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AN EFFECTIVE MINIMAL ENCODING OF UNCOUNTABLE SETS

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Abstract: We propose a method for encoding sets of the countable ordinals by generic reals which preserves cardinality and enjoys the property of minimality over the encoded set.

For $W \subseteq \omega_1$ there is a cardinal-preserving generic extension $\mathrm{L}[W][x]$ of the class $\mathrm{L}[W]$ by a generic real x such that W belongs to the class $\mathrm{L}[x]$, i.e., W is Gödel constructible with respect to x, while x itself is minimal over $\mathrm{L}[W]$.

Keywords: forcing, minimal encoding, relatively constructible set

§ 1. Introduction

Effective encoding of certain sets by some other sets is among the most important questions of the modern set theory. The encoding set must be somehow simpler than the encoded set. The concept of simplicity can be specified in various ways. For instance, an uncountable set can be encoded by a countable set, say, a real. The Borel sets and projective sets of reals in fact admit encodings of this type by definition. But, as for encoding an arbitrary set $W \subseteq \omega_1$ (of countable ordinals) by a real, basing on the axioms of Zermelo–Fraenkel set theory **ZFC** this seems impossible to implement. A set W of this type should be encoded by a *generic* real x, which in general is absent in the universe of this set theory, but which we can adjoin to the universe by forcing. Our main result is of the type:

Theorem 1. *Consider $W \subseteq \omega_1$ with $\mathrm{V} = \mathrm{L}[W]$; thus, all sets belong to $\mathrm{L}[W]$. There is a cardinal-preserving extension $\mathrm{L}[W][x]$ of $\mathrm{V} = \mathrm{L}[W]$ by a generic real x with*
(1) *$W \in \mathrm{L}[x]$ and W belongs to the class $\Delta_1^{\mathrm{HC}}(x)$ in $\mathrm{L}[x]$;*
(2) *if a real y belongs to $\mathrm{L}[x]$ then $x \in \mathrm{L}[y]$ or $y \in \mathrm{L}[W]$.*

Recall that $\mathrm{L}[z]$ is the class of all Gödel constructible sets relative to a given set z. In particular, the condition $\mathrm{V} = \mathrm{L}[W]$ means that all sets are constructible relative to W. This may happen, for instance, if the universe under consideration is an extension of the constructible universe L by adjoining a generic set W of ordinals, while the set of constraints must be such that W encodes the countability of every ordinal in W.

Furthermore, HC is the set of all sets whose transitive closures are at most countable, while $\Delta_1^{\mathrm{HC}}(x)$ is the collection of all sets $X \subseteq \mathrm{HC}$ definable in HC by both a Σ_1-formula and a Π_1-formula; both formulas should have a sole parameter x. Essentially, $W \in \Delta_1^{\mathrm{HC}}(x)$ means that, as we pass from countable ordinals (and W consists of those) to the reals encoding them, the resulting set of codes belongs to the class $\Delta_2^1(x)$ of effective projective hierarchy. For more details on this see [1, 2]; and for projective hierarchy, [3, 4].

Conclusion 1 of the theorem means that x effectively encodes W as regards both the Gödel constructibility and the effective projective hierarchy. Conclusion 2 of the theorem means the *minimality* of x over $\mathrm{L}[W]$; it asserts that there are no reals strictly between W and x as regards relative constructibility.

Encoding methods for the results like Theorem 1 are known in set theory for quite a long time. They include, for instance, the almost disjunctive encoding [5, 6, 7]. Using it, we can easily prove Theorem 1 without requirement 2 of minimality in the particular case that there is a real $z \in \mathrm{L}[W]$ with $\omega_1^{\mathrm{L}[z]} = \omega_1^{\mathrm{L}[W]}$. A more complicated particular case is when we require that every limit ordinal $\lambda < \omega_1^{\mathrm{L}[W]}$ strictly

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satisfies $\lambda < \omega_1^{\mathrm{L}[W\cap\lambda]}$. In this case the required forcing for Theorem 1 (without minimality) can also be based on the almost disjunctive encoding. For these results, see [8, 9].

Our approach is to use another encoding method, going back to the article of Jensen [10]. It consists in constructing a set of constraints $\mathbb{P}$ from perfect binary trees: a "subforcing" of the Sacks forcing. We construct $\mathbb{P}$ as $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$ in Section 5, following the general preliminary definitions of Section 2, introduction of the notation related to our encoding in Section 3, and the definitions of a special ω_1-sequence $\widehat{\varphi}$ of reals in Section 4, which amounts to a more convenient expression for W. This forcing $\mathbb{P}$, with a modification in Section 8, is sufficient for proving Theorem 1 in Section 8. But prior to that we establish the preservation of cardinals in Section 6 and consider the encoding properties of $\mathbb{P}$ in Section 7.

For more details on forcing, set theory models, constructibility, and related questions, see the book [1] (including the Appendix [7]), as well as our surveys [2, 11].

§ 2. On Perfect Trees

In the conventions of descriptive set theory, *reals* are the elements of the *Baire space* ω^ω consisting of all functions $x : \omega \to \omega$, i.e., all infinite sequences of nonnegative integers. Given a finite sequence $s \in \omega^{<\omega}$ of nonnegative integers, put $\mathscr{N}_s = \{x \in \omega^\omega : s \subset x\}$, where $s \subset x$ means that s starts an infinite sequence x. The sets of the form $\mathscr{N}_s$ generate a base for the topology on ω^ω.

The set 2^ω of all functions $x : \omega \to 2 = \{0, 1\}$, i.e., all infinite dyadic sequences, is a subset of ω^ω; this is Cantor's discontinuum. Accordingly, the set $2^{<\omega}$ of all finite sequences of numbers 0 and 1 is a subset of $\omega^{<\omega}$. Given $s \in 2^{<\omega}$, put $\mathscr{D}_s = \{x \in 2^\omega : s \subset x\}$.

The equivalence relation $\mathtt{E_0}$ on ω^ω is determined as follows: $x\mathtt{E_0}y$ when $x(n) = y(n)$ for almost all (i.e., all but finitely many) values of n (see [3]).

Given $x \in \omega^\omega$ and $n \in \omega$, put $(x)_n(k) = x(2^n(2k + 1) - 1)$, so that $(x)_n \in \omega^\omega$. Define $x_{\text{even}} \in \omega^\omega$ as $x_{\text{even}}(n) = x(2n)$ for all n.

We will consider perfect sets $P \subseteq 2^\omega$. Every set P of this type is uniquely determined by its *code*:

$$\mathbf{cod}P = \{s \in 2^{<\omega} : P \cap \mathscr{D}_s \neq \varnothing\},$$

so that if $\mathbf{cod}P = S$ then, conversely, P coincides with $[S] = \{a \in 2^\omega : \forall m(a{\restriction}m \in S)\}$. In this case $S = \mathbf{cod}P$ is a perfect tree in $2^{<\omega}$, i.e., Λ (the empty sequence) belongs to S, and moreover

(1) if $s \in S$ then at least one of the extensions $s^\wedge 0$ and $s^\wedge 1$ belongs to S, and conversely, if $s^\wedge i \in S$ (for $i = 0, 1$) then $s \in S$;

(2) if $s \in S$ then S has a branching point above s: there is $t \in S$ with $s \subseteq t$, and both extensions $t^\wedge 0$ and $t^\wedge 1$ belong to S.

Denote by **Perf** the set of all perfect trees $S \subseteq 2^{<\omega}$.

A set $A \subseteq \mathbf{Perf}$ is called an *antichain* whenever $[S] \cap [T] = \varnothing$ for every pair $S \neq T$ in A. A set $D \subseteq \mathbb{X} \subseteq \mathbf{Perf}$ is called *dense* in $\mathbb{X}$ whenever given $S \in \mathbb{X}$ there is a tree $T \in D$ with $T \subseteq S$. Observe that if $S, T \in \mathbf{Perf}$ then the relation $S \subseteq T$ is equivalent to $[S] \subseteq [T]$.

Given $S \in \mathbf{Perf}$ denote by $\mathrm{CO}(S)$ the set of all trees $T \in \mathbf{Perf}$ with $T \subseteq S$ such that $[T]$ is clopen in $[S]$. By compactness, in this case $[T]$ is the intersection of $[S]$ with a finite union of sets $\mathscr{D}_s$, $s \in 2^{<\omega}$, and so $\mathrm{CO}(S)$ is countable. Thus, if $\mathbb{X} \subseteq \mathbf{Perf}$ is at most countable then so is $\mathrm{CO}(\mathbb{X}) = \bigcup_{S\in\mathbb{X}} \mathrm{CO}(S)$. Refer to a set $\mathbb{X}$ as CO-*closed* whenever $\mathbb{X} = \mathrm{CO}(\mathbb{X})$, and CO-*dense* whenever $\mathbb{X}$ is dense in $\mathrm{CO}(\mathbb{X})$.

Lemma 2. *If a set $\mathbb{X} \subseteq \mathbf{Perf}$ is* CO*-dense and $S_1, \dots, S_n \in \mathbb{X}$ then there are trees $T_1, \dots, T_n \in \mathbb{X}$, with $T_i \subseteq S_i$ for every i and $[T_i] \cap [T_j] = \varnothing$ for $i \neq j$.*

Proof. Assume for simplicity that $n = 2$. There exist $x_1 \in [S_1]$ and $x_2 \in [S_2]$ with $x_1 \neq x_2$. There is $m \in \omega$ with $x_1{\restriction}m = u \neq v = x_2{\restriction}m$. Then the trees $T_1' = \{s \in S_1 : u \subseteq s \vee s \subseteq u\}$ and $T_2' = \{s \in S_2 : u \subseteq s \vee s \subseteq v\}$ belong to $\mathrm{CO}(\mathbb{X})$, and $[T_1'] \cap [T_2'] = \varnothing$. By CO-density there are trees $T_1, T_2 \in \mathbb{X}$ with $T_i \subseteq T_i'$ for $i = 1, 2$. □

If $S \in \mathbf{Perf}$ and $\mathscr{X} \subseteq \mathbf{Perf}$ then the expression $S \subseteq^{\text{fin}} \bigcup \mathscr{X}$ means that there is a *finite* set $\mathscr{X}' \subseteq \mathscr{X}$ with $[S] \subseteq \bigcup_{X\in\mathscr{X}'} [X]$.

Lemma 3. *Take a countable* CO*-dense set* $\mathbb{X} \subseteq \mathbf{Perf}$ *and a countable family* $\{D_n : n \in \omega\}$ *of dense sets* $D_n \subseteq \mathbb{X}$*. There exists an antichain* $A \subseteq \mathbf{Perf} \setminus \mathbb{X}$ *such that*

(1) $T \subseteq^{\text{fin}} \bigcup D_n$ *for all* $n \in \omega$*, and* $T \in A$*;*

(2) *given* $S \in \mathbb{X}$ *there is* $T \in A$ *with* $T \subseteq S$*.*

Proof. Fix an arbitrary enumeration of $\mathbb{X} = \{S_n : n \in \omega\}$. In order to prove the lemma, it suffices to construct a family of trees $T^n_s \in \mathbb{X}$ indexed by $n \in \omega$ and $s \in 2^{<\omega}$ so that

(i) $T^n_\Lambda = S_n$ for all n, where Λ is the empty sequence;

(ii) $T^n_{s^\wedge 0} \cup T^n_{s^\wedge 1} \subseteq T^n_s$ but $T^n_{s^\wedge 0} \cap T^n_{s^\wedge 1} = \varnothing$;

(iii) $\operatorname{diam} T^n_s = \frac{1}{\operatorname{lh} s}$, where $\operatorname{lh} s$ is the length of a sequence $s \in 2^{<\omega}$, while if $T \in \mathbf{Perf}$ then $\operatorname{diam} T$ stands for $\frac{1}{m+1}$, where m is the greatest number satisfying $a{\restriction}m = b{\restriction}m$ for all $a, b \in [T]$;

(iv) if $X^n_k = \bigcup_{\operatorname{lh} s = k} \left[T^n_s\right]$ then $X^n_n \cap X^m_n = \varnothing$ for $m < n$;

(v) if $n \in \omega$, $k \geq 1$, and $\operatorname{lh} s = k$ then $T^n_s \subseteq T$ for some $T \in D_{k-1}$.

(vi) if $n, k \in \omega$ then either there is $s \in 2^{<\omega}$ with $\operatorname{lh} s = k$ and $\left[T^n_s\right] \cap [S_k] = \varnothing$, or $X^n_k \subsetneqq [S_k]$.

We leave to the reader the details of the easy construction of a suitable system of sets. (In particular, to guarantee claim (iv), we have to use Lemma 2.) Upon completing it, take $A = \{T_n : n \in \omega\}$, where $T_n = \bigcap_{k \in \omega} \bigcup_{\operatorname{lh} s = k} T^n_s$. Observe that claim (vi) yields $A \cap \mathbb{X} = \varnothing$. □

§ 3. Notation Related to Encoding

We consider the sets

$$\mathbf{R}^0 = \{b \in 2^\omega : \forall n((b)_n = (b)_0)\} \quad \text{and} \quad \mathbf{R} = \{x \in 2^\omega : \exists b \in \mathbf{R}^0 (b \mathsf{E_0} x)\}.$$

For every $x \in \mathbf{R}$ there exists a unique $b = \mathbf{b}(x) \in \mathbf{R}^0$ with $x \mathsf{E_0} b$. In this case denote by $\mathtt{dif}(x)$ the smallest positive integer n with $x(i) = b(i)$ for all $i \geq n$. If $x \in \mathbf{R}^{\text{even}} = \{x \in 2^\omega : x_{\text{even}} \in \mathbf{R}\}$ then put $\mathbf{b}^{\text{even}}(x) = \mathbf{b}(x_{\text{even}})$ and $\boldsymbol{u}(x) = x{\restriction}(2\ \mathtt{dif}(x_{\text{even}}))$.

For $b \in 2^\omega$ and $n \geq 1$ denote by $U(b, n)$ the set of all finite sequences $s \in 2^{<\omega}$ of length $\operatorname{lh} s = 2n$ with $s(2n-2) \neq b(n-1)$. By contrast, $U(b, 0)$ consists of the unique empty sequence Λ. Put $U(b) = \bigcup_{n \in \omega} U(b, n)$. Given $u \in U(b, n)$ define

$$\widehat{Y}(b, u) = \{x \in \mathscr{D}_u : \forall k \geq n(x(2k = b(k))\} \quad \text{and} \quad \mathbf{T}(b, u) = \mathbf{cod}\widehat{Y}(b, u).$$

Then all trees $\mathbf{T}(b, u)$ with $u \in U(b)$ belong to $\mathbf{Perf}$, and the sets $[\mathbf{T}(b, u)] = \widehat{Y}(b, u) \subseteq 2^\omega$ are disjoint: $[\mathbf{T}(b, u)] \cap [\mathbf{T}(b, v)] = \varnothing$ for every $u \neq v$ in $U(b)$ (regardless of the same or different lengths).

We omit the proof the next lemma, as it is elementary.

Lemma 4. *We have*

$$\mathbf{R}^{\text{even}} = \bigcup_{b \in \mathbf{R}^0} \bigcup_{u \in U(b)} [\mathbf{T}(b, u)],$$

and if $x \in \mathbf{R}^{\text{even}}$ *then* $b = \mathbf{b}^{\text{even}}(x) \in \mathbf{R}^0$*,* $u = \boldsymbol{u}(x) \in U(b)$*, and* $x \in [\mathbf{T}(b, u)]$*.*

If a set $X \subseteq 2^\omega$ *is nonempty and open then there is* $u \in U(b)$ *with* $[\mathbf{T}(b, u)] \subseteq X$*.*

Now we relativize these definitions to an arbitrary perfect set $[P] \subseteq 2^\omega$, where $P \in \mathbf{Perf}$. Denote by $h_P : 2^\omega \xrightarrow{\text{on}} [P]$ the canonical homeomorphism resulting from the correspondence between the branching points of P and all finite sequences in $2^{<\omega}$. For $x \in \mathbf{R}^{\text{even}}_P = \left\{x \in [P] : h_P^{-1}(x) \in \mathbf{R}^{\text{even}}\right\}$ define

$$\mathbf{b}^{\text{even}}_P(x) = \mathbf{b}^{\text{even}}(x') \quad \text{and} \quad \boldsymbol{u}_P(x) = \boldsymbol{u}(x')\,, \quad \text{where} \quad x' = h_P^{-1}(x).$$

Accordingly, for $b \in \mathbf{R}^0$ and $u \in U(b)$ put

$$\widehat{Y}_P(b, u) = \{h_P(y) : y \in \widehat{Y}(b, u)\} \quad \text{and} \quad T_P(b, u) = \mathbf{cod}\widehat{Y}_P(b, u).$$

Corollary 5. *Take $P \in \mathbf{Perf}$. Then*

$$\mathbf{R}_P^{\text{even}} = \bigcup_{b \in \mathbf{R}^0} \bigcup_{u \in U(b)} [T_P(b,u)],$$

and if $x \in \mathbf{R}_P^{\text{even}}$ then $b = \mathbf{b}_P^{\text{even}}(x) \in \mathbf{R}^0$, $u = \boldsymbol{u}_P(x) \in U(b)$, and $x \in [T_P(b,u)]$. In addition,

(i) *if $b \in \mathbf{R}^0$ then the set $\mathbf{Next}_P(b) = \{T_P(b,u) : u \in U(b)\} \subseteq \mathbf{Perf}$ is an antichain: $[T_P(b,u)] \cap [T_P(b,v)] = \varnothing$, whatever $u \neq v$ in $U(b)$ may be;*

(ii) *if $X \subseteq [P]$ is nonempty and open in $[P]$ then there is $u \in U(b)$ with $[T_P(b,u)] \subseteq X$.* □

§ 4. The Auxiliary Function $\widehat{\varphi}$

Fix a recursive enumeration $\mathbb{Q} = \{r_n : n \in \omega\}$ without repetitions of the set $\mathbb{Q}$ of rationals. For $\xi < \omega_1$ (a finite or countable ordinal) denote by Word_ξ the collection of all $x \in 2^\omega$ such that the set $\{r_n : x(n) = 0\} \subseteq \mathbb{Q}$ is totally ordered by the natural order on the rationals in accordance with the type ξ. Put $\mathrm{Word} = \bigcup_{\xi<\omega_1} \mathrm{Word}_\xi$ (the codes of ordinals) and $|x| = \xi$ for $x \in \mathrm{Word}_\xi$. If $x \in \mathrm{Word}$ then for $|x| = \xi \geq \omega$ there exists a bijection $\mathbf{f}_x : \omega \xrightarrow{\text{on}} \xi$, determined by x, the details of whose construction we leave to the reader. But if $|x| < \omega$ then we simply put $\mathbf{f}_x(n) = n$ for every n.

Definition 6. Take $\varphi \in (2^\omega)^\lambda$ with $\lambda \leq \omega_1$. Define $\Omega_1[\varphi] = \{0\} \cup \{\xi+1 : \xi+1 < \lambda\}$ (all nonlimit ordinals up to $\lambda = \operatorname{dom}\varphi$). Denote by $\Omega_2[\varphi]$ the set of all ordinals $\xi \leq \lambda$, $\xi \notin \Omega_1[\varphi]$, with $\xi < \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$.

Given $\xi \in \Omega_2[\varphi]$ denote by $\mu_\xi[\varphi] = \mu_\xi[\varphi\restriction\xi]$ the smallest ordinal $\mu > \xi$ such that $\mathrm{L}_\mu[\varphi\restriction\xi]$ is a model of the theory $\mathbf{ZFC}^-$ (the minus means that we remove the power axiom), and the ordinal ξ is countable already in $\mathrm{L}_\mu[\varphi\restriction\xi]$. Put $M_\xi[\varphi] = \mathrm{L}_{\mu_\xi[\varphi]}[\varphi\restriction\xi]$. Then $\xi < \mu_\xi[\varphi] < \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$ and the set $M_\xi[\varphi]$ is a countable transitive model of $\mathbf{ZFC}^-$.

Finally, put $\Omega_3[\varphi] = \{\xi : \xi \leq \lambda\} \setminus (\Omega_1[\varphi] \cup \Omega_2[\varphi])$. □

Lemma 7. *If $\xi < \lambda \leq \omega_1$, $\varphi \in (2^\omega)^\lambda$, $\xi \in \Omega_2[\varphi]$, and M is a transitive model of $\mathbf{ZFC}^-$ containing $\varphi\restriction\xi$ and the ordinal $\mu_\xi[\varphi]$ then the ordinal $\mu_\xi[\varphi]$ is countable in M; consequently, $M_\xi[\varphi]$ belongs to M and is countable in M.*

Proof. Put $\kappa = M \cap \mathtt{Ord}$. We may assume that $M = \mathrm{L}_\kappa[\varphi\restriction\xi]$. If all ordinals $\vartheta < \kappa$ are countable in M then the result is obvious. Otherwise, $\xi < \vartheta = \omega_1^M$ since $\mu_\xi[\varphi] \in M = \mathrm{L}_\kappa[\varphi\restriction\xi]$. □

Recall that $\mathbf{R}^0$ consists of all $x \in 2^\omega$ with $(x)_m = (x)_0$ for every m. Introduce the smaller set $\mathbf{R}_0^+$ consisting of all $x \in \mathbf{R}^0$ such that $((x)_0)_0$ and $((x)_0)_1$ lie in Word. To each $x \in \mathbf{R}_0^+$ we can associate the sequence $\boldsymbol{\chi}_x \in (2^\omega)^{\lambda+1}$, where $\lambda = |((x)_0)_0|$, by putting $\boldsymbol{\chi}_x(\lambda) = x$, and if $\eta < \lambda$ then $\boldsymbol{\chi}_x(\eta) = ((x)_0)_{k+2}$, where $k \in \omega$ satisfies $\mathbf{f}_{((x)_0)_0}(k) = \eta$. We explain the role of $((b)_0)_1$ below.

Denote now by Φ_λ, where $\lambda \leq \omega_1$, the set of all functions $\varphi \in \left(\mathbf{R}_0^+\right)^\lambda$ (i.e., $\varphi : \lambda \to \mathbf{R}_0^+$) satisfying the following:

(1) $\varphi(0) \in 2^\omega$ satisfies $\varphi(0)(n) = 0$ for every n;

(2) if $0 < \xi < \lambda$ then $|((\varphi(\xi))_0)_0| = \xi$ and $\varphi\restriction(\xi+1) = \boldsymbol{\chi}_{\varphi(\xi)}$, so that every value $\varphi(\xi)$ encodes the restriction $\varphi\restriction\xi$,

(3) if $0 < \xi < \lambda$ and $\xi \in \Omega_2[\varphi]$ then $((\varphi(\xi))_0)_1 \in \mathrm{Word}_{\mu_\xi[\varphi]}$, so that the value $\varphi(\xi)$ also encodes the ordinal $\mu_\xi[\varphi\restriction\xi]$.

Put $\Phi = \bigcup_{\lambda\leq\omega_1} \Phi_\lambda$.

Lemma 8. *If $\lambda \leq \omega_1$, $\varphi \in \Phi_\lambda$, and $\xi \in \Omega_3[\varphi]$ then $\xi = \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$.*

Proof. If $\varphi \in \Phi_\lambda$ then every value $\varphi(\xi)$ encodes the at most countability of ξ, and so the inequality $\xi > \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$ is impossible.

At the same time, by definition $\xi < \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$ for $\xi \in \Omega_2[\varphi]$. □

Definition 9. In accordance with the hypotheses of Theorem 1, fix a set $W \subseteq \omega_1$ with $\mathrm{V} = \mathrm{L}[W]$. The definition of Φ_λ imposes no exceedingly heavy restrictions on the values of $\varphi(\xi)$, $\xi \in \Omega_1[\varphi]$, and so there is a function $\varphi \in \Phi_{\omega_1}$ with $W \in \mathrm{L}[\varphi]$; hence, certainly, $\mathrm{V} = \mathrm{L}[\varphi]$. Moreover, for the same reasons we can choose a function φ of this type satisfying the additional requirement that in every generic extension, if $x \in 2^\omega$ and φ belongs to the class Δ_1^{HC} then W is Δ_1^{HC} as well.

Fix $\widehat{\varphi} \in \Phi_{\omega_1}$ of this type as a base for our subsequent constructions. $\square$

§ 5. The Set of Constraints

We now construct a forcing $\mathbb{P}$ in order to prove Theorem 1. We solve a more general problem: given a sequence $\varphi \in \Phi_{\omega_1}$, define the forcing $\mathbb{P}[\varphi] = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi[\varphi]$; moreover, our goal is to meet the following:

(A) For $\xi < \omega_1$ the set $\mathbb{P}_\xi[\varphi] \subseteq \mathbf{Perf}$ is countable and nonempty.

(B) For $\eta < \xi$ and $S \in \mathbb{P}_\eta[\varphi]$ there is a tree $T \in \mathbb{P}_\xi[\varphi]$ with $T \subseteq S$.

(C) For $\xi \le \omega_1$ the subsequence $\{\mathbb{P}_\eta[\varphi]\}_{\eta<\xi}$, and thus $\mathbb{P}_{<\xi}[\varphi] = \bigcup_{\eta<\xi} \mathbb{P}_\eta[\varphi]$ as well, belongs to $\mathrm{L}[\varphi\restriction\xi]$.

(D) If $\xi \in \Omega_2[\varphi]$ then $\mathbb{P}_\xi[\varphi]$ is an antichain, and $T \subseteq^{\mathrm{fin}} \bigcup D$ holds for every tree $T \in \mathbb{P}_\xi[\varphi]$ and every set $D \subseteq \mathbb{P}_{<\xi}[\varphi]$, $D \in M_\xi[\varphi]$, which is dense in $\mathbb{P}_{<\xi}[\varphi]$.

The construction itself goes by induction on ξ.

1*. $\mathbb{P}_0[\varphi]$ consists of all trees of the form $T_s = \mathbf{cod}\mathscr{D}_s$, where $s \in 2^{<\omega}$.

2*. For $\xi = \eta + 1$ put $\mathbb{P}_\xi[\varphi] = \bigcup_{P\in\mathbb{P}_\eta[\varphi]} \mathbf{Next}_P(b)$, where $b = \varphi(\xi)$.

3*. For $\xi \in \Omega_3[\varphi]$ put $\mathbb{P}_\xi[\varphi] = \bigcup_{P\in\mathbb{P}_{<\xi}[\varphi]} \mathbf{Next}_P(b)$, where $b = \varphi(\xi)$.

4*. Take $\xi \in \Omega_2[\varphi]$. In other words, ξ is a limit ordinal which is countable in $\mathrm{L}[\varphi\restriction\xi]$; formally, $\xi < \omega_1^{\mathrm{L}[\varphi\restriction\xi]}$. Here the construction of $\mathbb{P}_\xi[\varphi]$ requires more work. First of all, according to (C) we must have $\mathbb{P}_{<\xi}[\varphi] \in \mathrm{L}[\varphi\restriction\xi]$. Formally, if this fails then we suppose for simplicity that $\mathbb{P}_\xi[\varphi] = \mathbb{P}_{<\xi}[\varphi]$; but actually (see below) this assumption will always be fulfilled. Consider the countable model $M = M_\xi[\varphi] = \mathrm{L}_{\mu_\xi[\varphi]}[\varphi\restriction\xi]$ (see Section 4). The set $\mathscr{D}$ of all sets $D \subseteq \mathbb{P}_{<\xi}[\varphi]$, $D \in M$, which are dense in $\mathbb{P}_{<\xi}[\varphi]$ is also countable, and, on assuming that $\mathbb{P}_{<\xi}[\varphi] \in \mathrm{L}[\varphi\restriction\xi]$ above, it belongs to the class $\mathrm{L}[\varphi\restriction\xi]$ and is countable in it by Lemma 7. By Lemma 3 applied in $\mathrm{L}[\varphi\restriction\xi]$, there is an antichain $E \in \mathrm{L}[\varphi\restriction\xi]$, $E \subseteq \mathbf{Perf} \setminus \mathbb{P}_{<\xi}[\varphi]$, countable in $\mathrm{L}[\varphi\restriction\xi]$ and satisfying

(1) $T \subseteq^{\mathrm{fin}} \bigcup D$ for all $D \in \mathscr{D}$, and $T \in E$;

(2) E is dense in $\mathbb{P}_{<\xi}[\varphi] \cup E$; i.e., for every tree $S \in \mathbb{P}_{<\xi}[\varphi]$ there is a tree $T \in E$ with $T \subseteq S$. Take as $\mathbb{P}_\xi[\varphi]$ the smallest (in the sense of the canonical Gödel total ordering $\le_{\varphi\restriction\xi}^{\text{Göd}}$ of the class $\mathrm{L}[\varphi\restriction\xi]$) of these sets E.

This completes Step 4* and the inductive construction of $\mathbb{P}_\xi[\varphi]$.

Property (C) follows since the construction is absolute: for every $\xi < \omega_1$ the initial segment $\{\mathbb{P}_\eta[\varphi]\}_{\eta<\xi}$ is reproduced in $\mathrm{L}[\varphi\restriction\xi]$. Property (D) is guaranteed by Step 4*. Property (B) in the nontrivial case $\xi \in \Omega_2[\varphi]$ is also guaranteed during the construction at Step 4*.

Choose $\widehat{\varphi} \in \Phi_{\omega_1}$ following Definition 9. Put $\widehat{\mathbb{P}}_\xi = \mathbb{P}_\xi[\widehat{\varphi}]$ for every ξ, and $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}] = \bigcup_{\xi<\omega_1} \widehat{\mathbb{P}}_\xi$. This set $\widehat{\mathbb{P}}$ is our forcing.

Using Corollary 5, it is easy to establish

Lemma 10. *The set $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ is* CO*-dense in the sense of Section 2.*

In addition, if $\vartheta < \omega_1$ then the set $\widehat{\mathbb{P}}^{\ge\vartheta} = \bigcup_{\vartheta\le\xi<\omega_1} \widehat{\mathbb{P}}_\xi$ is dense in $\widehat{\mathbb{P}}$, while if $\vartheta < \lambda < \omega_1$ then the set $\widehat{\mathbb{P}}^{\ge\vartheta}_{<\lambda} = \bigcup_{\vartheta\le\xi<\lambda} \widehat{\mathbb{P}}_\xi$ is dense in $\widehat{\mathbb{P}}_{<\lambda} = \bigcup_{\xi<\lambda} \widehat{\mathbb{P}}_\xi$. $\square$

The first claim implies that every $\widehat{\mathbb{P}}$-generic set $G \subseteq \widehat{\mathbb{P}}$ generates a $\widehat{\mathbb{P}}$-generic point $x_G \in 2^\omega$, i.e., a unique element of the intersection $\bigcap_{T\in G}[T]$, and then $G = \{T \in \widehat{\mathbb{P}} : x_G \in [T]\}$, so that $\mathrm{L}[G] = \mathrm{L}[x_G]$. The points of the form x_G (for $\widehat{\mathbb{P}}$-generic sets $G \subseteq \widehat{\mathbb{P}}$) are themselves called $\widehat{\mathbb{P}}$-*generic*.

Lemma 11. *If M is a countable transitive model of the theory $\mathbf{ZFC}^-$, $\lambda \in M$ is an ordinal, and $\varphi \in \Phi_\lambda \cap M$ then $\mathbb{P}_\xi[\varphi] = (\mathbb{P}_\xi[\varphi])^M$ for every $\xi < \lambda$. Moreover, if $\mu_\lambda[\varphi] \in M$ then in addition $\mathbb{P}_\lambda[\varphi] = (\mathbb{P}_\lambda[\varphi])^M$.*

PROOF. If $\xi < \lambda$ and $\xi \in \Omega_2[\varphi]$ then the ordinal $\mu_\xi[\varphi]$ is encoded by the point $\varphi(\xi)$ by the definition of Φ_ξ. Consequently, $\mu_\xi[\varphi] \in M$, so that the model $M_\xi[\varphi]$ belongs to M and is countable in M by Lemma 7. This implies the required absoluteness. $\square$

§ 6. Cardinals Are Preserved

We claim that *the forcing* $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ *preserves cardinals*. In other words, if two sets $X, Y \in \mathrm{L}[\varphi]$ have different cardinalities in $\mathrm{L}[\widehat{\varphi}]$ then their cardinalities remain different in the $\widehat{\mathbb{P}}$-generic extensions of the class $\mathrm{L}[\widehat{\varphi}]$. It suffices to prove that $\omega_1^{\mathrm{L}[\widehat{\varphi}]}$ remains a cardinal in the indicated generic extensions. In the following lemma we deduce a well-known sufficient condition for the preservation of $\omega_1^{\mathrm{L}[\widehat{\varphi}]}$ in our case.

Lemma 12. *Take a family* $\{D_n : n \in \omega\}$ *of dense subsets of* $\widehat{\mathbb{P}}$ *and* $S \in \widehat{\mathbb{P}}$. *Then there is a tree* $T \in \widehat{\mathbb{P}}$ *with* $T \subseteq S$ *and* $T \subseteq^{\mathrm{fin}} \bigcup D_n$ *for every* n.

PROOF. Let us prove the more general fact: if $\lambda \in \Omega_3[\widehat{\varphi}]$ or $\lambda = \omega_1$ then the set $\mathbb{P}_{<\lambda}$, which belongs to $\mathrm{L}[\widehat{\varphi}\restriction\lambda]$ according to (C) of Section 5, satisfies the same requirement of the hypotheses of the lemma inside $\mathrm{L}[\widehat{\varphi}\restriction\lambda]$. We establish this fact by transfinite induction: we deduce it for some ordinal $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$ on assuming that it holds for every ordinal $\lambda' \in \Omega_3[\widehat{\varphi}]$ with $\lambda' < \lambda$. This includes the cases that λ is the smallest ordinal in $\Omega_3[\widehat{\varphi}]$ or that $\lambda = \omega_1$ and $\Omega_3[\widehat{\varphi}]$ is empty.

Thus, suppose that $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$. Then $\lambda = \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$. Suppose that $S \in \mathbb{P}_{<\lambda}$, all sets $D_n \subseteq \mathbb{P}_{<\lambda}$ are dense in $\mathbb{P}_{<\lambda}$, and $\{D_n\}_{n\in\omega} \in \mathrm{L}[\widehat{\varphi}\restriction\lambda]$. This hypothesis, expressed as a formal proposition, is then valid in the model $\mathrm{L}_\delta[\widehat{\varphi}\restriction\lambda]$, where $\delta = \omega_2^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$. Taking in $\mathrm{L}[\widehat{\varphi}\restriction\lambda]$ a countable elementary submodel of the model $\mathrm{L}_\delta[\widehat{\varphi}\restriction\lambda]$ including S and the sequence D_n, and applying to it the Mostowski collapse (i.e., an $\in$-isomorphism onto a transitive set), we obtain some ordinals $\xi < \nu < \lambda$ with $S \in \mathbb{P}_{<\xi}$, while all sets $D'_n = D_n \cap \mathbb{P}_{<\xi}$ are dense in $\mathbb{P}_{<\xi}$ and belong to $N = \mathrm{L}_\mu[\widehat{\varphi}\restriction\xi]$. The two cases are possible here:

CASE 1. $\xi \in \Omega_3[\widehat{\varphi}]$. By the inductive assumption, the result holds for ξ, and so the tree $T \subseteq S$ with $T \subseteq^{\mathrm{fin}} \bigcup D'_n$ for all n exists in $\mathbb{P}_{<\xi}$.

CASE 2. $\xi \in \Omega_2[\widehat{\varphi}]$. The ordinal ξ is uncountable in N since $\lambda = \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$ is uncountable in $\mathrm{L}[\widehat{\varphi}\restriction\lambda]$, and even more so in $\mathrm{L}_\delta[\widehat{\varphi}\restriction\lambda]$. This implies that $\nu < \mu_\xi[\widehat{\varphi}]$; thus, $N \subseteq M_\xi[\widehat{\varphi}] = \mathrm{L}_{\mu_\xi[\widehat{\varphi}]}[\widehat{\varphi}\restriction\xi]$, so that all sets D'_n belong to $M_\xi[\widehat{\varphi}]$. According to (B) of Section 5, there is a tree $T \in \widehat{\mathbb{P}}_\xi$ with $T \subseteq S$. By (D), $T \subseteq^{\mathrm{fin}} D'_n$ holds for all n, and $T \subseteq^{\mathrm{fin}} D_n$ even more so. $\square$

§ 7. The Encoding Properties of Forcing

The following procedure for decoding and reconstructing the values of $\widehat{\varphi}(\xi)$ is based on Corollary 5: knowing a tree $P \in \mathbf{Perf}$ and a point $x \in [T_P(b,u)]$, we can reconstruct the values of b and u, and therefore, the set $[T_P(b,u)]$ itself.

DEFINITION 13. Take $x \in 2^\omega$. Using transfinite induction on $\alpha < \omega_1$, define the following objects:
(a) a tree $Q_\alpha(x) \in \mathbf{Perf}$ with $x \in [Q_\alpha(x)]$ and an ordinal $\boldsymbol{\lambda}_\alpha(x) < \omega_1$;
(b) a function $\boldsymbol{\psi}_\alpha(x) \in \Phi_{\boldsymbol{\lambda}_\alpha(x)}$; moreover, $\alpha < \beta \Longrightarrow \boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\psi}_\beta(x)$.
The construction either reaches ω_1 or stops at some ordinal $\xi^* < \omega_1$.

START. Put $Q_0(x) = 2^{<\omega}$, so that $[Q_0(x)] = 2^\omega$ and certainly $x \in [Q_0(x)]$ and $\boldsymbol{\psi}_\alpha(x) = \Lambda$ (the empty function), so that $\boldsymbol{\lambda}_0(x) = 0$.

STEP $\alpha \to \alpha + 1$. Assume that the objects $P = Q_\alpha(x)$, $\boldsymbol{\lambda}_\alpha(x)$, and $\boldsymbol{\psi}_\alpha(x)$ are already constructed and satisfy (a) and the condition $\boldsymbol{\psi}_\alpha(x) \in \Phi_{\boldsymbol{\lambda}_\alpha(x)}$ in (b). In particular, $x \in [P]$. If the condition

$(*)$ $x \in \mathbf{R}_P^{\mathrm{even}}$, $b = \mathbf{b}_P^{\mathrm{even}}(x) \in \mathbf{R}_0^+$, $|((b)_0)_0| = \boldsymbol{\lambda}_\alpha(x)$, and $\boldsymbol{\chi}_b$ belongs to Φ_λ for some $\lambda > \boldsymbol{\lambda}_\alpha(x)$ and satisfies $\boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\chi}_b$,

is violated then the construction is complete.

Assume that $(*)$ is fulfilled. Then $u = \boldsymbol{u}_P(x) \in U(b)$ and $x \in [T_P(b,u)]$ by Corollary 5. Put $Q_{\alpha+1}(x) = [T_P(b,u)]$, $\boldsymbol{\psi}_{\alpha+1}(x) = \boldsymbol{\chi}_b$, and $\boldsymbol{\lambda}_{\alpha+1}(x) = \lambda = \operatorname{dom}\boldsymbol{\psi}_{\alpha+1}(x)$.

The limit step. Take a limit ordinal $\gamma < \omega_1$ and assume that the values of $Q_\alpha(x)$, $\boldsymbol{\psi}_\alpha(x)$, and $\boldsymbol{\lambda}_\alpha(x)$ are already defined for all $\alpha < \gamma$ and satisfy (a) and (b) in this range. Put $\boldsymbol{\psi}_\gamma(x) = \bigcup_{\alpha<\gamma}\boldsymbol{\psi}_\alpha(x)$; Then $\psi = \boldsymbol{\psi}_\gamma(x) \in \Phi_\lambda$, where $\lambda = \boldsymbol{\lambda}_\gamma(x) := \sup_{\alpha<\gamma}\boldsymbol{\lambda}_\alpha(x)$. If the condition

(†) the function $\psi = \boldsymbol{\psi}_\gamma(x)$ satisfies $\omega_1^{\mathrm{L}[\psi]} > \lambda$, where $\lambda = \operatorname{dom}\psi = \boldsymbol{\lambda}_\gamma(x)$, i.e., formally, $\lambda \in \Omega_2[\psi]$,

is violated then the construction is complete.

Assume that (†) is satisfied. Make in $\mathrm{L}[\psi]$ the construction of Section 5 to Step λ. Namely, define the set $\mathbb{P}_\alpha[\psi] \subseteq \mathbf{Perf}$ for every $\alpha < \lambda$, define $\mathbb{P}_{<\lambda}[\psi]$, and adjust the antichain $\mathbb{P}_\lambda[\psi]$ according to 4*. (Observe that $\lambda \in \Omega_2[\psi]$.) If

(‡) there exists a unique tree $T \in \mathbb{P}_\lambda[\psi]$ with $x \in [T]$

then take this T as $Q_\gamma(x)$; otherwise, the construction stops. □

Lemma 14. *If M is a countable transitive model of $\mathbf{ZFC}^-$, $x \in 2^\omega \cap M$, $\alpha \in M$ is an ordinal countable in M, and the objects $(\boldsymbol{\psi}_\alpha(x))^M$ and $(Q_\alpha(x))^M$ are defined then $(\boldsymbol{\psi}_\alpha(x))^M = \boldsymbol{\psi}_\alpha(x)$ and $(Q_\alpha(x))^M = Q_\alpha(x)$.*

The proof goes by induction on α. The step $\alpha \to \alpha+1$ is trivial. Take a limit ordinal $\gamma \in M$ which is countable in M, and assume that the objects $(\boldsymbol{\psi}_\gamma(x))^M$ and $(Q_\gamma(x))^M$ exist in M. Then $(\boldsymbol{\psi}_\alpha(x))^M$ and $(Q_\alpha(x))^M$ exist in M for every $\alpha < \gamma$. By the inductive assumption, $(\boldsymbol{\psi}_\alpha(x))^M = \boldsymbol{\psi}_\alpha(x)$ and $(Q_\alpha(x))^M = Q_\alpha(x)$ for $\alpha < \gamma$.

It is clear that $(\boldsymbol{\psi}_\gamma(x))^M = \boldsymbol{\psi}_\gamma(x) = \bigcup_{\alpha<\gamma}\boldsymbol{\psi}_\alpha(x)$. Therefore, $\psi = \boldsymbol{\psi}_\gamma(x) \in M$. On the other hand, $\psi \in \Phi_\lambda$, where $\lambda = \boldsymbol{\lambda}_\gamma(x)$. Thus, the sequence $\{\mathbb{P}_\xi[\psi]\}_{\xi<\lambda}$ also belongs to M and coincides with $(\{\mathbb{P}_\xi[\psi]\}_{\xi<\lambda})^M$ by Lemma 11. Since $(Q_\gamma(x))^M$ exists in M, condition (†) is satisfied in M, so that $\lambda \in \Omega_2[\psi]$ in M; thus, $\lambda \in \Omega_2[\psi]$ also in the universe of all sets.

The existence of $(Q_\gamma(x))^M$ implies the existence of $(\mathbb{P}_\lambda[\psi])^M$ in M: essentially, $(Q_\gamma(x))^M$ is a unique tree $T \in (\mathbb{P}_\lambda[\psi])^M$ with $x \in [T]$. In addition, $(\mathbb{P}_\lambda[\psi])^M$ is the smallest in M, in the sense of the Gödel order, antichain in **Perf** in the already-defined family of antichains. This implies that $(\mathbb{P}_\lambda[\psi])^M$ coincides with $\mathbb{P}_\lambda[\psi]$; therefore, $(Q_\alpha(x))^M = Q_\alpha(x)$ as required. □

The following key lemma shows that Definition 13 correctly reconstructs $\widehat{\varphi}$ provided that we use the points x on sufficiently high levels $\widehat{\mathbb{P}}_\lambda$ of the forcing $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$.

Lemma 15. *Take $\lambda < \omega_1$, $T \in \widehat{\mathbb{P}}_\lambda = \mathbb{P}_\lambda[\widehat{\varphi}]$, and $x \in [T]$. Then there is an ordinal $\alpha \le \lambda$ such that the values of $Q_\alpha(x)$ and $\boldsymbol{\psi}_\alpha(x)$ are defined in accordance with Definition* 13*, $\operatorname{dom}\boldsymbol{\psi}_\alpha(x) = \boldsymbol{\lambda}_\alpha(x) = \xi$, $\widehat{\varphi}\restriction\lambda = \boldsymbol{\psi}_\alpha(x)$, and $T = Q_\alpha(x)$.*

The proof goes by induction on λ. If $\lambda = 0$ then $\widehat{\varphi}\restriction 0 = \Lambda = \boldsymbol{\psi}_0(x)$. Now the step of induction: let us verify the lemma for some $\lambda > 0$ on assuming that it holds for all ordinals $\xi < \lambda$.

Case 1. $\lambda \in \Omega_1[\widehat{\varphi}] \cup \Omega_3[\widehat{\varphi}]$. By definition, there are an ordinal $\xi < \lambda$ (for instance, $\lambda = \xi+1$ for $\lambda \in \Omega_1[\widehat{\varphi}]$) and a tree $P \in \widehat{\mathbb{P}}_\xi$ with $T = T_P(b,u)$ for some $u \in U(b)$, where $b = \widehat{\varphi}(\lambda)$. Then $T \subseteq P$, and so $x \in [P]$. By the inductive assumption, there exists an ordinal $\alpha \le \xi$ such that the values of $Q_\alpha(x) = P$ and $\boldsymbol{\psi}_\alpha(x) = \widehat{\varphi}\restriction\xi$ are defined in accordance with Definition 13. Since $x \in [T]$ and $T = T_P(b,u)$, it follows that $x \in \mathbf{R}_P^{\mathrm{even}}$ by Corollary 5, while $b = \widehat{\varphi}(\lambda) = \mathbf{b}_P^{\mathrm{even}}(x) \in \mathbf{R}_0^+$. Then $Q_{\alpha+1}(x) = T$ by construction. Moreover, since $b = \widehat{\varphi}(\lambda)$, we have $\boldsymbol{\chi}_b = \widehat{\varphi}\restriction(\lambda+1)$ by the choice of $\widehat{\varphi}$ in Section 4. Therefore, the condition $\boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\chi}_b$ is fulfilled, so that $\boldsymbol{\psi}_{\alpha+1}(x) = \boldsymbol{\chi}_b = \widehat{\varphi}\restriction(\lambda+1)$.

Case 2. $\lambda \in \Omega_2[\widehat{\varphi}]$. Then $\widehat{\mathbb{P}}_\lambda \in \mathrm{L}[\widehat{\varphi}\restriction\lambda]$ is an antichain in **Perf** countable in $\mathrm{L}[\widehat{\varphi}\restriction\lambda]$, and for every tree $T \in \widehat{\mathbb{P}}_\lambda$ we have $T \subseteq^{\mathrm{fin}} \bigcup D$, whatever a dense (in $\mathbb{P}_{<\lambda}$) set $D \subseteq \mathbb{P}_{<\lambda}$, $D \in M_\lambda[\widehat{\varphi}] = \mathrm{L}_{\mu_\lambda[\widehat{\varphi}]}[\widehat{\varphi}\restriction\lambda]$ may be, where $\mu_\lambda[\widehat{\varphi}] < \omega_1$ is the smallest ordinal ν so that λ is countable already in $\mathrm{L}_\nu[\widehat{\varphi}\restriction\lambda]$, and $\mathrm{L}_\nu[\widehat{\varphi}\restriction\lambda]$ satisfies $\mathbf{ZFC}^-$. But all sets of the form $\mathbb{P}_{<\lambda}^{\ge\vartheta} = \bigcup_{\vartheta\le\xi<\lambda}\widehat{\mathbb{P}}_\xi$ for $\vartheta < \lambda$ are dense in $\widehat{\mathbb{P}}_{<\lambda}$ by construction and belong to the model $M_\lambda[\widehat{\varphi}]$. (The sequence $\{\widehat{\mathbb{P}}_\xi\}_{\xi<\lambda}$ itself belongs to $M_\lambda[\widehat{\varphi}]$ by Lemma 11.)

Thus, $T \subseteq^{\text{fin}} \bigcup \mathbb{P}^{\geq\vartheta}_{<\lambda}$ for every $\vartheta < \lambda$; thus, since $x \in [T]$, for every $\vartheta < \lambda$ there are an ordinal ξ with $\vartheta \leq \xi < \lambda$ and a tree $S \in \widehat{\mathbb{P}}_\xi$ with $x \in [S]$. Then by the inductive assumption there is an ordinal $\alpha(\xi) \leq \xi$ with $Q_{\alpha(\xi)}(x) = S$ and $\boldsymbol{\psi}_{\alpha(\xi)}(x) = \widehat{\varphi}\restriction\xi$. Hence, $\boldsymbol{\psi}_\gamma(x) = \widehat{\varphi}\restriction\lambda \in \Phi_\lambda$ for some limit ordinal $\gamma \leq \lambda$.

Inspect the last paragraph of Definition 13 (prior to Lemma 15), taking $\psi = \boldsymbol{\psi}_\gamma(x) = \widehat{\varphi}\restriction\lambda$. Then the set $\mathbb{P}_{<\lambda}[\psi]$ coincides with our $\widehat{\mathbb{P}}_{<\lambda}$, and the antichain $\mathbb{P}_\lambda[\psi]$, with $\widehat{\mathbb{P}}_\lambda = \mathbb{P}_\lambda[\widehat{\varphi}]$. But $T \in \widehat{\mathbb{P}}_\lambda$ and $x \in [T]$. This by definition implies that $Q_\gamma(x) = T$, which completes the step of induction. □

Corollary 16. *Take a $\widehat{\mathbb{P}}$-generic point $x = x_G$ over $\mathrm{L}[\widehat{\varphi}] = \mathrm{L}[W]$. Then the elements of $Q_\alpha(x)$, $\boldsymbol{\psi}_\alpha(x)$, and $\boldsymbol{\lambda}_\alpha(x)$ are defined for all $\alpha < \omega_1^{\mathrm{L}[\varphi]}$ in accordance with Definition* 13, *and*

$$\widehat{\varphi} = \bigcup_{\alpha < \omega_1^{\mathrm{L}[\varphi]}} \boldsymbol{\psi}_\alpha(x) \in \mathrm{L}[x].$$

This yields $\omega_1^{\mathrm{L}[\varphi]} = \omega_1^{\mathrm{L}[\varphi,x]} = \omega_1^{\mathrm{L}[x]}$.

Proof. The sets $\widehat{\mathbb{P}}^{\geq\vartheta} = \bigcup_{\vartheta \leq \xi < \omega_1} \widehat{\mathbb{P}}_\xi$ for $\vartheta < \omega_1$ are dense in $\widehat{\mathbb{P}}$. Therefore, given $\vartheta < \omega_1$ there are an ordinal ξ with $\vartheta \leq \xi < \omega_1^{\mathrm{L}[\varphi]}$ and a tree $T \in \widehat{\mathbb{P}}_\xi$ with $x \in [T]$. Then

$$\widehat{\varphi} = \bigcup_{\alpha < \omega_1^{\mathrm{L}[\varphi]}} \boldsymbol{\psi}_\alpha(x)$$

by Lemma 15. Thus, $\widehat{\varphi} \in \mathrm{L}[x]$ since the construction of the functions $\boldsymbol{\psi}_\alpha(x)$ is absolute. □

§ 8. Minimality

To begin with, we prove Theorem 1 without the property of minimality.

Lemma 17. *Every point $x = x_G$ $\widehat{\mathbb{P}}$-generic over $\mathrm{L}[W] = \mathrm{L}[\widehat{\varphi}]$ satisfies Theorem* 1 *with the possible exception of minimality.*

Proof. The forcing $\widehat{\mathbb{P}}$ preserves cardinals (see Section 6). Furthermore, $\widehat{\varphi}$ and W belong to $\mathrm{L}[x]$ by Corollary 16. Let us verify that $\widehat{\varphi}$, and thus W as well, (see Definition 9) belong to Δ^{HC}_1 in $\mathrm{L}[x]$. Observe that $\widehat{\varphi}(\xi) = r$ is equivalent to each of the formulas

$$\exists M \exists \alpha \big(\alpha < \omega_1^M \wedge \xi, r \in M \wedge M \models \boldsymbol{\psi}_\alpha(x)(\xi) = r\big);$$
$$\forall M \forall \alpha \big(\alpha < \omega_1^M \wedge \xi, r \in M \Longrightarrow M \models (!\boldsymbol{\psi}_\alpha(x)(\xi) \Longrightarrow \boldsymbol{\psi}_\alpha(x)(\xi) = r)\big),$$

where M runs over all countable transitive models of $\mathbf{ZFC}^-$, and α are ordinals in M, while $!\boldsymbol{\psi}_\alpha(x)(\xi)$ means that the value $\boldsymbol{\psi}_\alpha(x)(\xi)$ is defined. The equivalence follows from Lemma 14 and Corollary 16. The first of these formulas yields the class $\Sigma^{\mathrm{HC}}_1(x)$ for $\widehat{\varphi}$, and the second, the class $\Pi^{\mathrm{HC}}_1(x)$. □

We now show how to achieve the minimality of $\widehat{\mathbb{P}}$-generic points.

We argue in the universe $\mathrm{V} = \mathrm{L}[A] = \mathrm{L}[\widehat{\varphi}]$ of Theorem 1.

Take an arbitrary forcing $\mathbb{Q} \subseteq \mathbf{Perf}$ not necessarily equal to $\widehat{\mathbb{P}}$ of Section 5. Denote by $\mathbf{N}_\mathbb{Q}$ the collection of all sets $t \subseteq \mathbb{Q} \times \omega$, i.e., all $\mathbb{Q}$-*names* for the subsets of ω. Given $t \in \mathbf{N}_\mathbb{Q}$, $k \in \omega$, and $T \in \mathbb{Q}$, say that T $\mathbb{Q}$-*implies* that $k \notin t$ whenever T is incompatible in $\mathbb{Q}$ with any $S \in t''k = \{S \in \mathbb{Q} : \langle S, k\rangle \in t\}$, and that T $\mathbb{Q}$-*implies* that $k \in t$ whenever every tree $T' \in \mathbb{Q}$ with $T' \subseteq T$ is compatible in $\mathbb{Q}$ with at least one $S \in t''k$.

Put $t[G] = \{k \in \omega : \exists T \in G(\langle T, k\rangle \in t)\}$ for every $\mathbb{Q}$-generic set $G \subseteq \mathbb{Q}$ and $t \in \mathbf{N}_\mathbb{Q}$. It is known that for every set $Y \in \mathrm{V}[G]$, $Y \subseteq \omega$, there is a name $t \in \mathbf{N}_\mathbb{Q}$ in V with $Y = t[G]$.

A name $t \in \mathbf{N}_\mathbb{Q}$ is called a $\mathbb{Q}$-*constant* on $T \in \mathbb{Q}$ if for every $k \in \omega$ either T $\mathbb{Q}$-implies that $k \in t$, or T $\mathbb{Q}$-implies that $k \notin t$ in the indicated sense. In this case T $\mathbb{Q}$-forces that $t[\underline{G}]$ belongs to this universe V, where $\underline{G}$ is the canonical name for G.

Refer as a *splittable system* in $\mathbb{Q}$ to a family $\{T_s\}_{s\in 2^{<\omega}}$ of trees $T_s \in \mathbb{Q}$ satisfying the following two conditions:

(i) $T_{s^\wedge 0} \cup T_{s^\wedge 1} \subseteq T_s^n$ but $T_{s^\wedge 0} \cap T_{s^\wedge 1}^n = \varnothing$;

(ii) $\operatorname{diam} T_s \le \frac{1}{\operatorname{lh} s}$, where $\operatorname{lh} s$ is the length of s and $\operatorname{diam} T = \frac{1}{m+1}$ with m being the greatest number such that $a{\restriction}m = b{\restriction}m$ for all $a, b \in [T]$;

In this case $S = \bigcap_n \bigcup_{\operatorname{lh} s = n} T_s \in \mathbf{Perf}$ but not necessarily $\in \mathbb{Q}$. But if $S \in \mathbb{Q}$ then we say that the system $\{T_s\}_{s\in 2^{<\omega}}$ *converges* to S in $\mathbb{Q}$.

A splittable system $\{T_s\}_{s\in 2^{<\omega}}$ is called *t-bijective* in $\mathbb{Q}$ (we still have $t \in \mathbf{N}_{\mathbb{Q}}$) whenever there exists a number $n \in \omega$ such that for every $s \in 2^{<\omega}$ with $\operatorname{lh} s \ge n$ there is a number $k \in \omega$ with either

(1) $T_{s^\wedge 0}$ $\mathbb{Q}$-implies that $k \in t$ and $T_{s^\wedge 1}$ $\mathbb{Q}$-implies that $k \notin t$, or

(2) $T_{s^\wedge 0}$ $\mathbb{Q}$-implies that $k \notin t$ and $T_{s^\wedge 1}$ $\mathbb{Q}$-implies that $k \in t$.

Then if the system $\{T_s\}_{s\in 2^{<\omega}}$ converges to some tree $S \in \mathbb{Q}$ then S $\mathbb{Q}$-forces $\underline{x} \in \mathrm{L}[t, \underline{G}]$, where $\underline{x}$ is the canonical name for a generic point x_G. Therefore, the following property (D) for the forcing $\widehat{\mathbb{P}}$ of Section 5 is sufficient for all $\widehat{\mathbb{P}}$-generic points to be minimal in the sense of Theorem 1.

(D) If $P \in \widehat{\mathbb{P}}$ and $t \in \mathbf{N}_{\widehat{\mathbb{P}}}$ then there is a splittable system $\{T_s\}_{s\in 2^{<\omega}}$ in $\widehat{\mathbb{P}}$ with $T_\Lambda \subseteq T$ converging in $\widehat{\mathbb{P}}$ and *rectifying* t in $\widehat{\mathbb{P}}$ in the sense that either t is $\widehat{\mathbb{P}}$-constant on T_Λ in $\widehat{\mathbb{P}}$ or the system is t-bijective in $\widehat{\mathbb{P}}$.

Therefore, we need to modify the construction of Section 5 and fulfill condition (D). The next lemma is rather obvious.

Lemma 18. *If a set $\mathbb{Q} \subseteq \mathbf{Perf}$ is* CO*-closed, $P \in \mathbb{Q}$, and $t_j \in \mathbf{N}_{\mathbb{Q}}$ for every $j < \omega$ then there exists a splittable system $\{T_s\}_{s\in 2^{<\omega}}$ in $\mathbb{Q}$ with $T_\Lambda \subseteq P$ rectifying every name t_j in $\mathbb{Q}$.*

(The lemma does not assert the convergence in $\mathbb{Q}$ of this system.)

Return to Step 4* of Section 5, where the set $\mathbb{P}_\xi[\varphi]$ is defined in the case that $\xi \in \Omega_2[\varphi]$ as the $\le^{\text{Göd}}_{\varphi\restriction\xi}$-smallest antichain $E \in \mathrm{L}[\varphi{\restriction}\xi]$, $E \subseteq \mathbf{Perf}$, with certain properties. Modify this definition as follows.

Definition 19. Define the set $E = \mathbb{P}_\xi[\varphi]$ as indicated (Step 4* in Section 5). Recall that $M_\xi[\varphi] = \mathrm{L}_{\mu_\xi[\varphi]}[\varphi{\restriction}\xi]$ is a transitive model of $\mathbf{ZFC}^-$ countable in $\mathrm{L}[\varphi{\restriction}\xi]$ by Lemma 7. Apply Lemma 18 in $\mathrm{L}[\varphi{\restriction}\xi]$ to $\mathbb{Q} = \mathbb{P}_{<\xi}[\varphi]$ and the family of names $\tau = M_\xi[\varphi] \cap \mathbf{N}_{\mathbb{P}_{<\xi}[\varphi]}$. For every $P \in E$ we obtain a splittable system $\sigma_P = \{T_s^P\}_{s\in 2^{<\omega}}$ in $\mathbb{P}_{<\xi}[\varphi]$ with $T_\Lambda^P \subseteq P$, which rectifies all names $t \in \tau$ in $\mathbb{P}_{<\xi}[\varphi]$. Choose this set of systems $\{\sigma_P\}_{P\in E}$ as the $\le^{\text{Göd}}_{\varphi\restriction\xi}$-smallest among all sets of this form in $\mathrm{L}[\varphi{\restriction}\xi]$. Put $P' = \bigcap_{n\in\omega} \bigcup_{\operatorname{lh} s = m} T_s^P$ for every $P \in E$, and then $\mathbb{P}_\xi[\varphi] = E' = \{P' : P \in E\}$. $\square$

Now we state the key property of the modified definition.

Lemma 20. *If $\varphi \in \Phi_{\omega_1}$, $\xi \in \Omega_2[\varphi]$, $t \in M_\xi[\varphi] \cap \mathbf{N}_{\mathbb{P}_{<\xi}[\varphi]}$, and $P \in \mathbb{P}_{<\xi}[\varphi]$ then in $\mathrm{L}[\varphi{\restriction}\xi]$ there exists a splittable system $\{T_s\}_{s\in 2^{<\omega}}$ in $\mathbb{P}_{<\xi}[\varphi]$ with $T_\Lambda \subseteq P$, which converges in $\mathbb{P}_\xi[\varphi]$ and rectifies t in $\mathbb{P}_{<\xi}[\varphi]$.*

Lemma 21. *The modified forcing $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ satisfies* (D)*, so that all $\widehat{\mathbb{P}}$-generic points are minimal in the sense of Theorem* 1.

Proof. By induction on λ we verify that if $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$ then the forcing $\widehat{\mathbb{P}}_{<\lambda} = \mathbb{P}_{<\lambda}[\widehat{\varphi}]$ satisfies (D) in $\mathrm{L}[\widehat{\varphi}{\restriction}\lambda]$.

Suppose that $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$; then, $\lambda = \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$ by Lemma 8. Arguing in $\mathrm{L}[\widehat{\varphi}{\restriction}\lambda]$, take $P \in \widehat{\mathbb{P}}_{<\lambda}$, $t \in \mathbf{N}_{\widehat{\mathbb{P}}_{<\lambda}}$, and $\vartheta = \omega_3^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$. Take a countable elementary submodel $N \subseteq \mathrm{L}_\vartheta[\widehat{\varphi}{\restriction}\lambda]$ containing $\widehat{\varphi}{\restriction}\lambda$, P, $\widehat{\mathbb{P}}_{<\lambda}$, and t, and consider the Mostowski collapse $h : N \xrightarrow{\text{on}} N'$ on the transitive $N' = \mathrm{L}_{\vartheta'}[\widehat{\varphi}{\restriction}\lambda']$, where $\lambda' < \vartheta' < \lambda = \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda]}$. It is easy to see that $h(\widehat{\varphi}{\restriction}\lambda) = \widehat{\varphi}{\restriction}\lambda'$, $h(P) = P \in \widehat{\mathbb{P}}_{<\lambda'}$, and $t' = h(t) = t \cap (\widehat{\mathbb{P}}_{<\lambda'} \times \omega) \in N' \cap \mathbf{N}_{\widehat{\mathbb{P}}_{<\lambda'}}$. Finally, $\lambda' = \omega_1^{N'}$.

Case 1. If $\lambda' \in \Omega_3[\widehat{\varphi}]$ then $\lambda' = \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda']}$ by Lemma 8. By the inductive hypothesis, the forcing $\widehat{\mathbb{P}}_{<\lambda'}$ satisfies (D) in $\mathrm{L}[\widehat{\varphi}{\restriction}\lambda']$, so that there is a splittable system $\{T_s\}_{s\in 2^{<\omega}} \in \mathrm{L}[\widehat{\varphi}{\restriction}\lambda']$ in $\widehat{\mathbb{P}}_{<\lambda'}$ with $T_\Lambda \subseteq T$, which converges in $\widehat{\mathbb{P}}_{<\lambda'}$ and rectifies t' in $\widehat{\mathbb{P}}_{<\lambda'}$. But h^{-1} is an elementary embedding; thus, the same system rectifies t in $\widehat{\mathbb{P}}_{<\lambda}$. Therefore, $\widehat{\mathbb{P}}_{<\lambda}$ satisfies (D).

Case 2. If $\lambda' \in \Omega_2[\widehat{\varphi}]$ then $\lambda' < \omega_1^{\mathrm{L}[\widehat{\varphi}\restriction\lambda']}$ and, moreover, λ' is countable in the model $M_{\lambda'}[\widehat{\varphi}] = \mathrm{L}_{\mu_{\lambda'}[\widehat{\varphi}]}[\widehat{\varphi}\restriction\lambda']$. On the other hand, $\lambda' = \omega_1^{N'}$, where $N' = \mathrm{L}_{\vartheta'}[\widehat{\varphi}\restriction\lambda']$. This implies that $\vartheta' < \mu_{\lambda'}[\widehat{\varphi}]$, $N' \in M_{\lambda'}[\widehat{\varphi}]$, and $t' \in M_{\lambda'}[\widehat{\varphi}]$. By Lemma 20, in $\mathrm{L}[\varphi\restriction\lambda']$ there is a splittable system $\{T_s\}_{s\in 2^{<\omega}}$ in $\mathbb{P}_{<\lambda'}[\widehat{\varphi}]$ with $T_\Lambda \subseteq P$, which converges to some $T \in \mathbb{P}_{\lambda'}[\widehat{\varphi}]$ and rectifies t in $\mathbb{P}_{<\lambda'}[\widehat{\varphi}]$. Then, since h^{-1} is an elementary embedding, the same splittable system rectifies t in $\widehat{\mathbb{P}}_{<\lambda}$ and, obviously, $T \in \widehat{\mathbb{P}}_{<\lambda}$. $\square$

It is easy to see that with the modified Definition 19 the result of Section 6 on the preservation of cardinals is retained. Moreover, the corresponding modification of the decoding procedure (Definition 7) preserves Lemma 15 and 17. This completes the proof of Theorem 1.

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