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STRUCTURE OF CONSTITUENTS OF Π_1^1-SETS

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1. INTRODUCTION. FORMULATION OF PROBLEMS AND MAIN RESULTS

Origin of the Problems. Problems, connected with the interrelation of the first uncountable cardinal number $\aleph_1$ and the cardinality of the continuum $c = 2^{\aleph_0}$, were in the center of attention in the descriptive set theory from the very beginning. The axiom of choice enables us to prove without any special difficulty the inequality $\aleph_1 \leqslant c$, i.e., to construct a subset of the continuum that has the cardinality $\aleph_1$. However, this construction does not give an "individual" example of a point set of cardinality $\aleph_1$ (individual in the same, e.g., of a transcendental number, say π, in counterpoise to the proof of the existence of transcendental numbers by analysis of cardinalities). Luzin turned repeatedly to the investigation of the problem, going back to Lebesgue, of construction of an "individual" point set of cardinality $\aleph_1$ and related problems in his works of the 1930s [1, pp. 552-682].

In particular, it has been observed in [1, p. 643] that "notwithstanding considerable efforts, extending over more than thirty years," it has not been possible to indicate an "individual" point set of cardinality $\aleph_1$ and, moreover, it has not been possible to construct an "individual" totality of $\aleph_1$ Borel sets of bounded rank, i.e., the complexity of these sets in the sense of position in the Borel hierarchy must be bounded above by a certain single countable ordinal. Considering various methods leading to the construction of "individual" totalities of $\aleph_1$ Borel sets, Luzin [1] posed a series of problems, which have the

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following common sense: Is there any method to construct a totality of this kind with bounded rank? These problems, the exact formulation of which is given below, have been considered by Luzin at the weakened forms of the continuum problem.

The present article is devoted to the solution of the problem, posed by Luzin. We show that some of these problems are solved negatively ("the construction is not possible") and others turn out to be unsolvable within the framework of the Zermelo–Fraenkel axiomatic set theory ZFC, but are solvable in a certain well-studied extension of this theory.

The Borel Hierarchy. Sets that can be obtained by repeated application of the operations of complementation and countable union (and countable intersection, which can be expressed in terms of the first two operations) from open sets in a given space are called Borel sets. The Borel sets form a hierarchy, consisting of the classes Σ^0_ρ, Π^0_ρ, and Δ^0_ρ, defined for each ordinal ρ, $1 \leqslant \rho < \omega_1$, by induction over ω in the following manner:

Σ^0_1 is the totality of all open sets;

Π^0_ρ is the totality of the complements of all sets from Σ^0_ρ;

Σ^0_ρ (for $\rho \geqslant 2$) is the totality of all countable unions of sets of the class $\mathbf{\Pi}^0_{<\rho} = \bigcup_{1\leqslant\alpha<\rho} \mathbf{\Pi}^0_\alpha$; $\mathbf{\Delta}^0_\rho = \mathbf{\Sigma}^0_\rho \cap \mathbf{\Pi}^0_\rho$.

Thus, Π^0_1 is the class of closed sets. Each Borel set belongs to one of the classes Δ^0_ρ (and, in this case, to an arbitrary Δ^0_ξ with $\xi > \rho$). The least ordinal ρ such that $X \in \mathbf{\Delta}^0_\rho$, is called the rank of the Borel set X. (Luzin uses the term "class" in place of "ran.") See [2, Chap. 8] for more details about Borel sets.

Sieves and Constituents. The Baire space I, consisting of all the functions defined on the set of natural numbers $\omega = \{0, 1, 2, \dots\}$ with values in ω and holomorphic to the set of all irrational points of the number line [3], is usually considered as the basic space in modern works on descriptive set theory. Let Q denote the set of all rational points of the number line.

Each subset R of I × Q is called a *sieve*. It is convenient to represent a sieve **R** in the form of a family $\langle \mathbf{R}_q : q \in Q\rangle$ of the sets $\mathbf{R}_q = \{x : \langle x, q\rangle \in \mathbf{R}\} \subseteq I$, situated in the ordinary Cartesian plane on the horizontal lines with ordinates q.

Each sieve R determines a partition of the space I into two sets $[\mathbf{R}]$ and $[R]_*$, called the outer and the inner sets:

$[\mathbf{R}] = \{x \in I$: the section $\mathbf{R}''x = \{q : x \in \mathbf{R}_q\}$ is well ordered in the sense of the natural order on the set Q};

$[\mathbf{R}]_* = \{x \in I : \mathbf{R}''$ is not well ordered}.

In its turn, each of these sets is decomposed into $\aleph_1$ pairwise disjoint sets $[R]_\nu$ and $[R]_{*\nu}$, called the *outer* and *the inner constituents*, respectively, and defined for each ordinal $\nu < \omega_1$ by the following equations:

$$[\mathbf{R}]_\nu = \{x \in [\mathbf{R}] : |\mathbf{R}''x| = \nu\};$$
$$[\mathbf{R}]_{*\nu} = \{x \in [\mathbf{R}]_* : |\mathbf{R}''x| = \nu\},$$

where, for each $S \subseteq Q$, the symbol $|S|$ denotes the order type of the greatest well-ordered initial segment of the set S, coinciding with S when S is itself well ordered.

If the sieve **R** is Borel (as a subset of the space I × Q with the discrete topology on Q), then all the constituents $[R]_\nu$ and $[R]_{*\nu}$ are Borel sets, $[\mathbf{R}]_*$ belongs to the class Σ^1_1 (i.e., is the projection of a certain closed subset P of I^2 on I), and [R] belongs to the class Π^1_1 (consisting of the complements of the Σ^1_1-sets).

To investigate the nature of the constituents, Luzin [1] introduced the notion of a derived sieve. The derived sieve **R'** of a given sieve **R** is defined by the following equation:

$$\mathbf{R}' = \{\langle x, q\rangle \in \mathbf{R} : \exists r < q (\langle x, r\rangle \in \mathbf{R})\}.$$

Thus, each section $\mathbf{R}'{}''x$ is obtained by deleting the least point from $\mathbf{R}''x$ if $\mathbf{R}''x$ has such a point; in the contrary case, $\mathbf{R}'{}''x$ and $\mathbf{R}''x$ coincide. The sequence $\langle \mathbf{R}^\nu : \nu < \omega_1\rangle$ of the derived sieves $\mathbf{R}^\nu$ is defined by induction over ν:

$$\mathbf{R}^0 = \mathbf{R},\ \mathbf{R}^{\nu+1} = (\mathbf{R}^\nu)' \text{ for all } \nu \text{ and } \mathbf{R}^\lambda \bigcap_{\nu<\lambda} \mathbf{R}^\nu \text{ for all limit } \lambda < \omega_1.$$

If the initial sieve $\mathbf{R}$ is Borel, then all sieves $\mathbf{R}^\nu$ are also Borel. It is clear that $\mathbf{R}^\nu \subseteq \mathbf{R}^\mu$ for $\mu \leqslant \nu$.

See [1] and [2, Translator's Appendix] for more details about sieves and constituents.

Formulation of Problems. We begin with the formulation of two problems, called "the fundamental problem of the theory of analytic classes" and "the fundamental problem of the theory of sieves" by Luzin [1, pp. 658, 661].

Problem 1. Does there exist an open (as a subset of $I \times Q$) sieve $\mathbf{R}$, the totality of all outer and inner constituents of which is bounded in rank (i.e., all the sets $[\mathbf{R}]_\nu$ and $[\mathbf{R}]_{*\nu}$ have rank less than a fixed ordinal $\rho < \omega_1$) and contains an uncountable number of nonempty sets?

Problem 2. Does there exist an open sieve $\mathbf{R}$, the sequence of the derived sieves $\mathbf{R}_\nu$ of which is bounded in rank and is nonstationary (i.e., $\mathbf{R}^{\nu+1}$ is strictly contained in $\mathbf{R}^\nu$ for each ν)?

An affirmative solution of Problem 1 would lead to the solution of the Luzin "narrow continuum problem," consisting in the construction of a concrete "individual" partition of the continuum into $\aleph_1$ nonempty Borel sets that is bounded in totality in rank.

In the following two problems we are concerned only with outer constituents, which occur significantly more often than the inner ones in the works of the 1920s and 1930s on descriptive set theory. The interest in outer constituents originates from their connection with the fundamental problem of perfect kernels of uncountable Π^1_1-sets. The fact is that if the sieve $\mathbf{R}$ is Borel, then the Π^1_1-set $[\mathbf{R}]$ has a perfect subset if and only if at least one of the constituents $[\mathbf{R}]_\nu$ is uncountable. Therefore, to construct an uncountable Π^1_1-set that does not have perfect subsets, it is sufficient (and also necessary) to construct a Borel sieve that has an uncountable number of nonempty outer constituents and is such that each outer constituent of it is at most countable. (Luzin has considered only open sieves.)

Problem 3. Does there exist an open sieve $\mathbf{R}$ that has an uncountable number of nonempty constituents $[\mathbf{R}]_\nu$ and is such that the sequence of all the outer constituents $[\mathbf{R}]_\nu$ of it is bounded in rank?

Problem 4. Does there exist an open sieve $\mathbf{R}$ that has an uncountable number of nonempty constituents $[\mathbf{R}]_\nu$ and is such that all the constituents $[\mathbf{R}]_\nu$ can be embedded in pairwise disjoint Borel sets, forming a bounded (in rank) sequence?

Problem 3 specially attracted the attention of Luzin (see [1, pp. 173, 555, 624, 672, etc.]). Problems 1-4 are mentioned among important unsolved problems of descriptive set theory in the survey articles [4-6].

Main Results. Before formulating theorems, giving the solutions of the formulated problems, we introduce one more ordinal characteristic of the constituents — the separability index. We call the least ordinal $\rho < \omega_1$ such that a Σ^0_ρ-set contains entirely a constituent $[\mathbf{R}]_\nu$, but does not intersect any constituent $[\mathbf{R}]_\mu$ with $\mu < \nu$, the *separability index* (in short, S.I.) *of the constituent* $[\mathbf{R}]_\nu$. We introduce an analogous notion for the inner constituents. It is clear that the rank of no constituent is less than its S.I., and therefore each sequence of constituents, bounded in rank, is also bounded in the separability index.

The following two theorems give negative solutions of Problems 1 and 2.

1.1. THEOREM. If a sequence of inner constituents $[\mathbf{R}]_{*\nu}$ of a Borel sieve $\mathbf{R}$ contains an uncountable number of nonempty sets, then this sequence is unbounded in S.I., i.e., for each $\rho < \omega_1$ there exists a number $\nu < \omega_1$ such that the S.I. of the constituent $[\mathbf{R}]_{*\nu}$ is greater than ρ.

1.2. THEOREM. If the sequence of the derived sieves $\mathbf{R}^\nu$ of a given Borel sieve $\mathbf{R}$ is nonstationary, then it is unbounded in rank.

A negative solution of Problem 1 ("such sieves do not exist") is deduced from Theorem 1.1 in the following manner. Let, on the contrary, an open, or even a Borel, sieve $\mathbf{R}$ satisfy the conditions of Problem 1. Then the number of nonempty constituents is at most countable by Theorem 1.1. Consequently, the set $[\mathbf{R}]_* = \bigcup_{\nu<\omega_1}[\mathbf{R}]_{*\nu}$ is Borel. Then $[\mathbf{R}] = I[\mathbf{R}]_*$ is also a Borel set. Hence, by a well-known theorem of descriptive set theory (see, e.g., [2, Appendix, Sec. 2]), the number of nonempty outer constituents $[\mathbf{R}]_\nu$ is also at most countable. Thus, R has in all an at most countable number of constituents of both types, which is a contradiction with the choice of **R**.

In distinction from Problems 1 and 2, Problems 3 and 4 are unsolvable in the classical sense, i.e., on the basis of the axioms of ZFC, according to the following theorem.

1.3. THEOREM. The following statements (a)-(g) are pairwise equivalent and undecidable by means of ZFC:

(a) Problem 3 has an affirmative solution ("such a sieve exists").

(b) Problem 4 has an affirmative solution.

(c) There exists an open sieve, each outer constituent of which contains at most one point and the number of nonempty outer constituents of which is uncountable.

(d) There exists an open sieve with an uncountable number of outer constituents, the ranks of the nonempty outer constituents of which do not converge to ω_1, i.e.,

$$\exists\rho<\omega_1\ \forall\nu<\omega_1\ \exists\mu\geqslant\nu([\mathbf{R}]_\mu\neq\varnothing\wedge\operatorname{rank}\cdot[\mathbf{R}]_\mu\leqslant\rho);$$

(e) There exists a Borel sieve with an uncountable number of nonempty outer constituents, the separability indices of the nonempty outer constituents of which do not converge to ω_1.

(f) Each statement lying between (c) and (e) [as, e.g., the statements (a) and (b)] is valid.

(g) There exists $z\in I$ such that $\omega_1^{L[z]}=\omega_1$.

We denote the ξ-th (in magnitude) uncountable cardinal in the class L[z] of all sets that are constructible with respect to z by $\omega_\xi^{L[z]}$ [2, 7]. The statement (g) has been studied quite well. In particular, we know that it is nondeducible in ZFC [8] and does not contradict ZFC (since it is a consequence of the noncontradictory axiom of constructibility). We also know that (g) is equivalent to each of the following two statements, which can therefore be added to Theorem 1.3:

(h) There exists an uncountable Π_1^1-set that does not have perfect subsets.

(i) There exists an open sieve, each outer constituent of which is at most countable and the number of nonempty outer constituents of which is uncountable.

Novikov [9] has proved that the axiom of constructibility V = L implies (h). It has been shown in [10] that the premise in this statement can be replaced by the statement (g), which is weaker than V = L. Thus, (g) $\to$ (h). The reverse implication has been proved by Solovay [11]. The equivalence (h) $\leftrightarrow$ (i) is one of the classical theorems (see [1] or [2, Appendix]). Finally, the equivalence (i) $\leftrightarrow$ (c) has been established in [12]. By virtue of all these results, to prove Theorem 1.3 it is sufficient to prove the implication (e) $\to$ (g), which we shall do.

As is obvious from Theorems 1.1 and 1.3, the inner and the outer constituents have various properties with respect to the considered problems. This is explained by the fact that nonempty inner constituents must occur quite often when their number is uncountable, whereas this does not hold for the outer constituents. We can indicate a series of additional conditions that ensure a sufficiently frequent appearance of nonempty outer constituents and enable us to prove theorems, similar to Theorem 1.1, for the outer constituents. The following theorem can serve as an example.

1.4. THEOREM. If all the outer constituents of a Borel sieve are nonempty, then their sequence is unbounded in separability index.

In the sequel we will indicate a weaker condition.

Estimation of the Rate of Growth of Separability Indices (S.I.). By Theorem 1.3, if under the supposition of $\neg$ (g) a Borel sieve R has an uncountable number of nonempty outer constituents $[R]_\nu$, then the separability indices of nonempty $[R]_\nu$ converge to ω_1. The following two theorems enable us to give a lower bound for the rate of growth of S.I. Before their formulation, let us observe that for each set or class M we introduce the part $\Delta_1^1(M)$ of the totality Δ_1^1 of all Borel sets that consists of all the sets which admit a Σ_1^1- or a Π_1^1-definition with parameters from M (see Sec. 2).

1.5. THEOREM. Let $z \in I$, $1 \leqslant \rho < \omega_1^{L[z]}$, $\omega_\rho^{L[z]} \leqslant \nu < \omega_1$, the sieve R belong to $\Delta_1^1(L[z])$, and $[\mathbf{R}]_\nu \neq \emptyset$. Then by S.I. of the constituent $[\mathbf{R}]_\nu$ is strictly greater than the ordinal $1 + \rho$. The same is valid for the inner constituents.

1.6. THEOREM. Let $z \in I$, $1 \leqslant \rho < \omega_1^{L[z]}$, $\omega_\rho^{L[z]} < \omega_1$, and the sieve $\mathbf{R} \in \Delta_1^1(L[z])$ have an uncountable number of nonempty outer constituents. Then there exists an outer constituent $[\mathbf{R}]_\mu$ with the index $\mu < \omega_1^{L[z]}$ whose S.I. is greater than $1 + \rho$. The same is valid for the inner constituents.

The proof of Theorems 1.1–1.6 is the content of the present article. The proof of Theorem 1.5 (Secs. 2–5), which is the main theorem of our article, occupies most of the space. It is organized in the following manner. In Sec. 2 we consider the problems of definability of the sequence of the constituents of Borel sieves and of definability of a special coding of Borel sets. In Sec. 3 we give a proof of the absoluteness of certain properties of constituents and Borel codes. In Sec. 4 we carry out proof of Theorem 1.5, leaving two auxiliary statements unproved; these are proved in Sec. 5. Finally, Sec. 6 contains the deduction of Theorems 1.1–1.4 and 1.6 from Theorem 1.5.

Some Remarks. The main results of the present article have been published by the author in the notes [12, 13].

Theorems 1.1–1.4 can be reformulated quite naturally in the language of the arithmetic of second order A_2, speaking only about natural numbers and points of the space I. The same can be done (although not in such a direct manner) for Theorems 1.5 and 1.6 also. Therefore, the question of the possibility to prove these theorems by means of the theory A_2 or its conservative extension ZFC^- (i.e., ZFC without the axiom of power) is quite natural. This possibility exists for all the classical theorems of descriptive set theory. An analysis of the proposed proofs shows that Theorems 1.3 and 1.5 (more precisely, their reformulations) can be proved in A_2. On the contrary, the possibility of proof in A_2 remains open for Theorems 1.1, 1.2, 1.4, and 1.6. The fact is that our proofs of the last four theorems use the principle of Borel determinacy [14], which is provable in ZFC but not in A_2. On the contrary, this principle can be removed from the proofs of Theorems 1.3 and 1.5.

By Theorem 1.3, the absence of uncountable bounded (in rank) sequences of Borel sets, given by determined constructions, is deduced in the theory ZFC + $\neg$ (g). Certain other statements of descriptive set theory, undecidable by means of ZFC (see [10, 11, and 2, Appendix]), are decidable in this theory. In the theory ZF + Axiom of Determinacy, we can prove [15] that, in general, there do not exist any bounded (in rank) sequences that contain $\aleph_1$ pairwise different sets. (This result, as the axiom of determinacy itself, contradicts the axiom of choice.) There are no such sequences in the well-known Levy–Solovay mode, where all the sets are Lebesgue-measurable (a result of Stern [16]; see also [2, Appendix]). Incidentally, a certain statement, formulated without proof in [16] (Theorem 4.2 below), plays an important role in the proof of Theorem 1.5.

The reader can find a series of interesting results, similar to the theme of the present article, in the articles [17, 18].

2. ANALYTIC EXPRESSIONS FOR CERTAIN RELATIONS

Analytic Formulas. Each Borel sieve, which is, in general, an uncountable object, can nevertheless be given by countable information. Below we set forth one of the methods of mounting this information, directly realizing a Borel structure. Another method uses the definition of Borel and projective sets with the help of analytic formulas, i.e., formulas of the language of the arithmetic of second order.

This language contains 1) variables i, j, k, l,... of type ω with the domain ω; 2) variables x, y, z, f, w,... of type I with the domain I; 3) the functional symbols of addition and multiplication for variables of type ω; and 4) the three-place predicate symbol $x(i) = j$.

Analytic formulas, not having connected variables of type ω, are said to be *arithmetical.* Formulas of the forms

$$\exists x\varphi,\ \forall x\varphi,\ \exists x\forall y\varphi,\ \forall x\exists y\varphi,$$

where φ is an arithmetical formula, are called, respectively, the Σ_1^1-formulas, the Π_1^1-formulas, the Σ_2^1-formulas, and the Π_2^1-formulas.

The free variables of the analytic formulas can, in the usual manner, be replaced by variables from ω and I. For an arbitrary set, or class, M we denote the totality of all subsets of the space of the form $I^m \times \omega^n$, defined by Σ^1_1-formulas, in which parameters of type ω from ω and of type I from $M \cap I$ can occur, by $\Sigma^1_1(M)$. The classes $\Pi^1_1(M)$, $\Sigma^1_2(M)$, and $\Pi^1_2(M)$ are defined analogously and $\Delta^1_k(M) = \Sigma^1_k(M) \cap \Pi^1_k(M)$. The totality of all Borel sets in the indicated spaces, denoted by $\underline{\Delta}^1_1$, coincides exactly with $\Delta^1_1(I)$.

Codes of Sieves. Having fixed once for always some natural enumeration $Q = \{q_k : k \in \omega\}$ of the set of rational numbers Q, we carry over the notions, connected with analytic definability, to sieves.

We agree to call each pair $R = \langle \varphi\psi \rangle$, consisting of a Σ^1_1-formula $\varphi(x, k)$ and a Π^1_1-formula $\psi(x, k)$, having the free variables x and k only such that

$$\forall k \in \omega \ \forall x \in I(\varphi(x, k) \leftrightarrow \psi(x, k)),$$

a code of sieve. Each such pair R defines the Borel sieve

$$\mathbf{R} = \{\langle x, q_k \rangle : \varphi(x, k)\} = \{\langle x, q_k \rangle : \psi(x, k)\}. \tag{1}$$

Conversely, by virtue of the equation $\underline{\Delta}^1_1 = \Delta^1_1(I)$, each Borel sieve **R** is defined in the indicated manner by a certain code of sieve R. We denote the codes of sieves by simple letters R, whereas the sieves themselves are denoted by semibold letters **R**.

If a sieve **R** is defined by the code of sieve R by the relation (1), then we set

$$[R] =]\mathbf{R}], \ ]R]_* = [\mathbf{R}]_*,$$
$$[R]_\nu = [\mathbf{R}]_\nu \text{ and } [R]_{*\nu} = [\mathbf{R}]_{*\nu}$$

for all $\nu < \omega_1$.

We say that a sieve **R** belongs to the class $\Delta^1_1(M)$ when there exists a code of sieve R = $\langle \varphi\psi \rangle$ such that all parameters of type I of the formulas φ and ψ belong to the set $M \cap I$ and **R** is obtained from R by Eq. (1). In other words, it is required that the set

$$\{\langle x, k \rangle : k \in \omega \wedge \langle x, q_k \rangle \in \mathbf{R}\} \subseteq I \times \omega$$

should belong to the class $\Delta^1_1(M)$.

Let us observe that a code of sieve is, in essence, a suite of points of the space I, natural numbers, and logical symbols, which can also be assumed to be (coded) natural numbers. Consequently, if M is a transitive model (TM) of the theory ZFC, then a sieve **R** would belong to $\Delta^1_1(M)$ if and only if there exists a code of sieve $R \in M$, giving **R** by Eq. (1).

Analytic Expression of Belongingness to Constituents. We fix a code of sieve $R = \langle \varphi\psi \rangle$ and write down the formulas, expressing belongingness to the constituents $[R]_\nu$ and $[R]_{*\nu}$, and also to the sets [R] and $[R]_*$ and to the "approximations"

$$[R]_{<\nu} = \bigcup_{\mu<\nu} [R]_\mu, \ [R]_{*<\nu} = \bigcup_{\mu<\nu} [R]_{*\mu}.$$

If $x \in I$, then we set

$$x'0 = \{q_k : x(k) = 0\} \text{ and } x^* = \{\langle q_i, q_j \rangle : x(i) = j\}.$$

Thus, $x'0 \subseteq Q$, and x^* is a function from Q into Q.

We write down the following formulas:

$\Phi_R(w, x) \leftrightarrow \exists f$ *(the restriction* $f^* \restriction w'0$ *of the function* f^* *to the set* $w'0$ *is an order-isomorphism (in short, OI) of* $w'0$ *onto* $R''x = \{q_k : \varphi(x, k)\}$);

$\Phi_{R*}(w, x) \leftrightarrow \exists f (f^* \restriction w'0)$ *is an OI of the set* $w'0$ *onto a certain initial segment (in short, IS)* u *of the sets* $R''x$ *such that the difference* $R''x - u$ *is nonempty and does not have any least element)*;

$\Psi_R(w, x) \leftrightarrow \forall f \forall y([(y'0$ *is an IS of the set* $w'0) \wedge (f^* \restriction y'0$ *is an OI of* $y'0$ *onto a certain IS* u *of the set* $R''x) \wedge$ *(at least one of the sets* $w'0 - y'0$ *and* $R''x - u$ *is either empty or does not have any least element)]* $\rightarrow w'0 = y'0 \wedge u = R''x)$;

$\Psi_{R*}(w, x) \leftrightarrow \forall f \forall y ([\ldots] \rightarrow w'0 = y'0 \wedge$ *the set* $R''x - u$ *is nonempty and does not have any least element)*.

(In the last formula, the same expression stands within the square brackets as in the preceding formula.) Further, if $w \in I$ and $m \in \omega$, then we define a function $w_{<m} \in I$ by the relations $w_{<m}(k) = w(k)$ for $q_k < q_m$ and $w_{<m}(k) = 1$ for $q_k \geqslant q_m$. We write down four more formulas:

$$\Phi_{R<}(w, x) \leftrightarrow \exists m(w(m)=0 \wedge \Phi_R(w_{<m}, x));$$
$$\Psi_{R<}(w, x) \leftrightarrow \exists m(w(m)=0 \wedge \Psi_R(w_{<m}, x)),$$

and exactly the same formulas for $\Phi_{R<*}(w, x)$, and $\Phi_{R<*}(w, x)$.

Finally, we write down the formula

$$\theta_R(x) \leftrightarrow \forall y(y'0 \subseteq R''x \wedge y'0 \neq \varnothing \rightarrow y'0 \text{ has a least element}).$$

These formulas have the following meaning. Let us set

$$WO = \{w \in I : w'0 \text{ is well ordered}\}.$$

and $WO_\nu = \{w \in WO : |w'0| = \nu\}$ for each ordinal $\nu < \omega_1$. The following lemma is valid.

2.1. LEMMA. Let $\nu < \omega_1$, $w \in WO_\nu$, and $x \in I$. Then

$$x \in [R]_\nu \leftrightarrow \Phi_R(w, x) \leftrightarrow \Psi_R(w, x);$$
$$x \in [R]_{*\nu} \leftrightarrow \Phi_{R*}(w, x) \leftrightarrow \Psi_{R*}(w, x);$$
$$x \in [R]_{<\nu} \leftrightarrow \Phi_{R<}(w, x) \leftrightarrow \Psi_{R<}(w, x);$$

$$x \in [R]_{*<\nu} \leftrightarrow \Phi_{R<*}(w, x) \leftrightarrow \Psi_{R<*}(w, x);$$
$$x \in [R] \leftrightarrow \theta_R(x) \text{ and } x \in [R]_* \leftrightarrow \neg\theta_R(x).$$

This lemma is proved by direct analysis of the definitions of Sec. 1. We leave its proof to the reader.

2.2. Remark. The formulas Φ_R, Φ_{R*}, $\Phi_{R<}$, and $\Phi_{R*<}$ are Σ^1_1-formulas and the formulas Ψ_R, Ψ_{R*}, $\Psi_{R<}$, $\Psi_{R*<}$, and θ_R are Π^1_1-formulas. All these formulas can be effectively obtained from the formulas φ and ψ, constituting the considered code of sieve R, and do not contain parameters of type I, except those which already occur in φ and ψ.

Indeed, all "verbal" fragments in the definitions of these formulas can be replaced by the corresponding arithmetical formulas, and the expressions of the type $k \in R''x$ can be eliminated with the help of the Σ^1_1-formula $\varphi(x, k)$ in some places and with the help of the Π^1_1-formula $\psi(x, k)$ in other places; moreover, this can be done such that the obtained expressions can be transformed into the desired form with the help of known rules [2, Appendix Sec. 2) and [19, Chap. 7]. Inequalities of the form $q_k < q_l$, which occur after passage from rational numbers to natural numbers with the help of the enumeration $k \to q_k$, can be replaced by suitable arithmetical formulas (which exist if the enumeration is chosen in a natural manner).

Borel Codes. The proposed method of coding of Borel sets is based on the fact that for $\rho \geqslant 2$ each set of the class Π^0_ρ in the space I is the complement of a countable union of sets of the class $\Pi^0_{<\rho}$, and conversely.

For each ordinal γ, we denote the totality of all finite sequences of ordinals less than γ, including the unique sequence Λ of length zero, by Seq_γ. Let us set $\mathrm{Seq} = \bigcup_{\gamma \in \mathrm{Ord}} \mathrm{Seq}_\gamma$, where Ord is the class of all the ordinals. The length of a sequence $\sigma \in \mathrm{Seq}$ is denoted by $\mathrm{dom}\,\sigma$. The relation $\sigma \subset \tau$ means that τ is a proper extension of σ. If $\sigma \in \mathrm{Seq}$ and $\alpha \in \mathrm{Ord}$, then we denote the sequences obtained by affixing the element α, respectively, to the right and to the left of the *elements of the sequence* σ by $\sigma^\wedge\alpha$ and $\alpha^\wedge\sigma$, respectively.

We will call each nonempty subset T of Seq, satisfying the following relation, a *tree*:

$$\sigma \in T \wedge \tau \in \mathrm{Seq} \wedge \tau \subset \sigma \rightarrow \tau \in T.$$

Each tree trivially contains the sequence Λ. We say that a tree T is *grounded* if there do not exist any infinite paths $\tau_0 \subset \tau_1 \subset \tau_2 \subset \ldots$, where each τ_i belongs to T. In this case, with each $\sigma \in T$ we can uniquely associate an ordinal $|\sigma|_T$ such that

$$|\sigma|_T = \sup_{\tau \in T,\, \sigma \subset \tau}(|\tau|_T + 1).$$

In addition, $|\sigma|_T = 0 \leftrightarrow \sigma \in \mathrm{Max}\, T$, where

$$\mathrm{Max}\, T = \{\sigma \in T : \neg\exists\tau \in T(\sigma \subset \tau)\}.$$

We call the ordinal $|T| = |\Lambda|_T$ *the height of the tree T* and denote the least ordinal γ such that $T \subseteq \mathrm{Seq}_\gamma$ by sup T.

We call each pair <T, d> such that T is a grounded tree with $\sup T < \omega_1$ and $d \subseteq T \times \omega$ a *Borel code*. With each such pair we associate a Borel set $[T, d] \subseteq I$ in the following manner. We fix once for always some natural enumeration $\langle I_k : k \in \omega \rangle$ of all basic closed–open sets of the space I. For each $\sigma \in T$ we define by induction over $|\sigma|_T$ the set $[T, d, \sigma] \subseteq I$ according to the equations:

$$[T, d, \sigma] = I - \bigcup_{\langle \sigma, k \rangle \in d} I_k \quad \text{for} \quad |\sigma|_T = 0;$$
$$[T, d, \sigma] \in I - \bigcup_{\sigma^\wedge \alpha \in T} [T, d, \sigma^\wedge \alpha] \quad \text{for} \quad |\sigma|_T \geqslant 1.$$

Now, let us set $[T, d] = [T, d, \Lambda]$. The sets $[T, d, \sigma]$ are all Borel sets. More precisely, $[T, d, \sigma] \in \mathbf{\Pi}^0_{1+|\sigma|_T}$, and therefore $[T, d] \in \mathbf{\Pi}^0_{1+|T|}$.

Conversely, if $\rho < \omega_1$, then each subset of I of the class $\Pi^0_{1+\rho}$ can be expressed in the form $[T, d]$, where the tree T has height ρ. We call a grounded tree $T \subseteq \mathrm{Seq}_\omega$ a *universal tree of height* ρ if $|T| = \rho$ and for each $\Pi^0_{1+\rho}$-set $X \subseteq I$ there exists a set $d \subseteq T \times \omega$ such that $X = [T, d]$.

2.3. LEMMA. Let $\rho < \omega_1$. Then there exists a universal tree $T_\rho \subseteq \mathrm{Seq}_\omega$ of height ρ.

We prove this lemma by induction over ρ. The tree $T_0 = \{\Lambda\}$ is universal of height zero, since each Π^0_1-set (i.e., closed) $X \subseteq I$ has the form $X = I - \bigcup_{k \in u} I_k$ (and now it is necessary to take $d = \{\Lambda\} \times u$).

Let us suppose that $\rho \geqslant 1$ and that for each ordinal $\alpha < \rho$ there exists a universal tree T_α of height α. We take a function $g : \omega$ onto $\rho = \{\alpha : \alpha < \rho\}$, taking each of its values $\alpha < \rho$ an infinite number of times. The tree

$$T_\rho = \{n^\wedge \sigma : n \in \omega \wedge \sigma \in T_{g(n)}\} \cup \{\Lambda\}$$

is universal of height ρ. Indeed, let $X \in \mathbf{\Pi}^0_{1+\rho}$. By the choice of the function g, we can write down $X = I - \bigcup_{i \in \omega} X_i$, where each X_i belongs to $\Pi^0_{1+g(i)}$. By virtue of the universality of the trees T_α, we have T_α, $X_i = [T_{g(i)}, d_i]$ for suitable sets $d_i \subseteq T_{g(i)} \times \omega$. Let us set

$$d = \{\langle n^\wedge \sigma, k \rangle : n \in \omega \wedge \langle \sigma, k \rangle \in d_n\}$$

and get $X = [T_\rho, d]$.

2.4. Remark. The inductive construction in the proof of Lemma 2.3 up to the step ρ inclusively becomes completely determined as soon as we fix a bijection of ω onto ρ. Therefore, Lemma 2.3 can be strengthened in the following manner.

If M is a TM of the theory ZFC and $\rho < \omega_1^M$, then there exists a grounded tree $T_\rho \in M$, such that $T_\rho \subseteq \mathrm{Seq}_\omega$ and in each TM, extending the model M (in particular, in M and in the universe of all sets), it is true that T_ρ is a universal tree of height ρ.

An Analytic Expression for Belongingness to [T, d]. Let M be a TM of the theory ZFC (a set or a class), the Borel code [T, d] belong to M, and $\sup T < \omega_1^M$.

2.5. THEOREM. Under these conditions, there exist a Σ^1_1-formula $\Phi_{Td}(x)$ and a Π^1_1-formula $\Psi_{Td}(x)$, both with parameters from M, such that the following equivalences are true for each $x \in I$ in each TM of the theory ZFC that extends M:

$$x \in [T, d] \leftrightarrow \Phi_{Td}(x) \leftrightarrow \Psi_{Td}(x).$$

Proof. Without loss of generality, we can suppose that $T \subseteq \mathrm{Seq}_\omega$. (If $\sup T > \omega$, then, since $\sup T < \omega_1^M$, there exists a bijection $b \in M$, $b : \omega$ onto $\sup T$ which belongs to M and whose action can "transfer" T and d into Seq_ω.) The formula

$$\begin{aligned}\Phi_{Td}(x) \leftrightarrow \exists g([(g : T \to \{0, 1\}) \wedge \forall \sigma \in \operatorname{Max} T(g(\sigma) = 1 \leftrightarrow \\ \leftrightarrow \forall k(\langle \sigma, k \rangle \in d \to x \notin I_k)) \wedge \forall \sigma \in T - \operatorname{Max} T(g(\sigma) = 1 \leftrightarrow \\ \leftrightarrow \forall n(\sigma^\wedge n \in T \to g(\sigma^\wedge n) = 0))] \wedge g(\Lambda) = 1)\end{aligned}$$

satisfies, as is easily verified, the desired equivalence. The formula Φ_{Td} is not formally a Σ^1_1-formula if only because it includes the parameter T not belonging, in general, to ω or I. However, using some bijection, belonging to M, of ω onto Seq_ω, and introducing parameters that belong to $M \cap I$, corresponding to the sets T and in a natural manner under this bijection, we can easily rewrite the definition of Φ_{Td} such that the variable g becomes a variable over I and the formula within the outer parentheses becomes an arithmetical formula.

The second desired formula can be defined as follows:

$$\Psi_{Td}(x) \leftrightarrow \forall g([\dots] \to g(\Lambda) = 1),$$

where the square brackets contain the same expression as in the formula $\Phi_{Td}(x)$. We leave the details to the reader.

3. ABSOLUTE PROPERTIES OF CONSTITUENTS AND BOREL CODES

Let us consider the following situation. Let M be a transitive model (TM) of the theory ZFC and the Borel code <T, d> and the code of sieve R belong to M, and, in addition, $\sup T < \omega_1^M$. How are the properties of <T, d> and R in the universe V of all sets related with their properties in M? The following theorem gives a list of the properties, whose validity in M and in V are interconnected.

3.1. THEOREM. (a) The following statement is valid in M: R is a code of sieve.

(b) If R has an uncountable number of nonempty inner constituents $[R]_{*\mu}$, then this is true in M also, i.e., for each $\nu < \omega_1^M$ there exists an ordinal $\mu > \omega_1^M$, $\mu \geqslant \nu$, such that $[R]_{*\mu} \neq \varnothing$ is true in M.

(c) If $\omega_1 \subseteq M$ and $\rho < \omega_1^M$, then the statements

"there exists a constituent $[R]_\nu$ with SI > 1 + ρ" and

"there exists a constituent $[R]_{*\nu}$ with SI > ρ" are absolute for M, i.e., each of them is true in M if and only if it is true in the universe V of all sets.

(d) If $x \in M \cap I$ and $\nu, \rho < \omega_1^M$, then the following statements and their analogs for inner constituents are absolute for M:

(1) $[R]_\nu \cap [T, d] \neq \varnothing$;

(2) $[R]_{<\nu} \subseteq [T, d]$;

(3) $x \in [R]_\nu$;

(4) the S.I. of $[R]_\nu$ does not exceed 1 + ρ.

(e) If $\omega_1 \subseteq M$, then the statement $[R] \cap [T, d] \neq \varnothing$ and $[R]_* \cap [T, d] \neq \varnothing$ are absolute for M.

All the statements of this theorem, except (c) and (d) (4), are proved by an elementary application of the following two principles:

The Mostowski Absoluteness Principle. All closed Σ_1^1-formulas and Π_1^1-formulas with parameters from M are absolute for M.

The Shoenfield Absoluteness Principle. If $\omega_1 \subseteq M$, then all closed Σ_2^1-formulas and Π_2^1-formulas with parameters from M are absolute for M.

A proof of the Shoenfield principle can be found in [19] or in [2, Appendix, Sec. 2]. The Mostowski principle is proved in the course of the standard proofs of the Shoenfield principle.

Proof of (a). The fact that $R = \langle \varphi, \psi \rangle$ is a code of sieve can be written down by the formula

$$\forall x \in I \forall k \in \omega(\varphi(x, k) \leftrightarrow \psi(x, k)),$$

i.e., by a Π_2^1-formula with parameters from M, since $R \in M$. (More precisely, this formula is reduced to Π_2^1-form with parameters from M with the help of the rules of [19, Chap. 7].) Now we use the Mostowski principle, according to which a Π_2^1-formula, true in the universe V, is also true in M.

Proof of (b). We fix an ordinal $\nu < \omega_1^M$. It is clear that there exists a $w \in WO_\nu \cap M$. Now the existence of a nonempty constituent $[R]_\mu$ with $\mu \geqslant \nu$ can (in M as well as in V) be expressed by the following Σ_1^1-formula with parameters from M:

$$\exists x(\neg \theta_R(x) \wedge \neg \Psi_{R<*}(w, x)).$$

After this, we apply the Mostowski principle.

Proof of (d) (1, 2, 3). It is necessary to use the same Mostowski principle by means of the following formulas with parameters from M:

$$\exists x(\Phi_R(w, x) \wedge \Phi_{Td}(x)) \quad (\Sigma_1^1\text{-formula });$$
$$\forall x(\Phi_R(w, x) \to \Psi_{Td}(x)) \quad (\Pi_1^1\text{- formula });$$
$$\Phi_R(w, x) \quad (\Sigma_1^1\text{-formula }),$$

where w is an arbitrary element of $WO_\nu \cap M$.

Proof of (e). We prove (e) with the help of the Shoenfield principle and the following Σ_2^1-formula: $\exists x(\theta_R(x) \wedge \Phi_{Td}(x))$. Incidentally, the condition $\omega_1 \subseteq M$ is introduced in (e) so that the Shoenfield principle can be applied.

The Borel Determinacy. To prove the statement (c) and the absoluteness of (d) (4), we must use the principle of Borel determinacy. We begin with some definitions, connected with this principle.

Each function $s:\mathrm{Seq}_\omega \to \omega$ ("strategy") determines two continuous mappings $x \mapsto x^\wedge s$ and $x \mapsto s^\wedge x$ from I into I in the following manner: If $x \in I$, then the points $y = x^\wedge s \in I$ and $z = s^\wedge x \in I$ are given by the relations

$$y(k) = s(x \upharpoonright (k+1)) \text{ and } z(k) = s(x \upharpoonright k) \text{ for all } k.$$

A subset A of I^2 is said to be *determined* if there exists a strategy $s:\mathrm{Seq}_\omega \to \omega$ such that

$$\forall y(\langle s^\wedge y, y\rangle \in A) \vee Ax(\langle x, x^\wedge s\rangle \notin A).$$

The Principle of Borel Determinacy [14]. Each Borel subset A of I^2 is determined.

Before using this principle in the proof of the remaining statements of Theorem 3.1, let us observe that for each ordinal $\rho < \omega_1$ there exists a subset U of I that belongs to the class $\Sigma_{1+\rho}^0$, but does not belong to $\Pi_{1+\rho}^0$, and therefore there exist a Σ_1^1-formula $\Phi(x)$ and a Π_1^1-formula $\Psi(x)$ such that

$$\{x \in I : \Phi(x)\} = \{x : \Psi(x)\} \in \Sigma_{1+\rho}^0 - \Pi_{1+\rho}^0.$$

The standard construction of these sets and formulas [2, Chap. 8] by induction over ρ is quite "effective" for the proof of the following lemma (for our model M).

3.2. LEMMA. If $\rho < \omega_1^M$, then there exist a Σ_1^1-formula $\Phi_\rho(x)$ and a Π_1^1-formula $\Psi_\rho(x)$, both with parameters from M only, such that the following statement is true in each TM of the theory ZFC that extends M (in particular, in M and in the universe V):

$$\{x : \Phi_\rho(x)\} = \{x : \Psi_\rho(x)\} \in \Sigma_{1+\rho}^0 - \Pi_{1+\rho}^0.$$

We fix the formulas $\Phi_\rho(x)$ and $\Psi_\rho(x)$, given by this lemma for the ordinal $\rho < \omega_1^M$ that occurs in the statements (c) and (d) of Theorem 3.1. We write down the following two formulas:

$$\Phi_{R\rho}(w) \leftrightarrow \exists s : \mathrm{Seq}_\omega \to \omega \forall y((\Phi_R(w, y) \to \Psi_\rho(s^\wedge y) \wedge (\Phi_{R<}(w, y) \to \neg\Phi_\rho(s^\wedge y)));$$

$$\Psi_{R\rho}(w) \leftrightarrow \exists s : \mathrm{Seq}_\omega \to \omega \forall x((\Phi_\rho(x) \to \Psi_{R<}(w, x^\wedge s)) \wedge (\neg\Psi_\rho(x) \to \Psi_R(w, x^\wedge s))).$$

3.3. LEMMA. The following statement is true in each TM of the theory ZFC that extends M: If $\nu < \omega_1^M$ and $w \in WO_\nu$, then

$$\Phi_{R\rho}(w) \leftrightarrow \Psi_{R\rho}(w) \leftrightarrow \mathrm{SI}\,[R]_\nu < 1 + \rho.$$

Proof. We reason in an arbitrary TM of the indicated form. Let us set $U = \{x : \Phi_\rho(x)\} = \{x : \Psi_\rho(x)\}$. If $\Phi_{R\rho}(w)$ is true, then the corresponding strategy s ensures the inequality SI × $[R]_\nu \leqslant 1 + \rho$, since in this case the set $\{y : s^\wedge y \in U\}$ belongs to $\Sigma_{1+\rho}^0$, being a continuous inverse image of the $\Sigma_{1+\rho}^0$-set U, and also contains $[R]_\nu$ entirely and does not intersect $[R]_{<\nu}$ (it is necessary to use Lemma 2.1).

If $\mathrm{SI}[R]_\nu \leqslant 1 + \rho$, then $\Psi_{R\rho}(w)$. Indeed, let us suppose that the $\Sigma_{1+\rho}^0$-set Z separates $[R]_\nu$ from $[R]_{<\nu}$ and, at the same time, the strategy s ensures $\neg\Psi_{R\rho}(w)$. Then $U = \{x : x^\wedge s \notin Z\}$, i.e., $U \in \Pi_{1+\rho}^0$ as a continuous inverse image of a $\Pi_{1+\rho}^0$-set (the complement of Z), which contradicts the choice of the formulas Φ_ρ and Ψ_ρ and the definition of U.

Finally, the implication $\Psi_{R\rho}(w) \to \Phi_{R\rho}(w)$ is proved by applying the principle of Borel determinacy to the Borel set

$$A = \{\langle x, y\rangle : (x \in U \wedge y \notin [R]_{<\nu}) \vee (x \notin U \wedge y \notin [R]_\nu)\}.$$

Proof of Statement (c) of Theorem 3.1. By virtue of Lemma 3.1, the fact of existence of a constituent $[R]_\nu$ with SI $> 1 + \rho$ can be expressed by the Σ^1_2-formula

$$\exists w(WO(w) \wedge \neg\Psi_{R\rho}(w))$$

with parameters from M, where $WO(w)$ is the Π^1_1-formula

$$\forall y(y'0 \subseteq w'0 \wedge y'0 \neq \varnothing \to y'0 \text{ has a least element}),$$

determining the set WO canonically. Now it remains to use the Shoenfield principle.

Proof of the Statement (d)(4). Since $\nu < \omega_1^M$, there exists a $w \in WO_\nu \cap M$. Let us observe that $\Phi_{R\rho}(w)$ is essentially a Σ^1_2-formula with parameters from M by Remark 2.2 and the choice of the formulas Φ_ρ and Ψ_ρ (the quantifier $\exists s$ can be replaced by a quantifier over I). In the same manner, $\Psi_{R\rho}(w)$ is a Π^1_2-formula with parameters from M. Thus, we have a Σ^1_2-formula and a Π^1_2-formula with parameters from M, each of which, by Lemma 3.3, is equivalent (in M and in the universe V) to the statement (d)(4), in which we are interested. Now the absoluteness of (d)(4) is given by the Mostowski principle.

The proof of Theorem 3.1 is complete.

Remark. We restrict ourselves to the proof of the statements (c), (d), and (e) for the outer constituents only. The proof for the inner constituents is carried out in exactly the same manner.

4. PROOF OF THEOREM 1.5

In accordance with the statement of this theorem, we fix $z \in I$, an ordinal ρ such that $1 \leqslant \rho < \omega_1^{L[z]}$, and an ordinal ν such that $\omega_\rho^{L[z]} \leqslant \nu < \omega_1$. Let us assume the contrary. According to what we have said at the beginning of Sec. 2, the assumption of the contrary gives us a code of sieve $R \in L[z]$ such that $[R]_\nu \neq \varnothing$ and the S.I. of the constituent [R] does not exceed $1 + \rho$. (We restrict ourselves to the consideration of outer constituents. The proof for the inner constituents is completely analogous.) We begin the deduction of a contradiction from the assumption of the contrary with some definitions.

For each ordinal $\lambda < \omega_1$ we denote the totality of all Borel codes $\langle T, d\rangle \in L[z]$ such that $|T| \leqslant \lambda$ and

$$\forall\sigma\forall\alpha\left(\sigma^\wedge\alpha \in T \to \alpha < \omega^{L[z]}_{|\sigma|_T}\right)$$

by C_λ. Each of the sets C_λ belongs to L[z] and has cardinality $\omega^{L[z]}_{\lambda+1}$ in L[z].

Further, if $\lambda \in \mathrm{Ord}$, then we denote the totality of all functions p such that $\mathrm{dom}\, p \subseteq \omega$ is finite and $\mathrm{ran}\, p \subseteq \lambda$ by P_λ. The set $P_\lambda \in L[z]$ (even belonging to L) is ordered by reverse inclusion: $p \leqslant q \leftrightarrow q \subseteq p$. (The rational numbers do not figure in Secs. 4 and 5, and the letters p, q, and r will be used for denoting elements of the sets P_λ.)

Finally, in accordance with Lemma 2.3 and Remark 2.4, we fix a tree $T_\rho \in L[z]$, $T_\rho \subseteq \mathrm{Seq}_\omega$, which is a universal tree of height ρ in an arbitrary TM of the theory ZFC that is an extension of L[z]. Let us set $|\sigma| = |\sigma|_{T\rho}$ for $\sigma \in T_\rho$.

Let there exist a set $t \in L[z]$ such that $t \subseteq P_\nu \times (T_\rho \times \omega)$. With each pair $p \in P_\nu$, and $\sigma \in T_\rho$ we associate a set $Z_{\sigma p} \subseteq I$ by induction over $|\sigma|$ such that

$$Z_{\sigma p} = I - \bigcup_{k \in S} I_k \quad \text{for} \quad |\sigma| = 0, \tag{1}$$

where $S = \{k \in \omega\colon \exists q, r \in P_\nu(r \leqslant p \wedge r \leqslant q \wedge \langle q, \langle\sigma, k\rangle\rangle \in t)\}$;

$$Z_{\sigma p} = I - \bigcup_{q \leqslant p, \sigma^\wedge n \in T_\rho} Z_{\sigma^\wedge n, q} \quad \text{for} \quad |\sigma| \geqslant 1. \tag{2}$$

The proof of the following theorem is given separately in Sec. 5.

4.1. THEOREM. There exist a $p_0 \in P_\nu$ and a set $t \in L[z]$, $t \subseteq P_\nu \times (T_\rho \times \omega)$, such that the set $Z_{\Lambda p_0}$, defined by (1) and (2), satisfies the following conditions:

$$[R]_\nu \cap Z_{\Lambda p_0} = \varnothing \text{ and } [R]_{<\nu} \subseteq Z_{\Lambda p_0}.$$

We fix the sets $p_0 \in P_\nu$ and t, given by this theorem, and prove a theorem, playing an important role in obtaining the contradiction. This theorem realizes an idea of [16].

4.2. THEOREM. For each pair $p \in P_\nu$ and $\sigma \in T_\rho$ there can be selected a code $\langle T_{\sigma p}, d_{\sigma p}\rangle \in C_{|\sigma|}$ such that $Z_{\sigma p} = [T_{\sigma p}, d_{\sigma p}]$.

We prove this theorem by induction over $|\sigma| = |\sigma|_{T_\rho}$.

A. Let $p \in P_\nu$ and $\sigma \in T_\rho$, be such that $|\sigma| = 0$. It is easily verified that the sets $T_{\sigma p} = \{\Lambda\}$ and $d_{\sigma p} = \{\Lambda\} \times S$ [where S is defined as in (1)] form the desired code.

B. Let us suppose that $p \in P_\nu$ and $\sigma \in T_\rho$, $|\sigma| \geqslant 1$, and let Theorem 4.2 be valid for all $\eta \in T_\rho$ such that $\sigma \subset \eta$. Thus, for each $q \in P_\nu$ and each η a code

$$\langle T_{\eta q}, d_{\eta q}\rangle \in C_{<|\sigma|} = \cup_{\gamma<|\sigma|} C_\gamma$$

such that $[T_{\eta q}, d_{\eta q}] = Z_{\eta q}$ has been constructed. Let us suppose that

$$\text{the function } \eta,\ q \mapsto \langle T_{\eta q}, d_{\eta q}\rangle \text{ belongs to } L[z]. \tag{3}$$

Under this supposition, which is justified below, we construct the desired code $\langle T_{\sigma p}, d_{\sigma p}\rangle \in C_{|\sigma|}$.

As is easily verified, the set $C_{<|\sigma|}$ has cardinality $\omega_{|\sigma|}^{L[z]}$ in L[z].

$$C_{<|\sigma|} = \{\langle T_{(\beta)}, d_{(\beta)}\rangle : \beta < \omega_{|\sigma|}^{L[z]}\}$$

be a certain enumeration of the set $C_{<|\sigma|}$ in L[z]. Let B denote the totality of all ordinals $\beta < \omega_{|\sigma|}^{L[z]}$ such that there exists an $n \in \omega$ and a $q \in P_\nu$ such that $q \leqslant p, \sigma^\wedge n \in T_\rho$, and

$$\langle T_{(\beta)}, d_{(\beta)}\rangle = \langle T_{\sigma^\wedge n, q}, d_{\sigma^\wedge n, q}\rangle.$$

Let us set $T_{\sigma p} = \{\beta\ ^\wedge\tau : \beta \in B \wedge \tau \in T_{(\beta)}\} \cup \{\Lambda\}$ and

$$d_{\sigma p} = \{\langle \beta^\wedge \tau, k\rangle : \beta \in B \wedge \langle \tau, k\rangle \in d_{(\beta)}\}.$$

The structured pair $\langle T_{\sigma p}, d_{\sigma p}\rangle$ belongs to the totality $C_{|\sigma|}$ by virtue of the choice of the enumeration of the set $C_{<|\sigma|}$ and according to the supposition (3), ensuring belongingness to L[z]. It remains to verify that $[T_{\sigma p}, d_{\sigma p}] = Z_{\sigma p}$.

By construction, we have $[T_{\sigma p}, d_{\sigma p}, \langle\beta\rangle] = [T_{(\beta)}, d_{(\beta)}]$ for all $\beta \in B$. Consequently,

$$[T_{\sigma p}, d_{\sigma p}] = I - \bigcup_{\beta \in B} [T_{(\beta)}, d_{(\beta)}]. \tag{4}$$

Let us observe that if $\beta \in B$, then, by definition, there exist an $n \in \omega$ and a $q \in P_\nu$, such that $q \leqslant p$, $\sigma^\wedge n \in T_\rho$, and $\langle T_{(\beta)}, d_{(\beta)}\rangle = \langle T_{\sigma^\wedge n, q}, d_{\sigma^\wedge n, q}\rangle$. Conversely, if $n \in \omega$ and $q \in P_\nu$ are such that $\sigma^\wedge n \in T_\rho$ and $q \leqslant p$, then there exists $\beta \in B$ such that

$$\langle T_{(\beta)}, d_{(\beta)}\rangle = \langle T_{\sigma^\wedge n, q}, d_{\sigma^\wedge n, q}\rangle,$$

since the enumeration of the codes $\langle T_{(\beta)}, d_{(\beta)}\rangle$ exhausts the whole set $C_{<|\sigma|}$. Finally, according to the choice of the codes $\langle T_{\eta q}, d_{\eta q}\rangle$, for $\sigma \subset \eta$ we have $[T_{\sigma^\wedge n, q}, d_{\sigma^\wedge n, q}] = Z_{\sigma^\wedge n, q}$ for all $n \in \omega$ and $q \in P_\nu$. Therefore, Eq. (4) can be rewritten in the form

$$[T_{\sigma p}, d_{\sigma p}] = I - \bigcup_{\sigma^\wedge n \in T_\rho, q \leqslant p} Z_{\sigma^\wedge n, q}.$$

By the definition (2), this means that $[T_{\sigma p}, d_{\sigma p}] = Z_{\sigma p}$, which was required to be proved.

It is interesting (and this is the main step of the proof) that the union in the right-hand side of Eq. (2), the number of whose terms can be very large in L[z] according to the magnitude of ν, can be replaced, in the process of the construction B, by the union in the right-hand side of (4), the number of whose terms does not depend directly on ν and cannot exceed the magnitude of the ordinal $\omega_{|\sigma|}^{L[z]}$ in L[z].

Finally, let us observe that the constructions A and B can be completely carried out in L[z], since $t \in L[z]$. This gives a justification for the assumption (3). Theorem 4.2 is proved.

We use this theorem to obtain a contradiction from the supposition of the contrary at the beginning of this section. By the choice of t (Theorem 4.1), there exists a $p_0 \in P_\nu$ such that the set $Z_{\Lambda p_0}$ satisfies the relation

$$[R]_\nu \cap Z_{\Lambda p_0} = \varnothing \wedge [R]_{<\nu} \subseteq Z_{\Lambda p_0}. \tag{5}$$

Theorem 4.2 gives us a code $\langle T', d'\rangle \in C_\rho$ such that $Z_{\Lambda p_0} = [T', d']$ (remark: $|\Lambda| = |\Lambda|_{T_\rho} = |T_\rho| = \rho$). By definition, we have the equation

$$[T', d'] = I - \bigcup_{\langle\alpha\rangle\in T'} [T', d', \langle\alpha\rangle].$$

Therefore, by (5) there exists an ordinal α such that $\langle\alpha\rangle \in T'$ and

$$[R] \cap [T', d', \langle\alpha\rangle] \neq \varnothing \text{ and } [R]_{<\nu} \cap [T', d', \langle\alpha\rangle] \neq \varnothing. \tag{6}$$

The Borel code [T, d], given by the relations

$$T = \{\tau : \alpha^\wedge\tau \in T'\} \text{ and } d = \{\langle\tau, k\rangle : \langle\alpha^\wedge\tau, k\rangle \in d'\},$$

belongs to the totality C_λ for a certain $\lambda < \rho$ and satisfies the equation $[T, d] = [T', d', \langle\alpha\rangle]$. Therefore, the statement (6) can be rewritten in the following form:

$$[R] \cap [T, d] \neq \varnothing \text{ and } [R]_{<\nu} \cap [T, d] = \varnothing. \tag{7}$$

Now we need the following theorem, whose proof, together with that of Theorem 4.1, will be given in Sec. 5.

4.3. THEOREM. In the situation under consideration, there exists a set G such that $\omega_\lambda^{L[z]} < \omega_1^{L[z][G]} \leqslant \nu$.

Let G satisfy this double inequality. Let us observe that $\sup T \leqslant \omega_\lambda^{L[z]}$ and $\langle T, d\rangle \in L[z]$, since $\langle T, d\rangle \in C_\lambda$. Therefore, $\langle T, d\rangle \in L[z][G]$ and $\sup T < \omega_1^{L[z][G]}$ by the choice of G. Let us also recall that $R \in L[z]$ and, all the more, $R \in L[z][G]$. Therefore, by (7) and 3.1(e), the relation $[R] \cap [T, d] \neq \varnothing$ is true in the model M = L[z][G], i.e., there exists an ordinal $\mu < \omega_1^{L[z][G]}$ such that $[T, d] \cap [R]_\mu \neq \varnothing$ in L[z][G] also. Consequently, $[R]_\mu \cap [T, d] \neq \varnothing$ in the universe V of all sets by 3.1(d)(1). We have obtained a contradiction with the second statement of (7), since $\mu < \nu$.

This contradiction refutes the assumption of the contrary at the beginning of Sec. 4 and completes the proof of Theorem 1.5 ("modulo" Theorems 4.1 and 4.3).

5. PROOF OF THEOREMS 4.1 AND 4.3

We begin with the first of these theorems. At first we prove that Theorem 4.1 is valid in an arbitrary countable TM of the theory ZFC. After this we will show how to transform this proof into a proof in the universe of all sets.

Thus, let M be a countable TM of the theory ZFC, $z \in M \cap I$, and $1 \leqslant \rho < \omega_1^{L^M[z]}$, where

$$L^M[z] = \{x \in M: \ x \in L[z]\} \text{ is true in M}\}.$$

Also, let $\omega_\rho^{L^M[z]} \leqslant \nu < \omega_1^M$ (then $P_\nu \in L^M[z]$) and $R \in L^M[z]$ be a code of sieve in M. Further, let a grounded tree $T_\rho \in L^M[z]$, $T_\rho \subseteq \mathrm{Seq}_\omega$, be a universal tree of height ρ in all TM that are extensions of $L^M[z]$. Finally, let us suppose that the following condition is fulfilled:

(1) (in M) the S. I. of $[R]_\nu$ does not exceed $1 + \rho$. In this situation, it is necessary to find a $p_0 \in P_\nu$ and a $t \in L^M[z]$, $t \subseteq P_\nu \times (T_\rho \times \omega)$, such that the set $Z_{\Lambda p_0}$, defined in M by Eqs. (1) and (2) of Sec. 4, satisfies the relations

(2) $(bM)\, [R]_\nu \cap Z_{\Lambda p_0} = \varnothing$ and $[R]_{<\nu} \subseteq Z_{\Lambda p_0}$.

The construction of the desired t and p_0 uses the method of forcing. As the set of the forcing conditions we consider the set P_ν ($p \leqslant q$ means that each inference that can be forced by q can also be forced by p). The models M and $L^M[z]$ will be the initial models for the generic extensions.

Let us recall some definitions and facts connected with forcing. Let K = M or K = $L^M[z]$. A subset G of P_ν is said to be P_ν-generic over K if

(a) $p \in P_\nu \wedge q \in G \wedge q \leqslant p \to p \in G$,

(b) $\forall p, q \in G \exists r \in G\ (r \leqslant p \wedge r \leqslant q)$ and

(c) the intersection of G with each dense subset $D \in K$, $D \subseteq P_\nu$, is nonempty.

(A subset D of P_ν is dense in P_ν if for each $p \in P_\nu$ there exists a $q \in D$, such that $q \leqslant p$.) The reader can find a proof of the properties of generic sets and forcing, set forth in

A-D, in [2, Chap. 4] and also (in somewhat other terminology) in [7, 19].

A. For each $p \in P_\nu$ there exists a P_ν-generic subset G of P_ν over K that contains p. This statement is easily proved, starting from the countability of the model K.

B. Let a subset G of P_ν be P_ν-generic over K. Then there exists a unique smallest countable TM of the theory ZFC that contains G and all the sets from K. We denote this model by K[G]. The ordinal series of K and K[G] coincide.

In this case, if $x \in K[G]$, $a \in K$, and $x \subseteq a$, then there exists a set $t \in K, t \subseteq P_\nu \times a$, such that $x = i_G(t)$, where

$$i_G(t) = \{y : \exists p \in G(\langle p, y\rangle \in t)\}.$$

The sets t such that $t \subseteq P_\nu \times a$ for a certain a are called P_ν-terms. Special P_ν-terms $a^* = P_\nu \times a$ play a crucial role. It is clear that $i_G(a^*) = a$ and $a^* \in K$ whenever $a \in K$.

C. The notion of forcing itself is as follows. Let $p \in P_\nu$, $\varphi(x_1, \ldots, x_n)$ be a $\in$-formula (i.e., a formula in the language of set theory), and $t_1, \ldots, t_n$ be P_ν-terms. We write $p \Vdash \varphi(t_1, \ldots, t_n)$, if for each P_ν-generic subset G of P_ν over M that contains p the statement

$$\varphi(i_G(t_1), \ldots, i_G(t_n))$$

is true in the model M[G]. The forcing is considered only in connection with the initial model M, but not in connection with $L^M[z]$.

D. The following property is a basic property of forcing. If a subset G of P_ν is P_ν-generic over M, $p \in G$, and $\varphi(i_G(t_1), \ldots, i_G(t_n))$, is true in M[G], then there exists $q \in G$, $q \leqslant p$, such that $q \Vdash \varphi(t_1, \ldots, t_n)$.

We continue the proof of Theorem 4.1 in M. We fix a (existing by virtue of A) P_ν-generic subset G of P over M. First of all, let us observe that the following statement is true in M[G]: R is a code of sieve. Indeed, as we have seen in the proof of Theorem 3.1(a), the fact that R is a code of sieve can be expressed by a Π^1_2-formula with parameters from M (in fact, $R \in M$). By the Shoenfield principle, such formulas are absolute. (The Shoenfield principle is applied in M[G]; the ordinal series of M and M[G] coincide.) Consequently, by Theorem 3.1(d)(4) (applicable in M[G]), the statement (1) is true in M[G].

We pass to the model $N = L^M[z][G] \subseteq M[G]$. The set $g = \cup G$ belongs to this model and is a mapping of ω onto ν by the choice of the ν-curtailing set P_ν of forcing conditions [7, 19]. Therefore, the ordinals ν and ρ are countable in N, and we can apply Theorem 3.1(d)(4) in M[G] to the TM N and obtain the truth of the statement (1) in N.

Further, let us recall that the tree T_ρ, fixed in the beginning of Sec. 5, is a universal tree of height ρ in each TM that is an extension of $L^M[z]$ and, in particular, in M. Consequently, by the truth of (1) in N there exists a set $d \in N$, $d \subseteq T_\rho \times \omega$, such that the following relations are true in N:

(3) $[R]_\nu \cap [T_\rho, d] = \varnothing$ and $[R]_{<\nu} \subseteq [T_\rho, d]$.

<u>Next Step.</u> According to B, there exists a set $t \in L^M[z]$, $t \subseteq P_\nu \times (T_\rho \times \omega)$, such that $d = i_G(t)$. With this choice of t the model $N = L^M[z][G]$ plays its own role in the proof. It makes possible the choice of t from $L^M[z]$, whereas the consideration of the model M[G] would give us only $t \in M$.

Let us observe that, by Theorem 3.1(d)(1, 2), the statement (3), true in N, is true in M[G] also. Therefore, by virtue of (d) there exists a forcing condition $p_0 \in G$ such that

(4) $p_0 \Vdash [T^*_\rho, t] \cap [R^*]_{\nu*} = \varnothing \wedge [R^*]_{<\nu*} \subseteq [T^*_\rho, t]$.

We show that the desired relation (2) is fulfilled for the constructed t and p_0.

<u>LEMMA 1.</u> The following relation is true for arbitrary $\sigma \in T_\rho$ and $p \in P_\nu$ in M:

$$Z_{\sigma p} = \{x \in M \cap I : p \Vdash x^* \in [T^*_\rho, t, \sigma^*]\}.$$

<u>Proof.</u> We prove this lemma by induction over $|\sigma| = |\sigma|_{T_\rho}$. Let $|\sigma| = 0$ and $x \in M \cap I$. Let us suppose that $x \notin Z_{\sigma p}$ in M. By a definition of Sec. 4, there exist q, $r \in P_\nu$ and k such that $r \leqslant p$, $r \leqslant q$, $x \in I_k$, and $\langle q, \langle \sigma, k\rangle\rangle \in t$. By the property A of generic sets there exists a P_ν-generic subset H of P_ν over M that contains r. Then $q \in H$ (see A), i.e., $\langle \sigma, k\rangle \in i_H(t)$. Consequently, $x \notin [T_\rho, i_H(t), \sigma]$ in M[H]. But p also belongs to H. This means that p cannot force the formula $x^* \in [T^*_\rho, t, \sigma^*]$.

Conversely, let p not force $x^* \in [T^*_\rho, t, \sigma^*]$. Then there exists a P_ν-generic subset H of P_ν over M that contains p and is such that $x \notin [T_\rho, i_H(t), \sigma]$ is true in M[H]. By a definition of Sec. 2, there exists a pair $\langle\sigma, k\rangle \in i_H(t)$ such that $x \notin I_k$. We have $\langle q, \langle\sigma, k\rangle\rangle \in t$ for a certain $b \in H$, by the definition of $i_H(t)$. But, since H is generic, there exists an $r \in P_\nu$ (even $\in H$) such that $r \leqslant p$ and $r \leqslant q$. Hence, it follows from Eq. (1) of Sec. 4 that $x \notin Z_{\sigma p}$ in M.

Induction Step. Let $|\sigma| \geqslant 1$ and suppose that the lemma has been proved for all $\eta \in T_\rho$ such that $\sigma \subset \eta$ (i.e., $|\eta| < |\sigma|$). Let $x \notin Z_{\sigma p}$ in M. By Eq. (2) of Sec. 4, there exist a $q \in P_\nu$ and an $n \in \omega$, such that $q \leqslant p$, $\sigma^\wedge n \in T_\rho$, and $x \in Z_{\sigma^\wedge n, q}$. By the induction hypothesis (applied to $\eta = \sigma^\wedge n$) we have $q \Vdash x^* \in [T^*_\rho, t, (\sigma^\wedge n)^*]$. Let us consider the P_ν-generic subset H of P_ν over M that contains q. By the definition of $\Vdash$, the statement $x \in [T_\rho, i_H(t), \sigma^\wedge n]$, and, by the same token, the statement $x \notin [T_\rho, i_H(t), \sigma]$, is true in M[H]. But $p \in H$, since $q \leqslant p$. Therefore, p cannot force the formula $x^* \in [T^*_\rho, t, \sigma^*]$.

Conversely, let p not force $x^* \in [T^*_\rho, t, \sigma^*]$. Then there exists a P_ν-generic subset H of P_ν over M that contains p and is such that $x \notin [T_\rho, i_H(t), \sigma]$ is true in M[H]. This means that $x \in [T_\rho, i_H(t), \sigma^\wedge n]$ in M[H] for a certain n such that $\sigma^\wedge n \in T_\rho$. Now, by virtue of the property D of forcing, we can choose a condition $q \in H$, $q \leqslant p$, that forces $x^* \in [T^*_\rho, t, (\sigma^\wedge n)^*]$. By the induction hypothesis, we have $x \in Z_{\sigma^\wedge n, q}$ in M. Now $x \in Z_{\sigma p}$ is fulfilled by virtue of Eq. (2) of Sec. 4. The induction step has been made and the lemma is proved.

Completing the proof of Theorem 4.1 in M, we verify that the relations (2) are true in M (see the beginning of Sec. 5). We prove the first of them. Let $x \in I \cap M$ and $x \in Z_{\Lambda p_0}$ in M. Then $p_0 \Vdash x^* \in [T^*_\rho, t, \Lambda]$, i.e., $p \Vdash x^* \in [T^*_\rho, t]$ by Lemma 1. Therefore, $x \in [T_\rho, d]$ is true in M[G], since $d = i_G(t)$ and $p_0 \in G$. But the statement (3) is also true in M[G] (see above). Therefore, $x \notin [R]_\nu$ in M[G] and, consequently, $x \notin [R]_\nu$ in M by Theorem 3.1(d)(3).

We prove the second relation of (2). Let $x \in [R]_{<\nu}$ in M. Again, by virtue of 3.1(d)(3) the relation $x \in [R]_{<\nu}$ is true in each model M[H]. Consequently, each $p \in P_\nu$, including our p_0, forms $x \in [R^*]_{<\nu^*}$; whence by (4) we have

$$p_0 \Vdash x^* \in [T^*_\rho, t], \quad \text{i.e.,} \quad p_0 \Vdash x^* \in [T_\rho, t, \Lambda].$$

Now $x \in Z_{\Lambda p_0}$ in H by Lemma 1, which was required to be proved.

The proof of Theorem 4.1 in the model M is complete.

In essence, the only reason for which we have preferred to give the proof of Theorem 4.1 at first in a countable model M is the possibility to consider P_ν-generic extensions of the model M, a possibility that obviously does not exist for the universe of all sets. However, we know a method that enables us to obtain information about the P_ν-generic extensions of the universe V, provided such extensions actually exist. This method of Boolean-valued models consists in the introduction of a special class of terms $V^{(P_\nu)}$, which plays the role of a P_ν-generic extension of the universe V of all sets. The mapping $a \to a^*$ is an embedding of V into $V^{(P_\nu)}$. Each standard fact, valid for the extensions $M \to M[G]$, turns out to be valid for the "extensions" $V \to V^{(P_\nu)}$ also. In particular, the set $t \in L[z]$, desired in the sense of Theorem 4.1, exists. See [7, 20] for more details about such a realization of forcing.

Proof of Theorem 4.3. The proof of this theorem also uses forcing. The class L[z] plays the role of the initial model, and the set P_γ, where $\gamma = \omega_\lambda^{L[z]}$, plays the role of the set of forcing conditions. Let us observe that the generalized continuum hypothesis is true in L[z]. (The verification of GCH in classes of the form L[z], $z \in I$, is completely analogous to the Gödel proof of GCH in the constructible universe L; see [2, Chap. 5] or [7].) Therefore, the totality S of all sets $D \subseteq P_\gamma, D \in L[z]$, has cardinality $\omega_{\lambda+1}^{L[z]} \leqslant \omega_\rho^{L[z]}$ in L[z]. Consequently, the totality S is countable in the universe V of all sets, since $\omega_\rho^{L[z]} \leqslant \nu < \omega_1$. Therefore, there exists a P_γ-generic subset G of P_γ over L[z] [2, Chap. 4, Sec. 2].

We show that the constructed set G is the desired one for Theorem 4.3. As in the proof of Theorem 4.3. As in the proof of Theorem 4.1, the set $g = \bigcup G$ maps ω onto γ. Therefore, $\gamma < \omega_1^{L[z][G]}$. On the other hand, the set P_γ, having cardinality γ in L[z], trivially does not contain any antichain of cardinality at least $\gamma^+ = \omega_{\lambda+1}^{L[z]}$ in L[z]. (A subset A of P_γ is called an *antichain* if any two different $p, q \in A$ are inconsistent in P_γ, i.e., there is no $r \in P_\gamma$

such that $r \leqslant p$ and $r \leqslant q$.) In other words, P_γ satisfies the γ^+-condition of antichains in $L[z]$ [2, Chap. 4, Sec. 3]. In this situation, γ^+ remains a cardinal in the class $L[z][G]$. Consequently, $\omega_1^{L[z][G]} \leqslant \omega_{\lambda+1}^{L[z]}$. But $\lambda + 1 \leqslant \rho$, since $\lambda < \rho$. Finally, we have

$$\omega_\lambda^{L[z]} < \omega_1^{L[z][G]} \leqslant \omega_{\lambda+1}^{L[z]} \leqslant \omega_\rho^{L[z]} \leqslant \nu,$$

which was required to be proved. Theorem 4.3 is proved.

By the same token, the proof of the main theorem 1.5 is complete. We have restricted ourselves to the proof of this theorem for outer constituents: The proof for inner constituents does not differ at all. We now return to Theorems 1.1–1.4 and 1.6.

6. PROOF OF REMAINING THEOREMS

Proof of Theorem 1.6. We again restrict ourselves to outer constituents. According to what we have said at the beginning of Sec. 2, we must prove that if $z \in I, 1 \leqslant \rho < \omega_1^{L[z]}$, $\omega_\rho^{L[z]} < \omega_1$, the code of sieve R belongs to $L[z]$, and the number of nonempty constituents $[R]_\nu$ is unstable, then there exists an ordinal $\mu < \omega_1^{L[z]}$ such that $SI[R]_\nu > 1 + \rho$.

By the condition, there exists a nonempty constituent $[R]_\nu$ with the index $\nu \geqslant \omega_\rho^{L[z]}$. According to the already-proved theorem 1.5 $SI[R]_\nu$ is greater than $1 + \rho$. Thus, R has an outer constituent with $SI > 1 + \rho$. This statement must be true in $L[z]$ also by Theorem 3.1-(c), i.e., there exists an ordinal $\mu < \omega_1^{L[z]}$ such that $SI[R]_\nu > 1 + \rho$ in $L[z]$. An application of 3.1(d)(4) completes the proof.

Let us observe that Theorem 1.6 has the following corollary.

6.1. COROLLARY. Let $z \in I, 1 \leqslant \rho < \omega_1^{L[z]}$, $\omega_\rho^{L[z]} < \omega_1 u$, and the code of sieve $R \in L[z]$ be such that the number of the nonempty constituents $[R]_\nu$ is uncountable. Then there exists a constituent $[R]_\mu$ with $SI > 1 + \rho$. The same is true for inner constituents.

Proof of Theorems 1.1 and 1.4. At first we carry out the proof under the following supposition:

1) *For each point $z \in I$ there exists a countable TM of the theory ZFC that contains z.*

To prove Theorem 1.1, we assume the contrary: $\rho < \omega_1$ and the code of sieve R is such that all $[R]_{*\nu}$ have $SI \leqslant 1 + \rho$, but the number of the nonempty inner constituents $[R]_{*\nu}$ is uncountable. Let us observe that R is a suite of sets of very simple nature (see Sec. 2) and $\rho < \omega_1$. Therefore, there exists a point $z \in I$ that effectively codes R and ρ. We choose, using (1), a countable TM M that contains z. We get:

2) *The statement $\rho < \omega_1^{L[z]}$ is true in M and $R \in L[z]$.*

Let $\lambda = \omega_\rho^{L^M[z]}$. Let us consider the model $N = M[G]$, obtained by adjoining the P_λ-generic subset G of P_λ over M to the model M. The statement (2) is true in N also, since $L^M[z] = L^N[z]$. Moreover, $\lambda < \omega_1$ is true in N, since $g = \cup G$ maps ω onto λ. Therefore, we can use Corollary 6.1 in N [the uncountability of the number of the nonempty inner constituents $[R]_{*\nu}$ in N is ensured by Theorem 3.1(b)]. As a result, we get an ordinal $\mu < \omega_1^N$ such that the following statement is true in N: $SI[R]_{*\mu} > 1 + \rho$. By 3.1(d)(4), we have $SI[R]_{*\mu} > 1 + \rho$ in the universe also, which contradicts the supposition of contrary. By the same token, the proof of Theorem 1.1 with the help of (1) is complete.

The proof of Theorem 1.4 is carried out in exactly the same manner. The only difference is that the uncountability of the number of the nonempty $[R]_\nu$ in N is ensured not by reference to 3.1(b), but by reference to the absoluteness of the statement $[R]_\nu \neq \emptyset$ (where $\nu < \omega_1^N$ is arbitrary), which is deduced from 3.1(d)(1) if as $\langle T, d\rangle$ we take the canonical code $\langle\{\Lambda\}, \emptyset\rangle$ of the set I.

As is obvious, the requirement of nonemptiness of all $[R]_\nu$ in the condition of Theorem 1.4 can be replaced by the following weaker requirement: For each TM N of the theory ZFC, containing a given code of sieve R, the number of nonempty outer constituents $[R]_\nu$ is uncountable in N, i.e., the following statement is true in N:

$$\forall \mu < \omega_1^N \exists \nu \geqslant \mu ([R_\nu] \neq \emptyset).$$

We now show how to bypass the assumption 1) in the proposed proofs. The principle of Borel determinacy of Sec. 3, Corollary 6.1, and the statement that the totality of all sets of set-theoretic rank $< \omega + \omega$ (see [7, 19]) is a set and forms a model of the Zermelo system ZC (obtained by replacement of the axiom of substitution in ZFC by the axiom of separation) are theorems of ZFC. Let A denote a finite fragment of ZFC, sufficient for the proof of these theorems. The theory A is sufficiently rich for the proof of the following two statements:

3) *Corollary 6.1 is true in each TM of the theory A.*

4) *Theorem 3.1 is true for all TM M of the theory A (and not only for models of the theory ZFC).*

In addition, since the theory A is finite, the following statement is a theorem of ZFC:

5) *If* $z \in I$, *then all the axioms of A and the equation* $L^V[z] = L^{V^{(P)}}[z]$ *are true in each Boolean-value extension* $V^{(P)}$ *of the universe V.*

Let B denote a finite fragment of ZFC that contains A and is sufficient for the proof of (5). We have the following statements:

6) *For each* $z \in I$ *there exists a countable TM of the theory B that contains z.*

7) *If M is a countable TM of the theory B and N is a generic extension of the model M, then N is a model of the theory A and* $L^N[z] = L^M[z]$ *for each* $z \in M \cap I$.

Since B is finite, the statement 6) is a consequence of the reflection principle [7, 19]. To prove 7), it is sufficient to note that from the point of view of M the model N is a Boolean-valued extension of the universe of all sets.

As is easily verified, the statements 3), 4), 6), and 7) enable us to eliminate the hypothesis 1) from the above proof of Theorems 1.1 and 1.4 by considering models of the theories A and B in place of models of the theory ZFC.

Proof of Theorem 1.2. Let us consider a Borel sieve $\mathbf{R}$ such that the sequence of the derived sieves $\mathbf{R}^\nu$, $\nu < \omega_1$, is bounded in rank. The following equation is valid for each ordinal ν:

$$[\mathbf{R}]_{*\nu} = \Big(\bigcup_{q \in Q} \mathbf{R}_q^\nu\Big) \cap \Big(\bigcap_{\mu<\nu} \bigcup_{q\in Q} \Big(\mathbf{R}_q^\mu - \bigcup_{r<q} \mathbf{R}_r^\mu\Big)\Big) - \bigcup_{q\in Q}\Big(\mathbf{R}_q^\nu - \bigcup_{r<q}\mathbf{R}_r^\nu\Big).$$

(The easy verification is left to the reader.) Therefore, the sequence of inner constituents is also bounded in rank. By the same token, the number of nonempty $[\mathbf{R}]_{*\nu}$ is countable by Theorem 1.1, proved already. In this case, the number of nonempty outer constituents $[\mathbf{R}]_\nu$ must also be countable (see Sec. 1, deduction of negative solution of Problem 1 from Theorem 1.1). Thus, there exists an ordinal $\nu < \omega_1$ such that $[\mathbf{R}]_{*\mu} = [\mathbf{R}]_\mu = \emptyset$ for all $\mu > \nu$. Hence it is easily deduced that $\mathbf{R}^\mu = \mathbf{R}^\nu$ for all $\mu \geqslant \nu$, which was required to be proved.

Proof of Theorem 1.3. By what we have said in Sec. 1, it is sufficient to prove that (e) $\to$ (g). We fix a code of sieve R and prove under the assumption of $\neg$(g) that the separability indices of nonempty $[R]_\nu$ converge to ω_1, if it is known that the number of nonempty $[R]_\nu$ is uncountable. It is sufficient to indicate for each $\rho < \omega_1$ an ordinal $\lambda < \omega_1$ such that $SI[R]_\nu > + \rho$ whenever $\nu \geqslant \lambda$ and $[R]_\nu \neq \emptyset$.

As we have seen at the beginning of Sec. 6, there exists a point $z \in I$ such that $R \in L[z]$ and $\rho < \omega_1^{L[z]}$. Then $\omega_\rho^{L[z]} < \omega_1$. Indeed, under the assumption of $\neg$(g), the "present" cardinal $\Omega = \omega_1$ is strongly inaccessible in L[z] for each $z \in I$ [8]. Hence $\Omega = \omega_\Omega^{L[z]}$. But $\rho < \omega_1 = \Omega$; whence

$$\omega_\rho^{L[z]} < \omega_\Omega^{L[z]} = \Omega = \omega_1.$$

Thus, the ordinal $\lambda = \omega_\rho^{L[z]}$ satisfies the inequality $\lambda < \omega_1$. It follows from Theorem 1.5 that if $\lambda \leqslant \nu < \omega_1$ and $[R]_\nu \neq \emptyset$, then $SI[R]_\nu > 1 + \rho$, which was required to be proved.

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STRONGLY HOMOGENEOUS TORSION-FREE ABELIAN GROUPS

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A torsion-free Abelian group G is called *strongly homogeneous* if the automorphism group of G acts transitively on the set of all pure subgroups of rank 1, i.e., if A and B are any pure subgroups of G of rank 1, then $\alpha A = B$ for some $\alpha \in \mathrm{Aut}\, G$.

Strongly homogeneous groups form an important and interesting class of groups. Closely connected with these groups are the strongly homogeneous torsion-free rings. An associative ring R with unity is called *strongly homogeneous* if each element is an integral multiple of some element that is invertible in R. The additive group R^+ of a strongly homogeneous torsion-free ring R is strongly homogeneous. Indeed, if A and B are pure subgroups of R^+ of rank 1, then there exist invertible elements $u \in A$ and $v \in B$. Left multiplication of R by the element $w = vu^{-1}$ is an automorphism of the group R^+ and $wA = B$.

In certain special cases, strongly homogeneous torsion-free groups of finite rank were described in [1-3]. Arnold [4] completed the description of strongly homogeneous groups of finite rank. In the present paper we study strongly homogeneous torsion-free groups of arbitrary rank and their endomorphism rings. We prove that a strongly homogeneous torsion-free

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