\gamma < \lambda$ such that p, $X \in X_\gamma$ and σ_γ is $(\exists y\, \Phi(x, y, p))$-complete. Prove that $Y = X_\gamma$ is as required.

Consider $x \in X$, hence $\in X_\gamma$. Suppose that there is $y \in \mathfrak{M}$ such that $\Phi(x, y, p)$ holds in $\mathfrak{M}$, and prove that such a set y exists in X_γ.

It follows from Lemma 8 that $\mathfrak{M}$ is the union of an increasing chain of $(\exists y\, \Phi(x, y, p))$-complete structures. Therefore, by Proposition 7 and an ordinary model-theoretic argument, $\langle \mathfrak{M} ; \in, \prec \rangle$ is an elementary extension of $\langle X_\gamma ; \in, <_\gamma \rangle$ with respect to the formula $\exists y\, \Phi(x, y, p)$ and all its subformulas. This proves the existence of y in X_γ.

To prove that $\mathbf{T}_\prec$ is innocuous, it suffices to show that $\mathbf{T}_\prec$ is definable in $\langle \mathfrak{M} ; \in, \mathbf{T}, C \rangle$. Let $\Phi(p_1, ..., p_k)$ be a closed $\mathcal{L}_{\in,<}$-formula with parameters $p_1, ..., p_k \in \mathfrak{M}$. Let n be the number of $\Phi(x_1, ..., x_k)$ (see Case 2 above). Take the least $\gamma < \lambda$ such that $p_1, ..., p_k \in X_\gamma$ and, in Case 2, $\gamma \ge n$. Arguing as above, we conclude that σ_γ is an elementary substructure of $\langle \mathfrak{M} ; \in, \prec \rangle$ w. r. t. Φ, hence $\Phi(p_1, ..., p_k)$ is either true or false simultaneously in both σ_γ and $\langle\mathfrak{M};\in,\prec\rangle$. It remains to recall that the sequence of structures σ_γ is definable in $\langle\mathfrak{M};\in,\mathbf{T},C\rangle$. ■

□ ((1) $\Longrightarrow$ (3) of Theorem 2)

2.2. Getting the model

This subsection not only proves implication (3) $\Longrightarrow$ (2) in Theorem 2, but also supplies some material useful for the metamathematical analysis of theories $\mathbf{IST}^+$ and **ISTGC** in Section 2.3. The key tool involved is the *adequate ultralimit* construction of Nelson [9], modified by Kanovei [4].

- A filter or ultrafilter $U \subseteq \mathcal{P}(\mathcal{P}_{\text{fin}}(A))$ is *A-adequate* iff it contains all sets of the form $\{i \in \mathcal{P}_{\text{fin}}(A) : a \in i\}$, where $a \in A$.

Recall that $U i\, \Phi(i)$ means "the set $\{i \in I : \Phi(i)\}$ belongs to U". (The quantifier: there exist U-many.) In this notation, the A-adequacy of U can be expressed as: "$U i\ (a \in i)$ for any $a \in A$".

We begin with a transitive set $\mathfrak{M}$ and a well-ordering $\prec$ of $\mathfrak{M}$ such that $\mathfrak{M}$ is a model of **ZF** (but, in general, will <u>not</u> assume that $\mathfrak{M} \models \mathbf{ZFC}$) while the pair of sets $\prec$ and $\mathbf{T} = \mathtt{Truth}^{\mathfrak{M}}_{\in,\prec}$ is innocuous for $\mathfrak{M}$, so that $\langle\mathfrak{M};\in,\prec,\mathbf{T}\rangle$ models Separation in the language $\mathcal{L}_{\in,<,T}$. In addition, we assume that every initial segment of $\mathfrak{M}$ in the sense of $\prec$ belongs to $\mathfrak{M}$.

The general aim is to embed $\mathfrak{M}$, as the class of all standard sets, in an "**ISTGC**-like" model $\mathbb{I}$.

Let $I = \mathcal{P}_{\text{fin}}(\mathfrak{M})$. Let $\mathcal{A}$ be the algebra of all sets $X \subseteq I$ which belong to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$, the collection of all sets $X \subseteq \mathfrak{M}$ definable in $\langle\mathfrak{M};\in,\prec\rangle$ by a formula of $\mathcal{L}_{\in,<}$ containing sets in $\mathfrak{M}$ as parameters.

PROPOSITION 10. *There exists an $\mathfrak{M}$-adequate ultrafilter $U \subseteq \mathcal{A}$ satisfying*

(A) *if a relation $P \subseteq \mathfrak{M} \times I$ belongs to the class $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$ then the relation $U i\, P(x,i)$ belongs to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$ as well;*

(B) *there is a set $\mathcal{U} \subseteq \mathfrak{M}$, definable in the structure $\langle\mathfrak{M};\in,\prec,\mathbf{T}\rangle$, such that $U = \{\mathcal{U}_x : x \in \mathfrak{M}\}$, where $\mathcal{U}_x = \{i \in I : \langle x,i\rangle \in \mathcal{U}\}$ for all x .*

PROOF. The family U_0 of all sets of the form

$$I_{a_1\dots a_m} = \{i \in I : a_1,\dots,a_m \in i\}, \quad \text{where} \quad a_1,..,a_m \in \mathfrak{M}\,,$$

obviously satisfies FIP (*the finite intersection property*).

Now suppose that a FIP family U_n of subsets of I has been constructed. Denote by $\chi_n(x,i)$ the n-th formula in a recursive enumeration, fixed beforehand, of all $\mathcal{L}_{\in,<}$-formulas with exactly two free variables.

We define $U_{n+1} = U_n \cup \{B^n_x : x \in \mathfrak{M}\}$, where B^n_x is equal to the set $A^n_x = \{i \in \mathbb{I} : \langle \mathfrak{M}; \in, \prec\rangle \models \chi_n(x,i)\}$ whenever $U_n \cup \{B^n_y : y \prec x\} \cup A^n_x$ still satisfies FIP, and $B^n_x = I \setminus A^n_x$ otherwise.

Let us prove that $U = \bigcup_n U_n$ satisfies both (A) and (B). We first prove, by induction on n, that the map

$$F_n(x) = \begin{cases} 1 & \text{if } B^n_x = A^n_x \\ 0 & \text{if } B^n_x = I \setminus A^n_x \end{cases}$$

belongs to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$. Let us fix n and suppose that all of F_k, $k < n$, already belong to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$. Let, for any $y \in \mathfrak{M}$, a *y-function* be a function f defined on $W_y = \{x : x \prec y\}$ and satisfying, on this domain, the definition of F_n. Note that, by the assumptions above, $W_y \in \mathfrak{M}$ for all y, and $\prec$ wellorders W_y. It easily follows that, for any y, there is at most one y-function. As for the existence, normally we would have to apply Replacement (which is not assumed here), but, as clearly any y-function, as well as any y'-function for $y' \prec y$, is a subset of $W_y \times \{0,1\}$ (which is a set in $\mathfrak{M}$), the ordinary definition of an y-function, by induction on the $\prec$-position of y, goes through in $\langle \mathfrak{M}; \in, \prec\rangle$ with the help of Separation only. (We apply, of course, the assumption that all maps F_k with $k < n$ already belong to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$, to adequately express the definition of B^n_x.)

Now let us check (A). Let $P \subseteq \mathfrak{M} \times I$ be definable in $\mathfrak{M}$ by a $\mathcal{L}_{\in,<}$-formula $\chi(x,i)$: we have to prove that the relation $Ui\; P(x,i)$ belongs to $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$. It is sufficient to consider the case when χ does not contain any sets as parameters. (Otherwise the parameters involved simply join x.) Then χ is χ_n for some n, and, by definition, $Ui\; P(x,i)$ is equivalent to $F_n(x) = 1$, so that the result follows from the definability of F_n.

Let us check (B). The reasoning above can be summarized as follows: there is a recursive sequence of parameter-free $\mathcal{L}_{\in,<}$-formulas $\varphi_n(x)$ such that, for all n and $x \in \mathfrak{M}$, $F_n(x) = 1$ iff $\varphi_n(x)$ holds in $\mathfrak{M}$. Now the set $\mathfrak{U}$ of all pairs $\langle y, i\rangle \in \mathfrak{M} \times I$, such that $y = \langle n, x\rangle \in \omega \times \mathfrak{M}$ and it is true in $\mathfrak{M}$ that $\varphi_n(x) \Longleftrightarrow \chi_n(x,i)$, is definable in the structure $\langle \mathfrak{M}; \in, \prec, \mathbf{T}\rangle$, and, by the construction, we have $U = \{\mathfrak{U}_y : y \in \mathfrak{M}\}$. ■

Let us fix such an ultrafilter $U \subseteq \mathcal{A}$.

For $r \geq 1$, we let $I^r = I \times ... \times I$ (r times I), and

$$\mathbf{F}_r = \{f \in \mathtt{Def}^{\mathfrak{M}}_{\in,\prec} : f \text{ maps } I^r \text{ to } \mathfrak{M}\}.$$

Let separately $I^0 = \{0\}$ and $\mathbf{F}_0 = \{\{\langle 0, x\rangle\} : x \in \mathfrak{M}\}$. We finally define $\mathbf{F} = \bigcup_{r \in \omega} \mathbf{F}_r$, and, for $f \in \mathbf{F}$, let $r(f)$ be the only r such that $f \in \mathbf{F}_r$.

Suppose that $f \in \mathbf{F}$, $q \geq r = r(f)$, and $\mathbf{i} = \langle i_1, ..., i_r, ..., i_q \rangle \in I^q$. Then we set $f[\mathbf{i}] = f(\langle i_1, ..., i_r \rangle)$. In particular $f[\mathbf{i}] = f(\mathbf{i})$ whenever $r = q$. Separately we put $f[\mathbf{i}] = x$ for any $\mathbf{i}$ whenever $f = \{\langle 0, x \rangle\} \in \mathbf{F}_0$.

Let $f, g \in \mathbf{F}$ and $r = \max\{r(f), r(g)\}$. Define

$$f \ {}^*\!\!= g \quad \text{iff} \quad U i_r \, U i_{r-1} \dots U i_1 \ (f[\mathbf{i}] = g[\mathbf{i}]),$$

(where $\mathbf{i}$ denotes $\langle i_1, ..., i_r \rangle$: note the order of quantifiers), and define $f \ {}^*\!\!\in g$ and $f \ {}^*\!\!\prec g$ similarly. The following is a routine statement.

Proposition 11. *The relation* ${}^*\!\!=$ *is an equivalence relation on* $\mathbf{F}$. *The relations* ${}^*\!\!\in$ *and* ${}^*\!\!\prec$ *on* $\mathbf{F}$ *are* ${}^*\!\!=$*-invariant in both arguments.* $\square$

Define $[f] = \{g \in \mathbf{F} : f \ {}^*\!\!= g\}$. Let $\mathbb{I} = \{[f] : f \in \mathbf{F}\}$ (the quotient). For $[f], [g] \in \mathbb{I}$, define $[f] \ {}^*\!\!\in [g]$ iff $f \ {}^*\!\!\in g$, and $[f] \ {}^*\!\!\prec [g]$ iff $f \ {}^*\!\!\prec g$. (This is independent of the choice of representatives by the proposition.)

For any $x \in \mathfrak{M}$, define ${}^*x = [\{\langle 0, x \rangle\}]$, the image of x in $\mathbb{I}$.

We finally define $\mathtt{st}\,[f]$ iff $[f] = {}^*x$ for some $x \in \mathfrak{M}$.

Theorem 12. *The map* $x \mapsto {}^*x$ *is a* $1-1$ *map from* $\mathfrak{M}$ *onto the class of all standard* (i.e. satisfying $\mathtt{st}$) *elements of* $\mathbb{I}$. *Moreover,* $x \mapsto {}^*x$ *is an elementary embedding of* $\langle \mathfrak{M}; \in, \prec \rangle$ *in* $\langle \mathbb{I}; {}^*\!\!\in, {}^*\!\!\prec \rangle$. *In addition the structure* $\langle \mathbb{I}; {}^*\!\!\in, {}^*\!\!\prec, \mathtt{st} \rangle$ *is a model of* Transfer, Standardization, *and* Idealization, *in their* **ISTGC** *versions* (that is, in the language $\mathcal{L}_{\in,<}$).

Proof. We begin with some formalism. Let $\Phi(f_1, ..., f_m)$ be a formula of $\mathcal{L}_{\in,<}$ with $f_1, ..., f_m \in \mathbf{F}$ as parameters. Put $r(\Phi) = \max\{r(f_1), ..., r(f_m)\}$. If $r \leq q$ and $\mathbf{i} \in I^q$ then let $\Phi[\mathbf{i}]$ denote the formula $\Phi(f_1[\mathbf{i}], ..., f_m[\mathbf{i}])$ (a formula of $\mathcal{L}_{\in,<}$ with parameters in $\mathfrak{M}$). Let finally $[\Phi]$ denote the formula $\Phi([f_1], ..., [f_m])$, which is a formula of $\mathcal{L}_{\in,<}$ with parameters in $\mathbb{I}$.

The proof of the next proposition goes on by induction on the complexity of the formulas involved, following usual patterns, which survive in this setup despite the fact that not all functions participate in the definition of the ultralimit. (Only those in $\mathtt{Def}^{\mathfrak{M}}_{\in,\prec}$!) The point is that $\prec$ is innocuous, so that $\langle \mathfrak{M}; \in, \prec \rangle$ models Separation in $\mathcal{L}_{\in,<}$, hence we have sufficient environment to carry out the ordinary Łoś argument.

Proposition 13. (Łoś) *Let* $\Phi = \Phi(f_1, ..., f_m)$ *be a formula of* $\mathcal{L}_{\in,<}$ *with functions* $f_1, ..., f_m \in \mathbf{F}$ *as parameters, and* $r \geq r(\Phi)$. *Then*

$$[\Phi] \text{ holds in } \langle \mathbb{I}; {}^*\!\!\in, {}^*\!\!\prec \rangle \quad \text{iff} \quad U i_r \, U i_{r-1} \dots U i_1 \ (\langle \mathfrak{M}; \in, \prec \rangle \models \Phi[\mathbf{i}]). \ \square$$

($\mathbf{i}$ denotes $\langle i_1, ..., i_r \rangle$ in the displayed line.)

Using functions in $\mathbf{F}_0$, we immediately conclude that the map $x \mapsto {}^*x$ is a $1-1$ map onto the class of all standard sets in $\mathbb{I}$ and an elementary embedding into $\langle \mathbb{I};\, {}^*\!\in, {}^*\!\prec \rangle$, which implies Transfer in $\langle \mathbb{I};\, {}^*\!\in, {}^*\!\prec, \mathtt{st}\,\rangle$.

Idealization. Let $\Phi(a,x)$ be an $\mathcal{L}_{\in,<}$-formula with two free variables, a and x, and some functions in $\mathbf{F}$ as parameters. We have to demonstrate

$$\langle \mathrm{II} \rangle \quad \forall^{\mathrm{stfin}} A\, \exists\, x\, \forall\, a \in A\, [\Phi](a,x) \implies \exists\, x\, \forall^{\mathrm{st}} a\, [\Phi](a,x)$$

in $\mathbb{I}$. (It is known that the implication $\Longleftarrow$ here is a corollary of Standardization.) The left–hand side of $\langle \mathrm{II} \rangle$ implies, by Proposition 13,

$$\forall \text{ finite } A \subseteq \mathfrak{M}\; U i_r\; U i_{r-1} \ldots U i_1\; \exists\, x\, \forall\, a \in A\; \Phi[\langle i_1, \ldots, i_r \rangle](a,x)$$

in $\mathfrak{M}$, where $r = r(\Phi)$. To simplify the formula note that the leftmost quantifier is a quantifier over I and define a function $\alpha \in \mathbf{F}_{r+1}$ by the equality $\alpha(i_1, \ldots, i_r, i) = i$. The last displayed formula takes the form

$$\forall\, i \in I\; U i_r\; U i_{r-1} \ldots U i_1\; (\exists\, x\, \forall\, a \in \alpha\; \Phi)[\langle i_1, \ldots, i_r, i \rangle](a,x)\,,$$

which implies $\exists\, x\, \forall\, a \in [\alpha]\, [\Phi](a,x)$ in $\mathbb{I}$ by Proposition 13. Now, by the definition of $\mathtt{st}$, it suffices to check ${}^*x \;{}^*\!\!\in [\alpha]$ in $\mathbb{I}$ for any $x \in \mathfrak{M}$. This is equivalent to $U i\; U i_r \ldots U i_1\; (x \in i)$, which holds by the choice of U.

Standardization. This is the point when the assumption, that the sets $\prec$ and $\mathbf{T} = \mathtt{Truth}^{\mathfrak{M}}_{\in,\prec}$ are innocuous for $\langle \mathfrak{M}; \in \rangle$, comes into play.

Since U is definable in $\langle \mathfrak{M}; \in, \prec, \mathbf{T} \rangle$ by (B) of Proposition 10, the model $\langle \mathbb{I};\, {}^*\!\in, \mathtt{st}\,\rangle$ is definable in $\langle \mathfrak{M}; \in, \prec, \mathbf{T} \rangle$ as well. Thus we have only to check that, given $x \in \mathfrak{M}$, any set $y \subseteq x$, which is definable in $\langle \mathfrak{M}; \in, \prec, \mathbf{T} \rangle$, belongs to $\mathfrak{M}$. It remains to recall that $\langle \mathfrak{M}; \in, \prec, \mathbf{T} \rangle$ models Separation.

□ (*Theorem 12*)

We can now complete the proof of implication (3) $\Longrightarrow$ (2) in Theorem 2. In addition to the setup in the beginning of this Subsection, let us assume that $\langle \mathfrak{M}; \in, \prec \rangle$ is a model of **ZFGC**. Then, by Theorem 12, $\langle \mathbb{I};\, {}^*\!\in, {}^*\!\prec, \mathtt{st}\,\rangle$ is a model of **ISTGC** while the map $x \mapsto {}^*x$ is a $1-1$ $\in$-embedding of $\mathfrak{M}$ onto the class of all standard elements of $\mathbb{I}$.

□ (*Theorem 2*)

2.3. Conservativity

The aim of this subsection is to prove that $\mathbf{IST}^+$ and **ISTGC** share the property of conservativity $\langle \mathrm{I} \rangle$ with **IST**. It suffices to verify this property for **ISTGC** only.

THEOREM 14. **ISTGC** *satisfies* $\langle \mathrm{I} \rangle$, *that is, every theorem of* **ISTGC** *in the* $\in$*-language is a theorem of* **ZFC**.

It immediately follows that $\mathbf{IST}^+$ satisfies ⟨I⟩ as well. Thus both **ISTGC** and $\mathbf{IST}^+$ are equi-consistent with **ZFC**. We would say that **ISTGC** and $\mathbf{IST}^+$ are "mild" extensions of **IST**, meaning both Theorem 14 and the observation that the theories do not differ from each other and from **IST** in matters of development of applied nonstandard mathematics.

PROOF. Suppose that φ is an $\in$-sentence, provable in **ISTGC**. We have to show that φ is a theorem of **ZFC**, too. Let Φ be a finite subtheory of **ISTGC**, sufficient to prove φ.

Proof of φ *in* **ZFC**. Let λ be a limit ordinal, such that $\mathfrak{M} = V_\lambda$ satisfies all those cases of Replacement, which occur in Φ, and is an elementary submodel of the universe with respect to φ. Let $\prec$ be any wellordering of $\mathfrak{M}$ such that any initial segment of $\mathfrak{M}$ in the sense of $\prec$ belongs to $\mathfrak{M}$. Then, as sets of the form V_λ are closed under subset formation, everything is innocuous for $\mathfrak{M}$, so we can apply Theorem 12. (Note that Theorem 12 does not require any instance of Replacement to hold in $\mathfrak{M}$.) The structure $\langle \mathbb{I};\, {}^*\!\in, {}^*\!\prec, \mathtt{st} \rangle$ is then a model of Φ, hence a model of φ. Therefore $\mathfrak{M}$ models φ as well. This proves φ in the universe by the choice of $\mathfrak{M}$. ∎

3. External nonstandard set theories

Hrbaček [2, 3] and Kawaï [7, 8] introduced several nonstandard theories of this type which differ in detail but have a common part which we will call *basic external set theory*, or **BEST**. This is a theory in the language $\mathcal{L}_{\in,\mathrm{st},\mathrm{int}}$ with three atomic predicates, $\in$, $\mathtt{st}$, and $\mathtt{int}$. (The last one, expressing the property of being *internal*, will be reduced to $\mathtt{st}$ in some cases by means of a special axiom, see below.)

The axioms of **BEST** naturally split in two groups.

1. ***Basic axioms for the "universe of discourse".*** This group consists of all **ZF** axioms (with Separation in $\mathcal{L}_{\in,\mathrm{st},\mathrm{int}}$), *except for* the Replacement and Regularity axioms. (Note the absence of Choice.) A weaker form of Regularity is added, see below.

2. ***Axioms for standard and internal sets.*** This group includes:

$\mathbf{ZFC}^{\mathrm{st}}$: all axioms of **ZFC** (in the $\in$-language) relativized to the standard universe $\mathbb{S} = \{x : \mathtt{st}\, x\}$;

Transitivity of $\mathbb{I}$: the internal subuniverse $\mathbb{I} = \{x : \mathtt{int}\, x\}$ is transitive;

Transfer: $\Phi^{\mathrm{int}} \Longleftrightarrow \Phi^{\mathrm{st}}$ – for any $\in$-formula $\Phi(x)$ with standard parameters, where Φ^{int} means the relativization to $\mathbb{I} = \{x : \mathtt{int}\, x\}$;

Restricted Standardization: $\forall^{\text{st}} S \, \forall X \subseteq S \, \exists^{\text{st}} Y \, (X \cap \mathbb{S} = Y \cap \mathbb{S})$;

Saturation: if $\mathcal{X}$ is a set of standard size, such that every $X \in \mathcal{X}$ is internal and $\bigcap \mathcal{X}' \neq \emptyset$ for any $\mathbb{S}$-finite non-empty $\mathcal{X}' \subseteq \mathcal{X}$, then $\bigcap \mathcal{X} \neq \emptyset$.

(Here, a set *of standard size* is any set of the form $Y = \{f(x) : x \in X \cap \mathbb{S}\}$, where X is standard. "$\mathbb{S}$-finite" is understood as equinumerous to some $n = \{0, 1, ..., n-1\} \in \omega$, where ω is the set of all $\mathbb{S}$-natural numbers.)

This completes the list of **BEST** axioms. Note that by Transfer $\mathbb{I}$ is an elementary extension of $\mathbb{S}$ in the $\in$-language, transitive and appropriately saturated by Saturation. Now let us consider several special extensions.

3.1. "Nonstandard set theory" of Hrbaček

The axioms of the "nonstandard set theory" **NST** (the original version was denoted by $\mathfrak{NS}_2(\mathbf{ZFC})$ in [2], see also [3]) include all of **BEST**, together with the **ZFC** axioms of Power Set and Choice (as: every set is well-orderable), and the following axiom:

Standardization: $\forall X \, \exists^{\text{st}} Y \, (X \cap \mathbb{S} = Y \cap \mathbb{S})$.

This axiom is obviously equivalent to $\forall X \, \exists^{\text{st}} Y \, (X \cap \mathbb{S} \subseteq Y)$ plus Restricted Standardization. It clearly implies that $\mathbb{S}$ is not a set, and the following:

Internal Boundedness: every internal set is an element of a standard set[4].

Thus, in **NST** we have $\mathtt{int}\, x \iff \exists^{\text{st}} y \, (x \in y)$, so $\mathbb{I} = \{x : \exists^{\text{st}} y \, (x \in y)\}$ and **NST** is a theory in the $\mathtt{st}$-$\in$-language.

We define, as $\mathbf{NST}^+$, the theory **NST** strengthened by the following three axioms:

Regularity over $\mathbb{I}$: for any $X \neq \emptyset$ there is $x \in X$ such that $X \cap x \subseteq \mathbb{I}$;

Transitive Hulls: every set is a subset of a transitive set;

Full Boundedness: every set $X \subseteq \mathbb{I}$ is a subset of a standard set.

The theory $\mathbf{NST}^+$ is strong enough to prove that for any set x there is a transitive set X satisfying $x \subseteq X$, and a standard set S satisfying $X \cap \mathbb{I} \subseteq S$, leading to a more precise general picture of the universe.

[4] Let x be internal. Let $\alpha \in \mathbb{I}$ be the $\mathbb{I}$-rank of x (an $\mathbb{I}$-ordinal). If all $\mathbb{S}$-ordinals are smaller than α then clearly all standard sets belong to the set V_α computed in $\mathbb{I}$, an easy contradiction with the fact that $\mathbb{S}$ is not a set. Therefore there is an $\mathbb{S}$-ordinal $\sigma \geq \alpha$. Then x belongs to the standard set V_σ computed in $\mathbb{I}$.

3.2. Kawaï's set theory

Kawaï's set theory **KST** was introduced by Kawaï [8] (under the name: *nonstandard set theory*). Unlike the Hrbaček theory **NST**, it describes the class $\mathbb{I}$ of internal sets as an **IST** universe. This does not allow to use the **NST** definition of internal sets as elements of standard sets. Thus, in Kawaï's system, $\mathtt{int}\, x$ (" x is internal") as an independent atomic predicate.

The axioms of **KST** include: all of **BEST**, the usual **ZFC** axioms of Power Set and Choice, the schema of Replacement in $\mathcal{L}_{\in,\mathrm{st},\mathrm{int}}$, together with the Regularity over $\mathbb{I}$ and two more axioms:

Set–existence of $\mathbb{I}$: $\mathbb{I}$ is a set and $\mathbb{S} \subseteq \mathbb{I}$;[5]

Strong Saturation: if $\mathcal{X}$ is a set of $\mathbb{S}$-size such that every $X \in \mathcal{X}$ is internal and $\bigcap \mathcal{X}' \neq \emptyset$ for any $\mathbb{S}$-finite $\mathcal{X}' \subseteq \mathcal{X}$ then $\bigcap \mathcal{X} \neq \emptyset$.

(A set *of* $\mathbb{S}$-*size* is by definition any set of the form $Y = \{f(x) : x \in \mathbb{S}\}$.)

The axioms of **KST** suffice to prove that $\langle \mathbb{I}; \in, \mathtt{st} \rangle$ models any particular axiom of **IST**. The whole universe is postulated to be something like a **ZFC** world over $\mathbb{I}$ as the set of "atoms", but internal sets do not behave exactly like "atoms" because they participate in the membership relation.

3.3. Two "minimal" nonstandard set theories

Theorems 2, 3, 4 show that, despite of $\langle \mathrm{I} \rangle$ as their common property, theories **IST**, **NST**, **KST** require quite a lot from standard models of **ZFC** to be extendable to a model of such a theory. However there are theories which require much less (if anything) but preserve useful principles of the theories above.

Bounded set theory **BST** is a version of **IST** containing Idealization in the following weakened form:

Bounded Idealization:

$$\forall^{\mathrm{stfin}} A \subseteq A_0 \, \exists x \, \forall a \in A \, \Phi(a,x) \iff \exists x \, \forall^{\mathrm{st}} a \in A_0 \, \Phi(a,x)$$

– for any standard set A_0 and any $\in$-formula $\Phi(x)$,

and the following extra axiom (incompatible with full Idealization):

Boundedness: every set is an element of a standard set.

This theory (defined by Kanovei [4], but implicitly contained in Hrbaček [2]) is fully equivalent to **IST** as a basis for applied nonstandard mathematics, but has some advantages (see Kanovei and Reeken [5]). A visible difference between the two is that **IST** proves the existence of sets X satisfying $\mathbb{S} \subseteq X$ while **BST** proves that every set is a subset of a standard set.

[5] It follows, by Separation, that then $\mathbb{S} = \{x \in \mathbb{I} : \mathtt{st}\, x\}$ is a set as well.

The *Hrbaček set theory* **HST** ($\mathfrak{NS}_1$(**ZFC**) in Hrbaček [2], the current version see Kanovei and Reeken [6]) differs from **NST** so that the axioms of Power Set and Choice are removed while Regularity over $\mathbb{I}$, Replacement in the st-$\in$-language, and two useful restricted forms of Choice are added.

Note that the axioms of **HST**, as well as those of **NST**, are strong enough to prove that the internal subuniverse $\mathbb{I}$ models all axioms of **BST**. The "universe of discourse" is postulated by **HST** to be a $\mathbf{ZF}^-$-like (minus the Power Set axiom) world over internal sets as "atoms", but, unlike **KST**, the class $\mathbb{I}$ of all "atoms" is now a proper class rather than a set.

THEOREM 15. (Kanovei and Reeken [5]) *Every transitive $\in$-model $\mathfrak{M}$ of* **ZFC**, *which admits a well-ordering $\prec$ such that $\langle \mathfrak{M}; \in, \prec \rangle \models$* **ZFGC** ,[6] *extends to a model of* **BST**. *Every model $\mathbb{I}$ of* **BST** *extends to a model of* **HST** (*i.e.* there is a model $\mathbb{H} \models$ **HST** having $\mathbb{I}$ as the class of all internal sets). □

The extendibility criteria in this theorem are weaker than those in theorems 2, 3, 4. Possibly they can be totally eliminated, at least by Felgner [1] every *countable* transitive model of **ZFC** extends to a model of **ZFGC**, hence to **BST** and **HST** by the theorem. It is an interesting problem to figure out whether uncountable models of **ZFC** admit such an extension.

4. Extendibility to NST

This section proves Theorem 3. The proof consists of two parts.

4.1. From left to right

Let a transitive model $\mathfrak{M} \models$ **ZFC** be the standard part of a model $\mathbb{H} = \langle \mathbb{H}; \epsilon, \mathtt{st} \rangle$ of **NST**, so that $\mathfrak{M} = \mathbb{S} = \{x \in \mathbb{H} : \mathtt{st}\, x\}$ in $\mathbb{H}$ and $\epsilon \restriction \mathfrak{M} = \in \restriction \mathfrak{M}$. We have to define a transitive model $\mathfrak{N}$ of **ZC** and a $\mathfrak{N}$-cardinal $\kappa \in \mathfrak{N}$, such that $\mathfrak{M} = \mathfrak{N} \cap V_\kappa$. The next lemma is based on ideas introduced in [2]. Note the absence of Replacement in **NST**!

LEMMA 16. *There is a* st-$\in$-*formula $\mathcal{O}(d, s)$ such that* **NST** *proves the existence of a set M such that $\mathcal{O}$ defines a bijection from M onto $\mathbb{S}$.*

PROOF. We argue in **NST**. Let $\mathbb{S} = \{x : \mathtt{st}\, x\}$ and $\mathbb{I} = \{y : \exists^{\mathrm{st}} x\, (y \in x)\}$ be the classes of all standard and internal sets, as usual.

For any standard x, let $\mathtt{TC}(x)$ denote the $\mathbb{S}$-transitive closure of x.

[6] In fact the existence of a well-ordering $\prec$ of $\mathfrak{M}$ such that $\langle \mathfrak{M}; \in, \prec \rangle$ models Separation (nothing about Replacement) suffices in this theorem.

Let $\mathbb{F}$ be the (internal) set of all $\mathbb{I}$-*hereditarily finite* internal sets.

We will prove that $(*)$ *for any standard* x, *the set* $S_x = \mathtt{TC}(x) \cap \mathbb{S}$ *is* $\in$-*isomorphic to a subset of* $\mathbb{F}$. Then, for any standard x, let d_x to be the set of all sets $d \subseteq \mathbb{F}$, $\in$-isomorphic to S_x. Put $M = \{d_x : x \in \mathbb{S}\}$ (a set in **NST** since $M \subseteq \mathcal{P}(\mathcal{P}(\mathbb{F}))$) and define $\mathcal{O}(d, x)$ to say that $d = d_x \in M$. (Note that S_x is not $\in$-isomorphic to S_y whenever $x \neq y$ are standard — an easy application of Standardization). The formula $\mathcal{O}$ proves the lemma.

To prove $(*)$, let W_x be the set of all $\mathbb{S}$-finite subsets of S_x. Easily both S_x and W_x are sets of standard size. Moreover any $w \in W_x$ is internal and $\mathbb{I}$-finite (holds in **NST** for every $\mathbb{S}$-finite subset of $\mathbb{I}$). Therefore, every $w \in W_x$ is $\in$-isomorphic in $\mathbb{I}$ to a subset of $\mathbb{F}$, as it is a standard **ZFC** fact that any finite set z is $\in$-isomorphic to a (finite) set consisting of hereditarily finite sets. It follows that the (internal) set Φ_x of all internal $1-1$ functions ϕ, which are $\in$-isomorphisms and satisfy $W_x \subseteq \mathtt{dom}\,\phi$ and $\mathtt{ran}\,\phi \subseteq \mathbb{F}$, is non-empty. Moreover, if $W \subseteq W_x$ is $\mathbb{S}$-finite then $\Phi_W = \bigcap_{w \in W} \Phi_w$ is non-empty because it includes Φ_w, where $w = \bigcup W \in W_x$.

Thus the set $\mathcal{X} = \{\Phi_w : w \in W_x\}$ is a standard size collection of non-empty internal sets, satisfying the finite intersection property. Hence $\Phi = \bigcap_{w \in W_x} \Phi_w$ is non-empty by Saturation. But any $\phi \in \Phi$ proves $(*)$. ■

Let $M \in \mathbb{H}$ be a set which satisfies the lemma in $\mathbb{H}$. For any $x \mathrel{\epsilon} M$, let $s(x)$ be the only element $s \in \mathfrak{M} = \mathbb{S}$ satisfying $\mathcal{O}(x, s)$ in $\mathbb{H}$, so that s is a $1-1$ map from $M_\epsilon = \{x \in \mathbb{H} : x \mathrel{\epsilon} M\}$ onto $\mathfrak{M}$.

Let, for any n, M_n be the n-th power set of M in $\mathbb{H}$. Let $\mathfrak{M}_n$ be the corresponding (in the s-sense) part of the full n-th power $\mathcal{P}^n(\mathfrak{M})$ in the universe, so that $\mathfrak{M}_{n+1} \subseteq \mathcal{P}(\mathfrak{M}_n)$. Then $\mathfrak{N} = \bigcup_n \mathfrak{M}_n$ is as required.

Indeed, the sequence $\langle M_n : n \in \omega \rangle$ is clearly $\mathtt{st}$-$\in$-definable in $\mathbb{H}$, hence $\mathfrak{N}$ models Separation because $\mathbb{H}$ models $\mathtt{st}$-$\in$-Separation. Power Set and Choice hold in $\mathfrak{N}$ by the construction. In addition, it easily follows from the **NST** Standardization that every set $Y \in \mathfrak{N}$, satisfying $Y \subseteq X$ for some $X \in \mathfrak{M}$, belongs to $\mathfrak{M}$. This property, together with the assumption that $\mathfrak{M}$ is a transitive model of **ZFC**, easily implies that the least ordinal $\kappa \notin \mathfrak{M}$ (= the union, in $\mathfrak{N}$, of all ordinals in $\mathfrak{M}$) is an $\mathfrak{N}$-cardinal, and $\mathfrak{M} = \mathfrak{N} \cap V_\kappa$.

4.2. From right to left: the internal universe

Suppose that $\mathfrak{M} \in \mathfrak{N}$ are transitive models of resp. **ZFC** and **ZC** and $\mathfrak{M} = \mathfrak{N} \cap V_\kappa$, where $\kappa \in \mathfrak{N}$ is a cardinal in $\mathfrak{N}$.

Prove that $\mathfrak{M}$ *extends to a model of* **NST** .

The first step is to extend $\mathfrak{M}$, within $\mathfrak{N}$, to a model $\mathbb{I}$, which will be the internal part of a more complicated extension to **NST**. To define $\mathbb{I}$ we apply a version of the ultralimit construction of Subsection 2.2. However, in order to fulfill Standardization and Saturation, we shall involve, in the construction of the ultralimit, only those functions whose ranges belong to $\mathfrak{M}$, but take the length of the ultralimit much longer than ω.

We argue in $\mathfrak{N}$. Thus κ is a cardinal and $\mathfrak{M} = V_\kappa$. Since $\mathfrak{N}$ is assumed to be a model of **ZC**, the cardinal κ^+ may not exist in $\mathfrak{N}$ in the usual sense (as an initial ordinal). However **ZC** is obviously strong enough to get a well-ordered set κ^+ with the necessary properties: 1) every proper initial segment of κ^+ has cardinality $\le \kappa$, and 2) $\mathtt{card}\,\kappa^+ > \kappa$. Since we have Choice, any set $X \subseteq \kappa^+$ of cardinality $\le \kappa$ is bounded in κ^+.

We take κ^+ as the length of the ultrapower construction.

Let $\alpha < \kappa^+$. Put $D_\alpha = [0, \kappa\alpha)$ ($\kappa\alpha$ means α times κ), $I_\alpha = \mathcal{P}_{\text{fin}}(D_\alpha)$, $D^\alpha = D_{\alpha+1} \setminus D_\alpha = [\kappa\alpha, \kappa(\alpha+1))$, and $I^\alpha = \mathcal{P}_{\text{fin}}(D^\alpha)$.

Let $\alpha < \beta$. If $u \subseteq I_\alpha$ then we define $u[\to \beta] = \{j \in I_\beta : j \cap D_\alpha \in u\}$. If $U \subseteq \mathcal{P}(I_\alpha)$ then let $U[\to \beta] = \{u[\to \beta] : u \in U\}$.

Using the **ZC** Choice in $\mathfrak{N}$, fix a D^α-adequate ultrafilter U^α for any $\alpha < \kappa^+$. Define, in $\mathfrak{N}$, a sequence of D_α-adequate ultrafilters U_α, by induction on $\alpha < \kappa^+$, as follows. *1st*, $U_1 = U^0$ (a κ-adequate ultrafilter).

2nd, $U_{\alpha+1}$ is the set of all sets $u \subseteq I_{\alpha+1}$ such that $U^\alpha i'\, U_\alpha i\, (i \cup i' \in u)$. Here i' ranges over I^α, i ranges over I_α, thus $i \cup i'$ is a typical element of $I_{\alpha+1}$, while $U i\, \psi(i)$ means $\{i \in I : \psi(i)\} \in U$ as usual. Then $U_{\alpha+1}$ is an $D_{\alpha+1}$-adequate ultrafilter and $U_\alpha[\to \alpha+1] \subseteq U_{\alpha+1}$.

3rd, if γ is a limit ordinal then U_γ is any D_γ-adequate ultrafilter which includes $\bigcup_{\alpha<\gamma} U_\alpha[\to \gamma]$ as a subset.

By the construction, we have

$$\langle \text{III} \rangle \quad (1) \quad U^\alpha i'\, U_\alpha i\, \Phi(i \cup i') \iff U_{\alpha+1} j\, \Phi(j) \quad \text{for all } \alpha, \qquad \text{and}$$
$$(2) \quad U_\alpha[\to \beta] \subseteq U_\beta \quad \text{whenever } \alpha < \beta.$$

For $\alpha < \kappa^+$ let $\mathbf{F}_\alpha$ denote the set of all functions f (in $\mathfrak{N}$) mapping I_α into some $R = \mathtt{ran}\, f \in \mathfrak{M}$. Put $\mathbf{F} = \bigcup_{\alpha<\kappa^+} \mathbf{F}_\alpha$ and, for $f \in \mathbf{F}$, let $\alpha(f)$ be the only α such that $f \in \mathbf{F}_\alpha$. Further if $f \in \mathbf{F}_\alpha$, i is finite and, perhaps, $i \notin I_\alpha$, then let $f[i] = f(i \cap D_\alpha)$. Note that $f[i] = f(i)$ whenever $i \in I_\alpha$. Separately define $f[i] = x$ for any i whenever $f = \{\langle 0, x\rangle\} \in \mathbf{F}_0$.

Let $f, g \in \mathbf{F}$ and $\alpha \ge \alpha_0 = \max\{\alpha(f), \alpha(g)\}$. Define $f \mathrel{{}^*{=}} g$ iff we have $U_\alpha i\, (f[i] = g[i])$. (It follows from $\langle \text{III} \rangle$ that this does not depend on the choice of $\alpha \ge \alpha_0$.) Define $f \mathrel{{}^*{\in}} g$ similarly.

A routine verification shows that ${}^*{=}$ is an equivalence relation on $\mathbf{F}$. Hence we can define $[f] = \{g \in \mathbf{F} : f \mathrel{{}^*{=}} g\}$ (the equivalence class of f).

Let $\mathbb{I} = \{[f] : f \in \mathbf{F}\}$ (the quotient). In particular, ${}^*x = [\{\langle 0, x\rangle\}] \in \mathbb{I}$ for any x (the image of x in $\mathbb{I}$). For $[f], [g] \in \mathbb{I}$, define $[f] \,{}^*{\in}\, [g]$ iff $f \,{}^*{\in}\, g$. (This is independent of the choice of representatives because ${}^*{\in}$ is ${}^*{=}$-invariant.) Finally put $\mathtt{st}\,[f]$ iff $[f] = {}^*x$ for some x.

This completes the definition of $\langle \mathbb{I}; {}^*{\in}, \mathtt{st}\,\rangle$ as a set in $\mathfrak{N}$.

For any formula $\Phi(f_1, ..., f_n)$ with functions $f_1, ..., f_n \in \mathbf{F}$ as parameters, define $\Phi[i]$ to be $\Phi(f_1[i], ..., f_n[i])$ and $[\Phi]$ to be $\Phi([f_1], ..., [f_n])$. Put $\alpha(\Phi) = \max_{l \le n} \alpha(f_l)$. In this notation, the Łoś lemma takes the form:

PROPOSITION 17. *Let* Φ *be an* $\in$*-formula, having parameters in* $\mathbf{F}$, *and* $\alpha(\Phi) \le \alpha < \kappa^+$. *Then* $[\Phi]$ *holds in* $\langle \mathbb{I}; {}^*{\in}\rangle$ *iff* $U_\alpha i\, (\mathfrak{M} \models \Phi[i])$. □

This immediately implies both $\mathbf{ZFC}^{\mathrm{st}}$ and Transfer for $\mathbb{I}$.

In addition, the map $x \mapsto {}^*x$ is an isomorphism of $\langle \mathfrak{M}; \in\rangle$ onto $\langle \mathbb{S}; {}^*{\in}\rangle$, where $\mathbb{S} = \{y \in \mathbb{I} : \mathtt{st}\, y\}$.

4.3. From right to left: the external universe

Now, it is not too difficult to "enlarge", in $\mathfrak{N}$, the model $\mathbb{I} = \langle \mathbb{I}; {}^*{\in}, \mathtt{st}\,\rangle$ to a model $\mathbb{H}$ of **NST**, defined as a sort of full type theoretic "superstructure" over $\mathbb{I}$. Specifically, one defines, in $\mathfrak{N}$, a sequence of pairwise disjoint sets H_n, $n \in \omega$, and a binary relation ϵ on $\mathbb{H} = \bigcup_n H_n$, satisfying conditions (h.1) through (h.5). Note that each H_n is assumed to be a member of $\mathfrak{N}$ while the sequence of them is, generally speaking, a definable class but not a set in $\mathfrak{N}$: recall that $\mathfrak{N}$ is a model of **ZC**, but perhaps not of **ZFC**.

(h.1) $H_0 = \mathbb{I}$ and $\epsilon \restriction H_0 = {}^*{\in}$.

(h.2) If $X \in H_{n+1}$ then the collection $X_\epsilon = \{x \in \mathbb{H} : x \,\epsilon\, X\}$ of all ϵ-elements of X is a subset of the set $H_{\le n} = \bigcup_{k \le n} H_k$, but $X \not\subseteq H_{\le n'}$ for any $n' < n$.

Next requirements involve the following definition. Define $\|A\| \subseteq \mathbb{I}$ for all $A \subseteq H_{\le n}$ by induction on n. If $n = 0$, so that $A \subseteq \mathbb{I}$, put $\|A\| = A$. If $A \subseteq H_{\le n+1}$, put $\|A\| = \bigcup_{X \in A} \|X_\epsilon\|$. (Here any X_ϵ is a subset of $H_{\le n}$.)

Say that a set $A \subseteq H_n$ is $\mathbb{S}$-*bounded* when there is a set $S \in \mathbb{S} = \mathfrak{M}$ such that $\|A\| \subseteq S_\epsilon$, where, as above, $S_\epsilon = S_{{}^*\in} = \{x \in \mathbb{I} : x \,\epsilon\, X\}$ is the set of all ϵ-elements (= the set of all ${}^*{\in}$-elements) of X in $\mathbb{I}$.

(h.3) If $X \in H_{n+1}$ then X is $\mathbb{S}$-bounded.

(h.4) If $A \subseteq H_{\le n}$ is $\mathbb{S}$-bounded and $A \not\subseteq H_{\le n'}$ for all $n' < n$ then there is unique $X \in H_{n+1}$ such that $A = X_\epsilon$ — with the following exception:

(h.5) If $A \subseteq H_0$ and already $A = X_\epsilon$ for some $X \in \mathbb{I}$ then there is no $X \in H_1$ such that $A = X_\epsilon$.

(The exceptional case is introduced to keep Extensionality. The restriction to $\mathbb{S}$-bounded sets is introduced in order to keep Standardization, for which sets in $\mathbb{H}$ needn't grow "too big".) Finally, define the standardness predicate $\mathtt{st}$ in $\mathbb{H}$ so that $\mathtt{st}\, X$ iff $X \in \mathbb{I}$ and $\mathtt{st}\, X$ in the sense of $\mathbb{I}$. This ends the definition of $\mathbb{H} = \langle \mathbb{H}; \epsilon, \mathtt{st} \rangle$.

To see that $\mathbb{H}$ is a model of **NST**, it clearly suffices to check Standardization and Saturation in $\mathbb{H}$. (Other axioms follow either from the construction, as, *e.g.*, Separation or Choice, or from the results in the end of Subsection 4.2).

Standardization. Consider a set $X \in H_{n+1}$. Then X is $\mathbb{S}$-bounded by (h.3), so that $\|X\| \subseteq S_\epsilon$ for some $S \in \mathbb{S}$. By definition (Subsection 4.2), we have $S = {}^*s$, for some $s \in \mathfrak{M}$. Let y be the set of all $x \in s$ such that ${}^*x$ is an ϵ-element of X. Then $Y = {}^*y \in \mathbb{S}$ and $Y = X \cap \mathbb{S}$ is true in $\mathbb{H}$.

Saturation. Essentially we have to prove that, given a $\mathbb{S}$-bounded set $\mathcal{X} \subseteq \mathbb{I}$ of cardinality $\mathbf{card}\, \mathcal{X} = \lambda < \kappa$ (in $\mathfrak{N}$), satisfying FIP in the sense that $\bigcap \mathcal{X}' \neq \emptyset$ in $\mathbb{I}$ for any $\mathbb{S}$-finite $\mathcal{X}' \subseteq \mathcal{X}$, the intersection $\bigcap \mathcal{X}$ is also non-empty in $\mathbb{I}$. By the boundedness, $\mathcal{X} \subseteq X$ for a standard X. Then there is (in $\mathfrak{N}$) a sequence of functions $f_\gamma \in \mathbf{F}$, $\gamma < \lambda$, satisfying $\mathbf{ran}\, f_\gamma \subseteq X$ for all γ and $\mathcal{X} = \{[f_\gamma] : \gamma < \lambda\}$.

Pick an ordinal $\alpha < \kappa^+$ big enough for $\alpha(f_\gamma) \leq \alpha$ for all $\gamma < \lambda$. Define $h_{\kappa\alpha+\gamma} = f_\gamma$ for any $\gamma < \lambda$, so h_δ is defined for all $\delta \in D = [\kappa\alpha, \kappa\alpha + \lambda)$.

Define $h \in \mathbf{F}_{\alpha+1}$ as follows. Let $j \in I_{\alpha+1}$. Then $i' = j \cap D$ is a finite subset of D. If $i' = \emptyset$ put $h(j) = \{\emptyset\}$. Otherwise set $h(j) = \bigcap_{\delta \in i'} h_\delta[j]$. Then $h(j) \neq \emptyset$ for all j, hence $[h] \neq \emptyset$ in $\mathbb{I}$ by Proposition 17.

It remains to check that $[h] \subseteq [h_\delta]$ for each $\delta \in D$. This is equivalent to $U_{\alpha+1}\, i\ (h(i) \subseteq h_\delta[i])$. To see that the last inclusion statement is true note that by definition $h(i) \subseteq h_\delta[i]$ for all i such that $\delta \in i$. However we have $\{i \in I_{\alpha+1} : \delta \in i\} \in U_{\alpha+1}$ as this is a $D_{\alpha+1}$-adequate ultrafilter.

$\square$ (*Theorem 3*)

5. Extendibility to KST

This section proves Theorem 4.

5.1. From left to right

Let a transitive model $\mathfrak{M} \models \mathbf{ZFC}$ be the standard part of a model $\mathbb{H} = \langle \mathbb{H}; \epsilon, \mathbb{S}, \mathbb{I} \rangle$ of **KST**, so that $\mathfrak{M} = \mathbb{S} = \{x \in \mathbb{H} : \mathtt{st}\, x\}$ and $\epsilon \restriction \mathfrak{M} = \in \restriction \mathfrak{M}$.

We argue in $\mathbb{H}$.

Define $\mathfrak{N}$ to be the class of all well-founded sets, *e.g.* those sets x which have $\in$-well-founded transitive closure. (**KST** is strong enough to define transitive closures. Moreover, as **KST** contains all of **ZFC** except for Regularity, it is a standard fact that then the model $\langle \mathfrak{N}; \in \restriction \mathfrak{N} \rangle$ satisfies all of **ZFC** including Regularity.)

Define, following Hrbaček[2] and Kawaï [8], for any $x \in \mathfrak{M} = \mathbb{S}$ a set $\hat{x} \in \mathfrak{N}$ by $\hat{x} = \{\hat{y} : y \in x \cap \mathbb{S}\}$.[7] Then $x \mapsto \hat{x}$ is a $1-1$ $\in$-preserving embedding of $\mathbb{S}$ onto an $\in$-transitive part $S = \{\hat{x} : x \in \mathbb{S}\}$ of $\mathfrak{N}$. Moreover, by Standardization, if $x \in S$ and $y \subseteq x$ (in the $\in$-sense) then $y \in S$. This easily implies that $S = (V_\kappa)^{\mathfrak{N}}$, where κ is the least $\mathfrak{N}$-ordinal which does not belong to S, as required.

5.2. From right to left: the internal universe

Suppose that $\mathfrak{M}$ is a transitive model of **ZFC**, isomorphic to $\langle (V_\kappa)^{\mathfrak{N}}; \in \rangle$, where $\mathfrak{N} = \langle \mathfrak{N}; \in \rangle$ is a model of **ZFC** while κ is a $\mathfrak{N}$-cardinal. We have to extend $\mathfrak{M}$ to a model of **KST**.

The whole construction is carried out in $\langle \mathfrak{N}; \in \rangle$, hence

$\langle$IV$\rangle$ we suppose (in **ZFC**) that $\mathfrak{M}$ is a transitive model, of the form $\mathfrak{M} = V_\kappa$, where κ is a cardinal,

and the goal is to define a structure $\langle \mathbb{H}; \in, \mathtt{st} \rangle$ ($\mathbb{H}$ will be a proper class) which models **KST** and has $\mathfrak{M}$ as its standard part.

The whole plan will be as in Section 4: we first embed $\mathfrak{M}$ in an internal model $\mathbb{I}$ (which will be a set) and then extend it to $\mathbb{H}$ (a proper class).

Let κ^+ be the next cardinal, as usual. Define D^α, D_α, I^α, I_α, U^α, U_α for all $\alpha < \kappa^+$ exactly as in Subsection 4.2. $\langle$III$\rangle$ holds in this case as well.

For any $\alpha < \kappa^+$ let $\mathbf{F}_\alpha$ denote the set of all functions which map $I_\alpha = \mathcal{P}_{\text{fin}}(D_\alpha)$ into $\mathfrak{M}$. After this we define $\mathbf{F}$, $\alpha(f)$, $f[i]$, the relations ${}^*\!\!=$ and ${}^*\!\!\in$, the classes $[f] = \{g \in \mathbf{F} : f \,{}^*\!\!= g\}$ (in particular ${}^*x = [\{\langle 0, x \rangle\}]$ for any $x \in \mathfrak{M}$), the model $\langle \mathbb{I}; {}^*\!\!\in, \mathtt{st} \rangle$, and the subclass $\mathbb{S} = \{x \in \mathbb{I} : \mathtt{st}\, x\}$, following the pattern of Subsection 4.2.

Routine verification shows that Proposition 17 survives, together with the schemata $\mathbf{ZFC}^{\text{st}}$ in $\mathbb{I}$ and Transfer.

[7] This definition goes by induction on the von Neumann rank of x in $\mathbb{S}$. To justify this kind of definitions note that by Standardization $\in \restriction \mathbb{S}$ is a well-founded relation.

5.3. From right to left: the external universe

We continue to argue assuming $\langle \mathrm{IV} \rangle$. We have defined a nonstandard extension $\langle \mathbb{I}; {}^*\!\in, \mathtt{st} \rangle$ of $\mathfrak{M}$, which now is to be extended to a model of **KST**.

Following the construction in Subsection 5.2, one easily defines (in $\mathfrak{N}$) an $\mathtt{Ord}$-long sequence of pairwise disjoint sets H_α, $\alpha \in \mathtt{Ord}$, and a binary relation ϵ on $\mathbb{H} = \bigcup_\alpha H_\alpha$, satisfying conditions (h:1) through (h:4).

(h:1) $H_0 = \mathbb{I}$ and $\epsilon \restriction H_0 = {}^*\!\in$.

(h:2) If $\alpha > 0$ and $X \in H_\alpha$ then the collection $X_\epsilon = \{x \in \mathbb{H} : x \,\epsilon\, X\}$ is a subset of $H_{<\alpha} = \bigcup_{\beta<\alpha} H_\beta$ but $X \not\subseteq H_{<\alpha'}$ for any $\alpha' < \alpha$.

(h:3) If conversely $A \subseteq H_{<\alpha}$ but $A \not\subseteq H_{<\alpha'}$ for all $\alpha' < \alpha$ then there is unique $X \in H_\alpha$ such that $A = X_\epsilon$ — with the following exception:

(h:4) If $A \subseteq H_0$ and already $A = X_\epsilon$ for some $X \in \mathbb{I}$ then there is no $X \in H_1$ such that $A = X_\epsilon$.

The only point of notable difference, from the reasoning in Subsection 5.2, in the proof that the structure $\langle \mathbb{H}; \epsilon, \mathbb{S}, \mathbb{I} \rangle$ models **KST**, is the verification of Strong Saturation. Since clearly $\mathbf{card}\, \mathbb{S} = \mathbf{card}\, \mathfrak{M} = \kappa$ (in $\mathfrak{N}$), consider a set $\mathfrak{X} = \{X_\gamma : \gamma < \kappa\} \subseteq \mathbb{I}$, satisfying the property that $\bigcap \mathfrak{X}' \neq \emptyset$ in $\mathbb{I}$ for any finite $\mathfrak{X}' \subseteq \mathfrak{X}$. Then $X_\gamma = [f_\gamma]$ for some $f_\gamma \in \mathbf{F}$ – for all γ. By the cardinality argument there is an ordinal α, $\kappa \leq \alpha < \kappa^+$, big enough for $\alpha(f_\gamma) \leq \alpha$ for all $\gamma < \kappa$. Now, the argument in the end of Subsection 5.2 can be applied, with obvious minor changes, to prove that $\bigcap \mathfrak{X} \neq \emptyset$.

□ (*Theorem 4*)

5.4. Extendibility to well-founded parts of models of KST

We would be interested to strengthen the "only if" part (that is, implication $\Longrightarrow$) of Theorem 4 by the requirement that $\mathfrak{N} = \langle \mathfrak{N}; \epsilon \rangle$ is a transitive $\in$-model, so that $\mathfrak{N}$ is a transitive set while $\epsilon = \in \restriction \mathfrak{N}$. A natural way to get such an improvement would be to prove the following:

$\langle \mathrm{V} \rangle$ Suppose that $\mathfrak{N} = \langle \mathfrak{N}; \epsilon \rangle$ is a non-standard model of **ZFC**, with the standard ω. Then the well-founded part of $\mathfrak{N}$ is a model of **ZFC**.

(The *well-founded part* of $\mathfrak{N}$ consists here of those sets $x \in \mathfrak{N}$ whose $\in$-ranks, defined in $\mathfrak{N}$, are true ordinals. An $\mathfrak{N}$-ordinal α is a *true ordinal* when the set of all ϵ-smaller $\mathfrak{N}$-ordinals is well-founded by ϵ in the universe. For instance by our assumption all $\mathfrak{N}$-natural numbers are true ordinals, *i. e.* just the usual natural numbers.) However this is not provable!

To see that $\langle V \rangle$ is false, consider a countable transitive model $\mathfrak{M} \models$ **ZFC** with the shortest possible $\mathtt{Ord} \cap \mathfrak{M}$. (If there are no transitive models of **ZFC**, we are done.) However the existence of a countable ω-model of **ZFC** is a Σ^1_1 statement. Therefore, there is an ω-model $\langle M; \epsilon \rangle \in \mathfrak{M}$ of **ZFC**. The well-founded part of M is then (isomorphic to) a transitive $\in$-structure, with $\mathtt{Ord} \cap M$ shorter than $\mathtt{Ord} \cap \mathfrak{M}$, hence, not a model of **ZFC**.

Perhaps, this argument[8] can be sharpened enough to demonstrate that the desired improvement is impossible.

On the other hand, there is a different version of extendibility to **KST**, which admits a stronger connection with transitive $\in$-models.

Let, in **KST**, $\mathbb{V}$ be the class of all well-founded sets – *i.e.*, those having well-founded transitive closures. Say that a transitive model $\mathfrak{M} = \langle \mathfrak{M}; \in \rangle$ $\mathbb{V}$-*extends* to a model of **KST** when $\mathfrak{M}$ is $\mathbb{V}$ of a model of **KST**.

THEOREM 18. *A transitive model* $\mathfrak{N} \models$ **ZFC** $\mathbb{V}$-*extends to a model of* **KST** *iff there is a* $\mathfrak{N}$-*cardinal* $\kappa \in \mathfrak{N}$ *such that* $\mathfrak{N}_\kappa = \mathfrak{N} \cap V_\kappa$ *is a model of* **ZFC**.

PROOF. (Sketch) Suppose that a transitive model $\mathfrak{N} \models$ **ZFC** is $\mathbb{V}$ of a model $\mathbb{H} = \langle \mathbb{H}; \in, \mathbb{S}, \mathbb{I} \rangle$ of **KST**. For any standard x, define a set $\hat{x} \in \mathbb{V}$ by $\hat{x} = \{\hat{y} : y \in x \cap \mathbb{S}\}$. As we saw in Subsection 5.1,the set $\mathfrak{M} = \{\hat{x} : x \in \mathbb{S}\}$ is equal to $\mathfrak{N} \cap V_\kappa$ for a $\mathfrak{N}$-cardinal κ.

Conversely, if $\mathfrak{N}$ is a transitive model of **ZFC** while κ is a $\mathfrak{N}$-cardinal and $\mathfrak{N} = \mathfrak{N} \cap V_\kappa$ also is a model of **ZFC**, the construction described in subsections 5.2and 5.3yields a model of **KST** having $\mathfrak{N}$ as $\mathbb{V}$. ∎

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