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ON THE 'DEFINABILITY OF DEFINABLE' PROBLEM OF ALFRED TARSKI, PART II

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Abstract. Alfred Tarski [J. Symbolic Logic 13 (1948), pp. 107–111] defined $\mathbf{D}_{pm}$ to be the set of all sets of type p, type-theoretically definable by parameterfree formulas of type $\le m$, and asked whether it is true that $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ for $m \ge 1$. Tarski noted that the negative solution is consistent because the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies $\mathbf{D}_{1m} \notin \mathbf{D}_{2m}$ for all $m \ge 1$, and he left the consistency of the positive solution as a major open problem. This was solved in our recent paper [Mathematics 8 (2020), pp. 1–36], where it is established that for any $m \ge 1$ there is a generic extension of $\mathbf{L}$, the constructible universe, in which it is true that $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$. In continuation of this research, we prove here that Tarski's sentences $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ are not only consistent, but also independent of each other, in the sense that for any set $Y \subseteq \omega \smallsetminus \{0\}$ in $\mathbf{L}$ there is a generic extension of $\mathbf{L}$ in which it is true that $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ holds for all $m \in Y$ but fails for all $m \ge 1$, $m \notin Y$. This gives a full and conclusive solution of the Tarski problem.

The other main result of this paper is the consistency of $\mathbf{D}_1 \in \mathbf{D}_2$ via another generic extension of $\mathbf{L}$, where $\mathbf{D}_p = \bigcup_m \mathbf{D}_{pm}$, the set of all sets of type p, type-theoretically definable by formulas of any type.

Our methods are based on almost-disjoint forcing of Jensen and Solovay [Some applications of almost disjoint sets, North-Holland, Amsterdam, 1970, pp. 84–104].

Contents

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Section I. Introduction

1. Preamble

Questions of definability of mathematical objects were raised in the course of discussions on the foundations of mathematics, set theory, and the axiom of choice in the early twentieth century. See, for example, the famous *Sinq lettres* [22] by Baire, Borel, Hadamard, and Lebesgue. Later, with the development of mathematical logic, Alfred Tarski [50][1] showed that the notion of 'being definable' can be

[1]And apparently Kurt Gödel, unpublished, even earlier. See Murawski [49] on the history of this discovery and the role of Gödel.

rigorously analyzed only by first specifying a particular formal context in which the notion of definability is considered.

Generally, Tarski's definability theory was applied and further developed in such classical works as Kleene [40, 41] and in more recent research such as Addison [2], Cegielski [7], Kossak [44] among many other papers. From a broader point of view, modern set-theoretic studies on definability are precisely characterized by the following definition by Yiannis Moschovakis [48, page xiv]:

> [...] *the central problem of* [...] *definability theory in general* [is] *to find and study the characteristic properties of definable objects.*

Various aspects of definability have been in the focus of modern set theoretic research, in particular those related to the broadest context of the set or class universe [3, 9, 46] or even *multiverse* as one of the modern concepts in the mathematical foundations [4], or particular models of set theory as *e.g.* a recent comprehensive study of the Cohen-generic model in [39], as well as those focused on the domain of reals and sets of reals typically studied by *descriptive set theory*. In the latter case, the context of definability can be limited to the *descriptive set theoretic definability*, or can be extended as wide as to the *ordinal definability in the set theoretic universe*. In the first direction, we may mention such recent papers as

- [13] on generic models with long projective wellorderings of the reals,
- [17] on a Π^1_2 violation of the schema of dependent choices $\mathbf{DC}$ in the presence of the full axiom of countable choice $\mathbf{AC}_\omega$,
- [18] on a model in which all projective sets of reals are Lebesgue measurable, but there is a Δ^1_3 set without the Baire property,

among many other research, while for the second direction we suggest the following selection of related results:

- [8] on the existence of ordinal definable (OD, for brevity) equivalence classes of $\mathbf{\Sigma}^1_1$ equivalence relations on the reals, containing no OD elements,
- [23] on *algebraic definability*,[2] notably extended in [20] and then in [12] where it is established that, in the Sacks and some other generic extensions of $\mathbf{L}$ by a real, there is an OD partition of the reals into two non-OD parts,
- [52] on *Russell non-typicality*, or *blurry definability*,[3] extended in [19, 38].

This study of ours touches upon the field of definability in *type theory*, which in a certain sense lies between the two directions indicated.

2. The problem

More specific, restricted types of definability, in particular, the type-theoretic definability, were considered by Tarski in his subsequent work [51]. The following key definition will be elaborated upon later in more detail in Subsection 6.

Definition 2.1 (Tarski [51]). If $m, p < \omega$ then $\mathbf{D}_{pm}$ is the set of all objects X of type p (type $0 = \omega$, type $1 = \mathscr{P}(\omega)$, type $2 = \mathscr{P}(\mathscr{P}(\omega)) = \mathscr{P}^2(\omega)$, etc.), definable by a parameter-free type-theoretic formula φ of type m (*i.e.*, all variables are over types $\le m$). Furthermore, $\mathbf{D}_p = \bigcup_m \mathbf{D}_{pm}$ is the set of all objects of type p, definable by a parameter-free formula of any type. □

[2] A set is algebraically definable iff it belongs to a finite ordinal definable set.

[3] A set is Russell non-typical, or blurry definable, iff it belongs to a countable OD set.

Studying the definability properties of these sets $\mathbf{D}_{pm}$ in the obligatory case $p \le m+1$ (otherwise $\mathbf{D}_{pm} = \varnothing$), Tarski noted that $\mathbf{D}_{pm} \in \mathbf{D}_{p+1,m+1}$, but $\mathbf{D}_{pm} \notin \mathbf{D}_{p+1,m}$ in all cases except for $p = 1$, $m \ge 1$.[4] This led Tarski to the following questions left in [51] as major open problems.

Problem 2.2 (Tarski [51])**.**

(A) Is it true that $\mathbf{D}_1 \in \mathbf{D}_2$?
(B) Let $m \ge 1$. Is it true that $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$?

Tarski observed in [51], with a reference to Gödel's work on constructibility, that the positive solution, *i.e.*, a straight proof of $\mathbf{D}_1 \in \mathbf{D}_2$ or $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$, is "very unlikely". Basically if the axiom of constructibility $\mathbf{V} = \mathbf{L}$ (or even a weaker hypothesis $\omega_1^{\mathbf{L}} = \omega_1$, Lemma 8.2) holds then $\mathbf{D}_1 \notin \mathbf{D}_2$ and $\mathbf{D}_{1m} \notin \mathbf{D}_{2m}$ for all m.[5] Thus the **negative** solution of Problem 2.2 does not contradict the **ZFC** axioms.

As for the **positive** solution, one naturally has to construct a generic extension of $\mathbf{L}$, the constructible universe, in which $\mathbf{D}_1 \in \mathbf{D}_2$ holds, or $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ holds for a given $m \ge 1$. Such stronger sentences as $\mathbf{D}_1 = \mathscr{P}(\omega) \cap \mathbf{L}$ or $\mathbf{D}_{1m} = \mathscr{P}(\omega) \cap \mathbf{L}$ can be considered as well; it's clear that $\mathscr{P}(\omega) \cap \mathbf{L} \in \mathbf{D}_{11}$.

3. Early studies and Harrington's statement

The existence of such models was widely discussed in early years of forcing, especially in the case $m = 1$ corresponding to analytic definability in second-order arithmetic. The early survey [47] by A. R. D. Mathias[6] contains Problem 3112, that requires finding a model of **ZFC** in which it is true that *the set of analytically definable reals is analytically definable*, that is, $\mathbf{D}_{11} \in \mathbf{D}_{21}$; *reals* in this context mean subsets of ω. Another problem P 3110 there suggests a sharper form of this statement, namely: find a model in which it is true that *analytically definable reals are precisely the constructible reals*, that is, $\mathbf{D}_{11} = \mathscr{P}(\omega) \cap \mathbf{L}$. The set $\mathscr{P}(\omega) \cap \mathbf{L}$ of all constructible reals is (lightface) Σ^1_2, and hence $\mathbf{D}_{21}$, so that the equality $\mathbf{D}_{11} = \mathscr{P}(\omega) \cap \mathbf{L}$ implies $\mathbf{D}_{11} \in \mathbf{D}_{21}$.

Somewhat later, Problem 87 in Harvey Friedman's survey *One hundred and two problems in mathematical logic* [14] requires to prove that for each $n \ge 3$ there is a model of **ZFC** + "the constructible reals are precisely the Δ^1_n reals". (For $n \le 2$ this is definitely impossible by the Shoenfield absoluteness theorem.) At the very end of [14], it is noted that Leo Harrington had solved this problem affirmatively. A similar remark is given in [47] (p. 166), a comment to P 3110. Meanwhile an independent construction of a generic model, in which it is true that $\mathbf{D}_{11} = \mathscr{P}(\omega) \cap \mathbf{L}$, was published in [29].

And indeed, Harrington's handwritten notes [24] include a sketch of a generic extension of $\mathbf{L}$ on pp. 2–4 in which it is true that $\mathscr{P}(\omega) \cap \mathbf{L} =$ all Δ^1_3 reals, and a few sentences on p. 5 to explain, without going into details, how Harrington planned to get some other claimed models, namely those of "$\mathscr{P}(\omega) \cap \mathbf{L} =$ all Δ^1_n reals", for

[4] If we slightly change the definition of $\mathbf{D}_{pm}$ by stipulating that only *quantified* variables in φ have type $\le m$ in Definition 2.1, then the case $m = 0$ of *arithmetical definability* becomes meaningful for $p = 2$ (sets of reals). Addison [1] established that $\mathbf{D}_{01} \notin \mathbf{D}_{02}$ holds with such a modification, in other words, the set of all arithmetical reals is not arithmetically definable. See [6, Section 23.2] for the proof of the Addison result, and see further similar results, related rather to the ramified type theoretic hierarchy, in [43].

[5] See [42] on more consequences of the axiom of constructibility related to the Tarski problem.

[6] The original typescript of [47] had been in circulation among set theorists since 1968.

a given n, and those of "$\mathscr{P}(\omega) \cap \mathbf{L} = \mathbf{D}_{11}$". Unfortunately no detailed proofs of these results have ever emerged in Harrington's published mathematical works. An article by Harrington, entitled *Consistency and independence results in descriptive set theory*, which apparently might have contained these results, was announced "to appear in *Ann. of Math.*, 1978" in the References list in Hinman's book [25, p. 462], but in fact this or similar article has never appeared.

Anyway, a relevant part of Harrington's statement was (re)proved in our recent papers [34, 35], with complete proofs of Theorems 3.1 and 3.2.

Theorem 3.1 (Theorem 1 in [35]). *Let $n \geq 3$. There is a generic extension of $\mathbf{L}$ in which it is true that $\mathscr{P}(\omega) \cap \mathbf{L} =$ all Δ^1_n reals.*

Theorem 3.2 (Theorem 2 in [34]). *Let $M \geq 1$. There is a generic extension of $\mathbf{L}$ in which it is true that $\mathbf{D}_{1M} = \mathscr{P}(\omega) \cap \mathbf{L}$, and hence, $\mathbf{D}_{1M} \in \mathbf{D}_{2M}$, and on the other hand $\mathbf{D}_{1n} \notin \mathbf{D}_{2n}$ for all $n \neq M$.*

Thus, for every particular $M \geq 1$, there exists a generic extension of $\mathbf{L}$ in which the Tarski sentence $\mathbf{D}_{1M} \in \mathbf{D}_{2M}$ *holds* (whereas $\mathbf{D}_{1n} \notin \mathbf{D}_{2n}$ for all $n \neq M$).

4. The main results of this paper

This paper extends our research project on the issues of definability in models of set theory that was started in [21, 30, 31] (among other papers). The paper is a continuation of [34], and is devoted to the proof of two new theorems (Theorem 4.1 and Theorem 4.2) on the existence of generic models related to Problem 2.2.

Theorem 4.1. *Assume that $\Upsilon \subseteq \omega \smallsetminus \{0\}$ is a set in $\mathbf{L}$. There is a generic extension of $\mathbf{L}$ in which it is true that for every $m \geq 1$: $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ iff $m \in \Upsilon$.*

Thus not only it is consistent with **ZFC** that $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ holds for all $m \geq 1$ rather than for a single pre-selected $M \geq 1$ as in Theorem 3.2, but any finite, or say recursively-infinite conjunction of sentences of the form $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ and their negations, non-contradictory in an elementary sense, is still consistent with **ZFC**.

Theorem 4.2. *There is a generic extension of $\mathbf{L}$ in which it is true that $\mathbf{D}_1 = \mathbf{L} \cap \mathscr{P}(\omega)$, therefore $\mathbf{D}_1 \in \mathbf{D}_2$.*

These results will be established by applications of the methods sketched by Harrington and further developed in our previous paper [34] on the Tarski problem.

We may note that sentences $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ in Theorem 4.1 cannot be strengthened to $\mathbf{D}_{1m} = \mathbf{L} \cap \mathscr{P}(\omega)$, as in Theorem 4.2 or Theorem 3.2, because we have $\mathbf{D}_{1m} \subsetneq \mathbf{D}_{1,m+1}$ strictly for all m. In fact it is not hard to check that if $\mathbf{D}_{1m} = \mathbf{L} \cap \mathscr{P}(\omega)$ for some m then $\mathbf{D}_{1n} \notin \mathbf{D}_{2n}$ holds for all $n < m$.

5. The structure of the paper

Section II of the paper includes some preliminary and mostly well-known material related to definability, constructibility, and forcing. In particular we consider the Cohen-style collapse forcing $\mathbb{C}_{\omega+1} = \mathbf{Fn}(\omega, \Omega_1) \in \mathbf{L}$ that adjoins a generic collapse map $\omega \xrightarrow{\text{onto}} \Omega_1 = \omega^{\mathbf{L}}_{\omega+1}$. The collapse is necessary since any model for Theorem 4.1 (with an infinite set Υ) has to satisfy the inequality $\omega^{\mathbf{L}}_{\omega+1} \leq \omega_1$ (see Corollary 8.3), and we have to collapse $\omega^{\mathbf{L}}_{\omega+1}$, too, to avoid complications related to the singularity of ω_ω.

We also make use of smaller collapse forcings $\mathbb{C}_n = \mathbf{Fn}(\omega, \omega_n^{\mathbf{L}})$. The finite support product $\mathbb{C}^\Upsilon = \mathbb{C}_{\omega+1} \times \prod_{n\in\Upsilon} \mathbb{C}_n \in \mathbf{L}$ is our first major forcing notion. Each generic set $\boldsymbol{\zeta} \subseteq \mathbb{C}^\Upsilon$ naturally splits into a generic collapse map $\boldsymbol{\zeta}_{\omega+1} : \omega \overset{\text{onto}}{\longrightarrow} \Omega_1$ and an array of generic maps $\boldsymbol{\zeta}_n : \omega \overset{\text{onto}}{\longrightarrow} \omega_n^{\mathbf{L}}$, $n \in \Upsilon$. If $M \in \Upsilon$ then we consider finite generic strings $\boldsymbol{\zeta}^\Upsilon_{<M} = \{\boldsymbol{\zeta}_k\}_{k\in\Upsilon, k<M}$ and $\boldsymbol{\zeta}^\Upsilon_{\le M} = \boldsymbol{\zeta}^\Upsilon_{<M+1}$, and the corresponding submodels

$$\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}] \subseteq \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{\le M}] = \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}, \boldsymbol{\zeta}_m] \subseteq \mathbf{L}[\boldsymbol{\zeta}],$$

and define a set $\boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}] \in \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{\le M}]$, $\boldsymbol{w}^\Upsilon_m[\boldsymbol{\zeta}] \subseteq \omega \times \omega$, whose *slices* $(\boldsymbol{w}^\Upsilon_m[\boldsymbol{\zeta}])_n$, $n < \omega$, provide a simple enumeration of reals in $\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]$ (Definition 10.4).

Section III introduces the main coding tool used in this paper, *the almost disjoint forcing* of Jensen and Solovay [27].

The model for Theorem 4.1 is defined in Section IV. We make use of a product forcing of the form $\mathbb{C}^\Upsilon \times \prod_{M\in\Upsilon} \mathbb{P}^M$ in $\mathbf{L}$, where each $\mathbb{P}^M$ is itself equal to a product $\mathbb{P}^M = \prod_{n,i<\omega} \mathbb{P}^M(n,i) \in \mathbf{L}$, with each factor $\mathbb{P}^M(n,i)$ being an almost-disjoint type forcing determined by a set $\mathbb{U}^M(n,i) \in \mathbf{L}$, $\mathbb{U}^M(n,i) \subseteq \mathbf{Fun}_M = ({\Omega_M}^{\Omega_M}) \cap \mathbf{L}$, where $\Omega_M = \omega^{\mathbf{L}}_{\omega+M}$. These sets $\mathbb{U}^M(n,i)$ are introduced by Theorem 16.1.

Any corresponding generic extension of $\mathbf{L}$ has the form $\mathbf{L}[\boldsymbol{\zeta}, \{G_M\}_{M\in\Upsilon}]$, where $\boldsymbol{\zeta} \subseteq \mathbb{C}^\Upsilon$ is $\mathbb{C}^\Upsilon$-generic and each $G_M \subseteq \mathbb{P}^M$ is $\mathbb{P}^M$-generic over $\mathbf{L}$, so that $G_M = \{G_M(n,i)\}_{n,i<\omega}$, where each factor $G_M(n,i) = \{p(n,i) : p \in G_M\}$ is $\mathbb{P}^M(n,i)$-generic over $\mathbf{L}$. Such an extension is a "basic product model" (Subsection 18).

Further, using the sets $\boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]$ as above, we define *the key submodel*

$$\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^\Upsilon[\boldsymbol{\zeta}]] = \mathbf{L}[\boldsymbol{\zeta}, \{G_M(n,i)\}_{M\in\Upsilon \wedge \langle n,i\rangle \in \boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]}]$$

of the basic product model $\mathbf{L}[\boldsymbol{\zeta}, \{G_M\}_{M\in\Upsilon}]$ in Subsection 19. This will be a model for Theorem 4.1, in particular, if $M \in \Upsilon$ then $\mathbf{D}_{1M} = \mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}] \in \mathbf{D}_{2M}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^\Upsilon[\boldsymbol{\zeta}]]$. This will be established by Theorem 20.2 and Remark 20.4.

Theorem 20.2 contains three claims, and one of them will be dealt with in the two last parts of the paper. In particular, we introduce *forcing approximations* in Section V, an auxiliary forcing-like relation that admits invariance under some transformations, including the permutations of the index set $\omega \times \omega$, see Subsection 32. The actual forcing notions $\mathbb{P}^M = \mathbf{P}[\mathbb{U}^M] = \prod_{n,i<\omega} \mathbb{P}^M(n,i)$ are definitely not invariant under permutations, but the completeness property (ii) of Theorem 16.1, maintained through the inductive construction of systems $\mathbb{U}^M$ in $\mathbf{L}$, allows us to prove that the auxiliary forcing is in the same relation to the truth in the generic extensions considered, as the true $\mathbb{P}^M$-forcing relation (Theorem 28.1).

Finally, Section VI presents the proof of the hidden invariance theorem (Theorem 30.1) for forcing approximations, and completes the proof of Theorem 4.1.

Theorem 4.2 is established in Subsection 23 on the basis of the proof of Theorem 4.1, but the material of Sections V, VI is not involved.

Section II. Preliminaries

For the convenience of the reader, this section contains a brief account of key definitions and results of our earlier paper [34], related to definability, constructibility, and forcing, and instrumental in the proof of Theorems 4.1 and 4.2.

6. Type-theoretic definability and Lévy hierarchy

The type-theoretic language deals with variables x^k, y^k, ... of types $k < \omega$, and includes the Peano arithmetic language for type 0 and the atomic predicate $\in$ of membership used as $x^k \in y^{k+1}$. The *type* of a formula φ is equal to the highest type of all variables in φ, and each quantifier $\exists\, x^k$, $\forall\, x^k$ is assumed to be relativized to the k-th iterated power set

$$\mathscr{P}^k(\omega) = \underbrace{\mathscr{P}(\dots\mathscr{P}(\mathscr{P}(\omega))\dots)}_{k \text{ times the powerset operation}} .$$

If $k, m < \omega$, $k \ge 1$, then $\mathbf{D}_{km}$ is the set of all $x^k \in \mathscr{P}^k(\omega)$, definable in the form

$$x^k = \{x^{k-1} \in \mathscr{P}^{k-1}(\omega) : \varphi(x^{k-1})\}$$

by a parameter-free formula φ of type $\le m$. We let $\mathbf{D}_k = \bigcup_m \mathbf{D}_{km}$, the set of all $x^k \in \mathscr{P}^k(\omega)$ definable by a parameter-free type theoretic formula of any type.

We'll occasionally extend the definition of $\mathbf{D}_{1m}$ to sets $x \subseteq \omega \times \omega$ and the definition of $\mathbf{D}_{2m}$ to sets $X \subseteq \mathscr{P}(\omega) \times \mathscr{P}(\omega)$, with the natural understanding that e.g. a set $X \subseteq \mathscr{P}(\omega) \times \mathscr{P}(\omega)$ (say a wellordering of $\mathscr{P}(\omega)$) belongs to $\mathbf{D}_{2m}$ iff it is definable by a parameter-free formula φ of type $\le m$.

We refer to e.g. [5, Part B, 5.4], or [26, Chap. 13] in matters of *the Lévy hierarchy* of $\in$-formulas and definability classes Σ_n^H, Π_n^H, Δ_n^H for any transitive set H as the domain of definability. In particular, we put

$$\begin{aligned} \Sigma_n^H &= \text{all sets } X \subseteq H, \text{ definable in } H \text{ by a parameter-free } \Sigma_n \text{ formula;} \\ \Sigma_n(H) &= \text{all sets } X \subseteq H \text{ definable in } H \text{ by a } \Sigma_n \text{ formula with parameters in } H. \end{aligned}$$

Writing $\Sigma_n^H(x)$, $x \in H$, we understand that x is admitted as a parameter. If $P \subseteq H$ then $\Sigma_n^H(P)$ is understood so that all $x \in P$ are admitted as parameters. Π-classes Π_n^H, $\Pi_n^H(x)$, $\Pi_n^H(P)$ are defined similarly, and $\Delta_n^H = \Sigma_n^H \cap \Pi_n^H$, *etc.*

We let $\Sigma_\infty^H = \bigcup_{n<\omega} \Sigma_n^H$ = all sets $X \subseteq H$ parameter-free definable in $\langle H; \in\rangle$.

Note that if $1 \le k \le m$ and $X \in \mathscr{P}^k(\omega)$ then $X \in \Sigma_\infty^{\mathscr{P}^m(\omega)}$ iff $X \in \mathbf{D}_{km}$. Theorem 6.1 demonstrates that $\in$-definability even over much bigger sets can be reduced to the type-theoretic definability. If κ is an infinite cardinal then

$$\mathbf{H}\kappa = \{x : \mathtt{card}\,(\mathrm{TC}\,(x)) < \kappa\}, \quad \text{where TC is } \textit{the transitive closure.}$$

In particular, $\mathbf{HC} = \mathbf{H}\omega_1$, all heredidarily-countable sets.

Theorem 6.1 (Theorem 3 in [34])**.** *If $m \ge 1$, the generalized continuum hypothesis $2^\vartheta = \vartheta^+$ holds for all infinite cardinals $\vartheta < \omega_{m-1}$, and $X \subseteq \omega$ or $X \subseteq \mathscr{P}(\omega)$, then X belongs to $\mathbf{D}_{1m}$, resp., $\mathbf{D}_{2m}$, iff $X \in \Sigma_\infty^{\mathbf{H}\omega_m}$.*

In case $m = 1$ (then $\mathbf{H}\omega_m = \mathbf{H}\omega_1 = \mathbf{HC}$ and the GCH premise is vacuous), this result was mentioned, in [28, p. 281], a detailed proof see [26, Lemma 25.25].

7. The pairing function and definability

Let Ω be an infinite cardinal. Then $\kappa = \Omega^+$ is the next cardinal. Theorem 7.3 reduces definability in $\mathbf{H}\kappa$ to definability in a smaller set $\mathscr{P}(\Omega) = \{X : X \subseteq \Omega\}$ with a pairing function as an extra atomic element.

To define the pairing function, let $\preccurlyeq$ be the wellordering of $\mathbf{Ord} \times \mathbf{Ord}$ defined so that $\langle \alpha, \beta\rangle \preccurlyeq \langle \alpha', \beta'\rangle$ iff

$$\langle \mathtt{max}\{\alpha, \beta\}, \alpha, \beta\rangle \ \leqslant_{\mathtt{lex}}\ \langle \mathtt{max}\{\alpha', \beta'\}, \alpha', \beta'\rangle$$

lexicographically. Let $\mathbb{p} : \mathbf{Ord} \times \mathbf{Ord} \xrightarrow{\text{onto}} \mathbf{Ord}$ be the order preserving map: $\langle\alpha, \beta\rangle \preccurlyeq \langle\alpha', \beta'\rangle$ iff $\mathbb{p}(\alpha, \beta) \le \mathbb{p}(\alpha', \beta')$; $\mathbb{p}$ is *the canonical pairing function.*

Lemma 7.1 (routine). *If κ is an infinite cardinal then $\mathbb{p}$ maps $\kappa \times \kappa$ onto κ bijectively, and the restriction $\mathbb{p}{\restriction}(\kappa \times \kappa)$ is constructible and $\Delta_1^{\mathbf{H}\kappa}$.* □

It is perhaps an interesting question whether the restricted map $\mathbb{p}{\restriction}(\Omega \times \Omega)$ is definable in $\langle\Omega; \in\rangle$ or in a bigger (but still much smaller than $\mathbf{H}\Omega^+$) structure $\langle\mathscr{P}(\Omega); \in\rangle$. As explained by James Hanson at Mathoverflow question 384194, this question likely answers in the negative. Yet we fortunately can circumwent this obstacle by simply allowing $\mathbb{p}$ as a background parameter.

Definition 7.2. Let a $\mathbb{p}$-*formula* be any $\in$-formula containing occurrences of $\mathbb{p}(\cdot, \cdot)$.

Let Ω be a cardinal. If φ is a closed $\mathbb{p}$-formula with sets in $\mathscr{P}(\Omega)$ as parameters, then $\mathscr{P}(\Omega) \models \varphi$ is understood in the natural sense, and we say that φ is Ω-*true.* By $\Sigma_\infty^{\mathscr{P}(\Omega),\mathbb{p}}$ we denote the collection of all sets $X \subseteq \mathscr{P}(\Omega)$, definable in $\mathscr{P}(\Omega)$ by a parameter-free $\mathbb{p}$-formula in this sense. □

The following result will be of key importance. It shows that under the assumptions of Lemma 7.1 the definability in $\mathbf{H}\kappa$ is reducible to the definability in a much smaller set $\mathscr{P}(\Omega)$, with only a minor help of the pairing function.

Theorem 7.3 (Theorem 4 in [34]). *If $x, y \subseteq \omega, \Omega$ is a regular cardinal, $\kappa = \Omega^+$, and $x \in \Sigma_\infty^{\mathbf{H}\kappa}(y)$, then $x \in \Sigma_\infty^{\mathscr{P}(\Omega),\mathbb{p}}(y)$.* □

8. Constructibility and definability issues

Recall that $\mathbf{L}$ is the constructible universe and $<_{\mathbf{L}}$ is the Gödel wellordering of $\mathbf{L}$. Let κ be an infinite cardinal. The following are some well-known facts.

(i) The set $\mathbf{H}\kappa \cap \mathbf{L}$ is an initial segment of $\mathbf{L}$ w.r.t. $<_{\mathbf{L}}$;
(ii) The set $\mathbf{H}\kappa \cap \mathbf{L}$ belongs to $\Sigma_1^{\mathbf{H}\kappa}$ and is equal to $(\mathbf{H}\kappa)^{\mathbf{L}}$;
(iii) The restriction $<_{\mathbf{L}}{\restriction}(\mathbf{H}\kappa \cap \mathbf{L})$ is a $\Sigma_1^{\mathbf{H}\kappa}$ relation.
(iv) If $n \ge 1$, $\mathbf{H}\kappa \subseteq \mathbf{L}$, $p \in \mathbf{H}\kappa$ is any parameter, and $R(x, y, z, \dots)$ is a $\Delta_n^{\mathbf{H}\kappa}(p)$ relation on $\mathbf{H}\kappa$, then the following derived relations are $\Delta_n^{\mathbf{H}\kappa}(p)$, too:
$$\exists\, x <_{\mathbf{L}} y\, R(x, y, z, \dots) \quad \text{and} \quad \forall\, x <_{\mathbf{L}} y\, R(x, y, z, \dots).$$

Lemma 8.1 (Theorem 5(i) in [34]). *Assume that $\mathbf{L}[F]$ is a generic extension of $\mathbf{L}$, $\kappa > \omega$ is a cardinal in $\mathbf{L}[F]$, and $Y \in \mathbf{L}$, $Y \subseteq H = (\mathbf{H}\kappa)^{\mathbf{L}}$, $n \ge 1$, Y belongs to Σ_n^H, and $H[F] = (\mathbf{H}\kappa)^{\mathbf{L}[F]}$. Then $Y \in \Sigma_n^{H[F]}$, too.* □

Lemma 8.2 (essentially Tarski [51]). *If $n \ge 1$ and $\mathbf{D}_{1n} \in \mathbf{D}_{2n}$ then $\mathbf{L} \cap \mathscr{P}(\omega) \subseteq \mathbf{D}_{1n}$, $\omega_1^{\mathbf{L}} < \omega_1$ strictly, and there is a real $z \in \mathbf{D}_{1,n+1}$ such that $\omega_1^{\mathbf{L}[z]} > \omega_1^{\mathbf{L}}$.*

Proof. Suppose to the contrary that $X = \mathbf{L} \cap \mathscr{P}(\omega) \not\subseteq \mathbf{D}_{1n}$. and let x be the $<_{\mathbf{L}}$-least element of $X \smallsetminus \mathbf{D}_{1n}$. Note that X is Σ_2^1 by (iii) and Theorem 6.1, hence $X \in \mathbf{D}_{21}$. It follows that the difference $X \smallsetminus \mathbf{D}_{1n}$ belongs to $\mathbf{D}_{2n}$ as $\mathbf{D}_{1n} \in \mathbf{D}_{2n}$. The relation $<_{\mathbf{L}}$ belongs to $\mathbf{D}_{21}$ via (iii) and Theorem 6.1. It easily follows that x itself is $\mathbf{D}_{1n}$, which is a contradiction.

The inequality $\omega_1^{\mathbf{L}} < \omega_1$ is an immediate consequence as $\mathbf{D}_{1n}$ is countable.

And finally by [51] there is a universal formula of type $n{+}1$ for all (parameterfree) formulas of type n. This yields a $\mathbf{D}_{1,n+1}$ set $z \subseteq \omega \times \omega$ such that $\mathbf{D}_{1n}$ is equal to the collection of all slices $(z)_n = \{k : \langle n, k\rangle \in z\}$. Such a z is as required. □

Corollary 8.3. *If* $\mathbf{D}_{1n} \in \mathbf{D}_{2n}$ *holds for infinitely many* n *then* $\omega_1 \geq \omega_{\omega+1}^{\mathbf{L}}$.

Proof. Suppose that *e.g.* $\mathbf{D}_{13} \in \mathbf{D}_{23}$ and $\mathbf{D}_{16} \in \mathbf{D}_{26}$. By the lemma, there is a real $z \subseteq \omega$ in $\mathbf{D}_{14}$ such that $\omega_1^{\mathbf{L}[z]} > \omega_1^{\mathbf{L}}$. Then both the set $\mathbf{L}[z] \cap \mathscr{P}(\omega)$ and the canonical wellordering $<_{\mathbf{L}[z]}$ on $\mathbf{L}[z] \cap \mathscr{P}(\omega)$ belong to $\mathbf{D}_{24}$. Using $<_{\mathbf{L}[z]}$ and $\mathbf{L}[z] \cap \mathscr{P}(\omega)$ instead of $<_{\mathbf{L}}$ and $\mathbf{L} \cap \mathscr{P}(\omega)$, we obtain a real $w \in \mathbf{D}_{17}$ such that $\omega_1^{\mathbf{L}[w]} > \omega_1^{\mathbf{L}[z]}$. Then $\omega_1 \geq \omega_1^{\mathbf{L}[w]} > \omega_1^{\mathbf{L}[z]} > \omega_1^{\mathbf{L}}$, in other words, $\omega_1 \geq \omega_3^{\mathbf{L}}$. □

9. Forcing issues

Let $\varkappa$ is an infinite ordinal. By [26, Chap. 15], a forcing notion $P = \langle P; \leq \rangle$:

- is *$\varkappa$-closed*, if any $\leq$-decreasing sequence $\{p_\alpha\}_{\alpha<\lambda}$ in P, of length $\lambda \leq \varkappa$, has a lower bound in P;
- is *$\varkappa$-distributive*, if the intersection of $\varkappa$-many open dense sets is open dense, and a set $D \subseteq P$ is *open*, iff $q \leq p \in D \Longrightarrow q \in D$, and *dense*, iff for any $p \in P$ there is $q \in D$, $q \leq p$;
- satisfies *$\varkappa$-chain condition*, or *$\varkappa$-CC*, if every antichain $A \subseteq P$ has cardinality strictly less than $\varkappa$;

We'll make use of the following general results in forcing theory.

Lemma 9.1 (Lemma 5 in [34]). *Assume that, in* $\mathbf{L}$, $\vartheta < \Omega = \vartheta^+$ *are regular cardinals, and* $Q, P \in \mathbf{L}$ *are forcing notions,* Q *satisfies* Ω*-CC in* $\mathbf{L}$, *and* P *is* ϑ*-closed in* $\mathbf{L}$. *Let* $\langle F, G \rangle$ *be a pair* $(Q \times P)$*-generic over* $\mathbf{L}$. *Then*

(i) P *remains* ϑ*-distributive in* $\mathbf{L}[F]$,

(ii) Ω *is still a cardinal in* $\mathbf{L}[F, G]$ *and every set* $X \in \mathbf{L}[F, G]$, $X \subseteq \Omega$, *bounded in* Ω, *belongs to* $\mathbf{L}[F]$. □

Definition 9.2 (see [11]). A forcing notion $P = \langle P; \leq \rangle$ is *cone homogeneous*, if for any conditions $p_0, q_0 \in P$ there exist stronger ones $p \leq p_0$, $q \leq q_0$, such that the cones $P_{\leq p} = \{p' \in P : p' \leq p\}$ and $P_{\leq q}$ of stronger conditions are order-isomorphic. □

Lemma 9.3. *Assume that* K *is any set,* $\varkappa > \omega$ *is a cardinal in* $\mathbf{L}[K]$, $H[K] = (\mathbf{H}\varkappa)^{\mathbf{L}[K]}$, $Q \in H[K]$ *is a forcing cone homogeneous in* $\mathbf{L}[K]$, *a set* $F \subseteq Q$ *is* Q*-generic over* $\mathbf{L}[K]$, $\varkappa$ *remains a cardinal in* $\mathbf{L}[K, F]$, $H[K, F] = (\mathbf{H}\varkappa)^{\mathbf{L}[K,F]}$, *and a set* $X \in \mathbf{L}[K, F]$, $X \subseteq H[K]$, *belongs to* $\Sigma_\infty^{H[K,F]}$ (parameterfree $\in$-definable in $H[K, F]$). *Then* $X \in \Sigma_\infty^{H[K]}(Q)$ (with Q as the only parameter), *hence* $X \in \mathbf{L}[K]$.

Proof. Suppose that $X = \{x \in H[K] : \varphi(x)\}$ in $H[K, F]$, φ being a parameterfree formula. We have $X = \{x \in H[K] : \exists\, q \in F\, (q \Vdash \varphi(\dot{x}))\}$, where $\dot{x}$ is a canonical Q-name of $x \in H[K]$. But if some q forces $\varphi(\dot{x})$ then any other $q' \in Q$ also forces $\varphi(\dot{x})$ by the cone homogeneity of Q. Therefore $X = \{x \in H[K] : \exists\, q \in Q\, (q \Vdash \varphi(\dot{x}))\}$. Finally as $Q \in H[K]$, the Q-forcing relation $\Vdash$ is $\in$-definable in $H[K]$ with Q as the only parameter. □

10. Collapse forcing

It follows from Corollary 8.3 that if $\mathbf{D}_{1n} \in \mathbf{D}_{2n}$ holds for all (or even infinitely many) integers $n \geq 1$, then $\omega_{\omega+1}^{\mathbf{L}} \leq \omega_1$. Technically, we'll need even $\omega_{\omega+1}^{\mathbf{L}} < \omega_1$. To maintain such a collapse in a way compatible with its applications below, we make use of a suitable product of the ordinary Cohen-style collapse forcing notions

$\mathbf{Fn}(\omega, X)$ (also denoted by $\mathbf{Coll}(\omega, X)$). Recall that by [45] $\mathbf{Fn}(\omega, X)$ consists of all functions $c : \mathtt{dom}\, c \to X$, where $\mathtt{dom}\, c \subseteq \omega$ is finite, ordered by inclusion (if $c \subseteq c'$ then c' is stronger). We let $\mathbb{C}_n = \mathbf{Fn}(\omega, \omega_n^{\mathbf{L}})$, $\mathbb{C}_{\omega+1} = \mathbf{Fn}(\omega, \Omega_1)$ (recall that $\Omega_1 = \omega_{\omega+1}^{\mathbf{L}}$), and consider some forcing products of these sets.

Definition 10.1. For the remainder, fix a set $\Upsilon \subseteq \omega \smallsetminus \{0\}$ in $\mathbf{L}$ as in Theorem 4.1. □

Definition 10.2. We define

$$\mathbb{C}^{\Upsilon} = \mathbb{C}_{\omega+1} \times \prod_{n \in \Upsilon} \mathbb{C}_n \quad (\textit{finite-support}\text{ product}), \qquad \mathbb{C}^{\Upsilon}_{<m} = \prod_{n \in \Upsilon, n<m} \mathbb{C}_n,$$

and $\mathbb{C}^{\Upsilon}_{\le m} = \mathbb{C}^{\Upsilon}_{<m+1}$. Thus $\mathbb{C}^{\Upsilon} \in \mathbf{L}$, $\mathtt{card}\, \mathbb{C}^{\Upsilon} = \Omega_1 = \omega_{\omega+1}^{\mathbf{L}}$ in $\mathbf{L}$, and Ω_1 is countable in $\mathbb{C}^{\Upsilon}$-generic extensions. Note that $\mathbb{C}^{\Upsilon}_{\le m} = \mathbb{C}^{\Upsilon}_{<m} \times \mathbb{C}_m$ in case $m \in \Upsilon$. □

Each set $\boldsymbol{\zeta} \subseteq \mathbb{C}^{\Upsilon}$, $\mathbb{C}^{\Upsilon}$-generic over $\mathbf{L}$, naturally adjoins a $\mathbb{C}_{\omega+1}$-generic map $\boldsymbol{\zeta}_{\omega+1} : \omega \xrightarrow{\text{onto}} \Omega_1$ and, for each $n \in \Upsilon$, a $\mathbb{C}_n$-generic map $\boldsymbol{\zeta}_n : \omega \xrightarrow{\text{onto}} \omega_n^{\mathbf{L}}$. We put $\boldsymbol{\zeta}^{\Upsilon}_{<m} = \{\boldsymbol{\zeta}_k\}_{k \in \Upsilon, k<m}$ and $\boldsymbol{\zeta}^{\Upsilon}_{\le m} = \boldsymbol{\zeta}^{\Upsilon}_{<m+1}$; if $m \in \Upsilon$ then $\boldsymbol{\zeta}^{\Upsilon}_{\le m} = \langle \boldsymbol{\zeta}^{\Upsilon}_{<m}, \boldsymbol{\zeta}_m \rangle$.

Lemma 10.3 (routine)**.** *If a set $\boldsymbol{\zeta} \subseteq \mathbb{C}^{\Upsilon}$ is $\mathbb{C}^{\Upsilon}$-generic over $\mathbf{L}$, and $m \in \Upsilon$, then*

$$\omega_\gamma^{\mathbf{L}[\boldsymbol{\zeta}]} = \omega_\gamma^{\mathbf{L}[\boldsymbol{\zeta}_{\omega+1}]} = \omega_{\omega+1+\gamma}^{\mathbf{L}} \quad \textit{and} \quad \omega_\gamma^{\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le m}]} = \omega_\gamma^{\mathbf{L}[\boldsymbol{\zeta}_m]} = \omega_{m+\gamma}^{\mathbf{L}}$$

for any ordinal $\gamma \ge 1$. Moreover, $\boldsymbol{\zeta}_m$ and $\boldsymbol{\zeta}^{\Upsilon}_{<m}$ are resp. $\mathbb{C}_m$-generic map and $\mathbb{C}^{\Upsilon}_{<m}$-generic tuple over $\mathbf{L}$. □

Definition 10.4 works within a broader framework.

Definition 10.4. Recall that $(z)_n = \{i : \langle n, i \rangle \in z\}$ for any $z \subseteq \omega \times \omega$. Assume that $M \in \Upsilon$, and a pair $\langle \vec{\mathbf{s}}, \boldsymbol{\theta} \rangle$ is $(\mathbb{C}^{\Upsilon}_{<M} \times \mathbb{C}_M)$-generic over $\mathbf{L}$. Then $\omega_1^{\mathbf{L}[\vec{\mathbf{s}}]} \le \omega_M^{\mathbf{L}}$; let

$$\mathscr{P}(\omega) \cap \mathbf{L}[\vec{\mathbf{s}}] = \{A_\nu[\vec{\mathbf{s}}] : \nu < \omega_M^{\mathbf{L}}\} \tag{1}$$

be a canonical Gödel enumeration (perhaps with repetitions) definable in $\mathbf{L}_{\omega_M^{\mathbf{L}}}[\vec{\mathbf{s}}]$ with $\vec{\mathbf{s}}$ as the only parameter. We put $\boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}] := \{\langle n, i \rangle : n, i < \omega \wedge i \in A_{\boldsymbol{\theta}(n)}[\vec{\mathbf{s}}]\}$, so $(\boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}])_n = A_{\boldsymbol{\theta}(n)}[\vec{\mathbf{s}}]$ and

$$\mathscr{P}(\omega) \cap \mathbf{L}[\vec{\mathbf{s}}] = \{A_{\boldsymbol{\theta}(n)}[\vec{\mathbf{s}}] : n < \omega\} = \{(\boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}])_n : n < \omega\} \tag{2}$$

by (1) and because $\boldsymbol{\theta} : \omega \xrightarrow{\text{onto}} \omega_M^{\mathbf{L}}$ by the genericity.

If $\boldsymbol{\zeta}$ is a set $\mathbb{C}^{\Upsilon}$-generic over $\mathbf{L}$ then these definitions are applicable for $\vec{\mathbf{s}} = \boldsymbol{\zeta}^{\Upsilon}_{<M}$ and $\boldsymbol{\theta} = \boldsymbol{\zeta}_M$. We define then $\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}] = \boldsymbol{w}[\boldsymbol{\zeta}^{\Upsilon}_{<M}, \boldsymbol{\zeta}_M]$. □

Lemma 10.5 (by (2) above)**.** *If $\boldsymbol{\zeta}$ is $\mathbb{C}^{\Upsilon}$-generic over $\mathbf{L}$ and $M \in \Upsilon$ then $\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}] \in \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}]$ and $\mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}] = \{(\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}])_n : n < \omega\}$.* □

Section III. **Almost disjoint forcing**

This is the main coding tool used in the proof of Theorem 4.1, an uncountable version of almost disjoint forcing of Jensen and Solovay [27], which depends on the choice of an uncountable regular cardinal in $\mathbf{L}$.

11. Almost disjoint forcing

Definition 11.1 introduces the cardinals and related sets which we'll systematically deal with in the proof of our main results.

Definition 11.1. If $M < \omega$ then we define an $\mathbf{L}$-cardinal $\Omega_M = \omega^{\mathbf{L}}_{\omega+M}$, and put

$$\begin{aligned} \mathbb{H}_M &= (\mathbf{H}\Omega_{M+1})^{\mathbf{L}} &&= \{x \in \mathbf{L} : \mathtt{card}\,(\mathrm{TC}\,(x)) < \Omega_{M+1} \text{ in } \mathbf{L}\}, \quad \text{and} \\ \mathbb{H}_M[G] &= (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[G]} &&= \{x \in \mathbf{L}[G] : \mathtt{card}\,(\mathrm{TC}\,(x)) < \Omega_{M+1} \text{ in } \mathbf{L}[G]\}, \end{aligned}$$

provided $\mathbf{L}[G]$ is a generic extension of $\mathbf{L}$ and Ω_{M+1} remains a cardinal in $\mathbf{L}[G]$. □

The cardinals Ω_M, $M \ge 1$, will be our *working cardinals.*

Corollary 11.2 (of Lemma 10.3)**.** *If a set $\boldsymbol{\zeta} \subseteq \mathbb{C}^{\Upsilon}$ is $\mathbb{C}^{\Upsilon}$-generic over $\mathbf{L}$, then Ω_1 is countable in $\mathbf{L}[\boldsymbol{\zeta}]$, whereas if $M \ge 2$ then $\Omega_M = \omega^{\mathbf{L}[\boldsymbol{\zeta}]}_{M-1}$ remains a cardinal.* □

The next definition of $\mathbf{Seq}_M$, $\mathbf{Fun}_M$, ${}^*P_M$ corresponds to $\mathbf{Seq}_\Omega$, $\mathbf{Fun}_\Omega$, ${}^*P_\Omega$ in [34, Section 2.4] in case $\Omega = \Omega_M$. In the context of [34], Ω was any regular $\mathbf{L}$-cardinal; here we restrict the generality to the cardinals $\Omega_M = \omega^{\mathbf{L}}_{\omega+M}$, $M \ge 1$.

- $\mathbf{Seq}_M = ({\Omega_M}^{<\Omega_M} \smallsetminus \{\Lambda\}) \cap \mathbf{L}$, the set of all constructible non-$\varnothing$ sequences s of ordinals $< \Omega_M$, of length $\mathtt{lh}\, s = \mathtt{dom}\, s < \Omega_M$, called *strings.* We stress that $\mathbf{Seq}_M \in \mathbf{L}$, and Λ, the empty string, does not belong to $\mathbf{Seq}_M$.
- $\mathbf{Fun}_M = {\Omega_M}^{\Omega_M} \cap \mathbf{L} \in \mathbf{L}$, all constructible Ω_M-sequences of ordinals $< \Omega_M$.
- If $X \subseteq \mathbf{Fun}_M$, put $X^{\vee} = \{f \restriction \nu : f \in F_p \wedge 1 \le \nu < \Omega_M\}$, a tree in $\mathbf{Seq}_M$.
- A set $X \subseteq \mathbf{Fun}_M$ is *dense* iff $X^{\vee} = \mathbf{Seq}_M$, *i.e.* for any $s \in \mathbf{Seq}_M$ there is $f \in X$ such that $s \subset f$.
- If $S \subseteq \mathbf{Seq}_M$, $f \in \mathbf{Fun}_M$ then let $S/f = \mathtt{sup}\{\nu < \Omega_M : f \restriction \nu \in S\}$. If S/f is unbounded in Ω_M then say that S *covers* f, otherwise S *does not cover* f.

Definition 11.3 (in $\mathbf{L}$)**.** Let $M \ge 1$. ${}^*P_M$ is the set of all pairs $p = \langle S_p ; F_p \rangle \in \mathbf{L}$ of sets $F_p \subseteq \mathbf{Fun}_M$, $S_p \subseteq \mathbf{Seq}_M$ of cardinality strictly less than Ω_M in $\mathbf{L}$. Elements of ${}^*P_M$ will be called (forcing) *conditions.*

Let $p, q \in {}^*P_M$. Define $q \leqslant p$ (q is *stronger*) iff $S_p \subseteq S_q$, $F_p \subseteq F_q$, and the difference $S_q \smallsetminus S_p$ does not intersect $F_p^{\vee}$, that is, $S_q \cap F_p^{\vee} = S_p \cap F_p^{\vee}$. Here $F_p^{\vee} = (F_p)^{\vee}$. Clearly $q \leqslant p$ iff $S_p \subseteq S_q$, $F_p \subseteq F_q$, and $S_q \cap F_p^{\vee} = S_p \cap F_p^{\vee}$.

If $u \subseteq \mathbf{Fun}_M$ then put $P[u] = \{p \in {}^*P_M : F_p \subseteq u\} \in \mathbf{L}$. □

By definition, the sets $\mathbf{Seq}_M$, $\mathbf{Fun}_M$, ${}^*P_M$ belong to $\mathbf{L}$ and $\mathtt{card}\,(\mathbf{Seq}_M) = \Omega_M$ whereas $\mathtt{card}\,(\mathbf{Fun}_M) = \mathtt{card}\,{}^*P_M = \Omega_{M+1}$ in $\mathbf{L}$.

If $u \subseteq \mathbf{Fun}_M$ in $\mathbf{L}$, and $G \subseteq P[u]$ then put $S_G = \bigcup_{p \in G} S_p$; thus $S_G \subseteq \mathbf{Seq}_M$.

Lemma 11.4 (Lemma 13 in [34])**.** *Suppose that $M \ge 1$ and, in $\mathbf{L}$, $u \subseteq \mathbf{Fun}_M$ is dense. Let $G \subseteq P[u]$ be a set $P[u]$-generic over $\mathbf{L}$. Then*

(i) *if $f \in \mathbf{Fun}_M$ in $\mathbf{L}$ then $f \in u \iff S_G$ does not cover f;*
(ii) *$G = \{p \in P[u] : S_p \subseteq S_G \wedge (S_G \smallsetminus S_p) \cap F_p^{\vee} = \varnothing\}$, hence $\mathbf{L}[G] = \mathbf{L}[S_G]$.* □

12. Homogeneity

Let $M \ge 1$. Assume that conditions $p, q \in {}^*P_M$ satisfy

$$F_p = F_q \quad \text{and} \quad S_p \cup S_q \subseteq F_p^{\vee} = F_q^{\vee}. \tag{3}$$

Following [35, § 2.4], we define a transformation h_{pq} acting as follows.

If $p = q$ then define $h_{pq}(r) = r$ for all $r \in {}^*P_M$ — the identity.

Suppose that $p \neq q$. Then $S_p \neq S_q$, and p, q are incompatible by (3) and Definition 11.3. We put $d_{pq} = \{r \in {}^*P_M : r \leqslant p \vee r \leqslant q\}$, the *domain* of h_{pq}. Let $r \in d_{pq}$. We define $h_{pq}(r) = r' := \langle S_{r'}, F_{r'} \rangle$, where $F_{r'} = F_r$ and

$$S_{r'} = \begin{cases} (S_r \smallsetminus S_p) \cup S_q & \text{in case} \quad r \leqslant p, \\ (S_r \smallsetminus S_q) \cup S_p & \text{in case} \quad r \leqslant q. \end{cases} \tag{4}$$

Thus, assuming (3), the difference between S_r and $S_{r'}$ lies entirely within the set $X = F_p^\vee = F_q^\vee$, so that if $r \leqslant p$ then $S_r \cap X = S_p$ but $S_{r'} \cap X = S_q$, while if $r \leqslant q$ then $S_r \cap X = S_q$ but $S_{r'} \cap X = S_p$.

Lemma 12.1 (in $\mathbf{L}$). *Assume that $M \geq 1$. Then:*

(i) *if $u \subseteq \mathbf{Fun}_M$ is dense and $p_0, q_0 \in P[u]$ then there exist conditions $p, q \in P[u]$ with $p \leqslant p_0$, $q \leqslant q_0$, satisfying* (3)*;*
(ii) *if $p, q \in {}^*P_M$ satisfy* (3) *then h_{pq} an order automorphism of the set $d_{pq} = \{r \in {}^*P_M : r \leqslant p \vee r \leqslant q\}$, satisfying $h_{pq}(p) = q$ and $h_{pq} = (h_{pq})^{-1} = h_{qp}$;*
(iii) *if $u \subseteq \mathbf{Fun}_M$ and $p, q \in P[u]$ satisfy* (3) *then h_{pq} maps the set $P[u] \cap d_{pq}$ onto itself order-preserving.*

Proof. (i) We argue in $\mathbf{L}$. By the density of u there is a set $F \subseteq u$ satisfying $\mathtt{card}\, F < \Omega_M$, $F_p \cup F_q \subseteq F$, and $S_p \cup S_q \subseteq F^\vee = \{f \restriction \nu : f \in F \wedge 1 \leq \nu < \omega_1\}$. Put $p = \langle S_p, F \rangle$ and $q = \langle S_q, F \rangle$. Claims (ii), (iii) are routine. □

Corollary 12.2 (in $\mathbf{L}$). *Assume that $M \geq 1$ and a set $u \subseteq \mathbf{Fun}_M$ is dense. Then $P[u]$ is cone homogneous in the sense of Definition* 9.2. □

13. Product almost disjoint forcing

Arguing under the assumptions and notation of Definition 11.1, we consider $\mathcal{I} = \omega \times \omega$, the cartesian product, as the *index set* for a product forcing.

Definition 13.1 (in $\mathbf{L}$). Let $M \geq 1$. We define ${}^*\mathbf{P}_M = ({}^*P_M)^{\mathcal{I}}$ to be the $\mathbf{L}$-product of $\mathcal{I}$ copies of ${}^*P_M$ ordered componentwise: $p \leqslant q$ (p is stronger) iff $p(n,i) \leqslant q(n,i)$ in ${}^*P_M$ for all $n, i < \omega$. Thus ${}^*\mathbf{P}_M \in \mathbf{L}$ consists of all maps $p \in \mathbf{L}$, $p : \mathcal{I} \to {}^*P_M$.

Let $p \in {}^*\mathbf{P}_M$. If $n, i < \omega$ then put $\mathbf{S}_p(n,i) = S_{p(n,i)}$, $\mathbf{F}_p(n,i) = F_{p(n,i)}$, and

$$\mathbf{F}_p^\vee(n,i) = F_{p(n,i)}^\vee = \{f \restriction \nu : f \in \mathbf{F}_p(n,i) \wedge 1 \leq \nu < \Omega_M\}.$$

Thus $p(n,i) = \langle \mathbf{S}_p(n,i); \mathbf{F}_p(n,i) \rangle \in {}^*P_M$, and $\mathbf{F}_p(n,i) \subseteq \mathbf{Fun}_M$, $\mathbf{S}_p(n,i) \subseteq \mathbf{Seq}_M$ are sets of cardinality $< \Omega_M$ in $\mathbf{L}$. Let $|p| = \{\langle n, i \rangle : p(n,i) \neq \langle \varnothing; \varnothing \rangle\}$. □

Let $M \geq 1$. An M**-system** is any map $U \in \mathbf{L}$, $U : \mathcal{I} \to \mathscr{P}(\mathbf{Fun}_M)$ such that each set $U(n,i)$ is empty or dense in $\mathbf{Fun}_M$. In this case, let

$$\mathbf{P}[U] = \{p \in {}^*\mathbf{P}_M : \forall\, n, i\, (\mathbf{F}_p(n,i) \subseteq U(n,i))\},$$

the $\mathbf{L}$-product of the sets $P[U(n,i)]$, $n, i < \omega$.

We take the next easy lemma from [34, Lemma 14].

Lemma 13.2 (in $\mathbf{L}$). *Let $M \geq 1$ and U be an M-system. Then it holds in $\mathbf{L}$ that* $\mathtt{card}\, \mathbf{P}[U] = \Omega_{M+1}$ *and*

(i) *the forcing notion $\mathbf{P}[U]$ is Ω_{M-1}-closed, hence Ω_{M-1}-distributive;*
(ii) *the forcing notion $\mathbf{P}[U]$ satisfies Ω_{M+1}-CC, both in $\mathbf{L}$ and in any generic extension of $\mathbf{L}$ in which Ω_{M+1} remains a cardinal.* □

Definition 13.3. Suppose that $M \geq 1$ and $z \subseteq \mathcal{I}$. If $p \in {}^*\mathbf{P}_M$ then define $p' = p{\restriction}z$ to be the usual restriction, so that $\mathtt{dom}\,(p{\restriction}z) = z$ and $p'(n,i) = p(n,i)$ for all $\langle n,i\rangle \in z$. Several special cases of restriction: if $m, i < \omega$ then let

$$\begin{aligned} p{\restriction}_{\neq\langle m,i\rangle} &= p{\restriction}z, \quad \text{where} \quad z = (\mathcal{I} \smallsetminus \{\langle m,i\rangle\}),\\ p{\restriction}^{<m} &= p{\restriction}z, \quad \text{where} \quad z = \{k : k < m\} \times \omega,\\ p{\restriction}^{\geq m} &= p{\restriction}z, \quad \text{where} \quad z = \{k : k \geq m\} \times \omega,\\ p{\restriction}^{m} &= p{\restriction}z, \quad \text{where} \quad z = \{m\} \times \omega. \end{aligned}$$

If U is an M-system then define $U{\restriction}z$, $U{\restriction}^{<m}$, $U{\restriction}^{\geq m}$, $U{\restriction}^{m}$ similarly to the above. Finally, if $Q \subseteq {}^*\mathbf{P}_M$ then let $Q{\restriction}z = \{p{\restriction}z : p \in Q\}$; $Q{\restriction}z \subseteq {}^*\mathbf{P}_M{\restriction}z$. This will be applied, *e.g.*, in case $Q = \mathbf{P}[U]$, where $U \in \mathbf{L}$ is a system, and then we get $\mathbf{P}[U]{\restriction}z = \{p{\restriction}z : p \in \mathbf{P}[U]\}$, $\mathbf{P}[U]{\restriction}_{\neq\langle n,i\rangle}$, $\mathbf{P}[U]{\restriction}^{\geq m}$, *etc.* □

Remark 13.4. Suppose that $z \in \mathbf{L}$ in Definition 13.3. If $p \in {}^*\mathbf{P}_M$, then $p{\restriction}z$ can be identified with a condition $q \in {}^*\mathbf{P}_M$ such that $q{\restriction}z = p{\restriction}z$ and $q(n,i) = \langle\varnothing;\varnothing\rangle$ for all $\langle n,i\rangle \in \mathcal{I} \smallsetminus z$. For instance, this applies w.r.t. $p{\restriction}_{\neq\langle n,i\rangle}$, $p{\restriction}^{\geq m}$, $p{\restriction}^{<m}$, $p{\restriction}^{m}$.

With such an identification, we have ${}^*\mathbf{P}_M{\restriction}z \subseteq {}^*\mathbf{P}_M$, and $Q{\restriction}z \subseteq {}^*\mathbf{P}_M$ for $Q \subseteq {}^*\mathbf{P}_M$ (in case $z \in \mathbf{L}$). But if $z \notin \mathbf{L}$ then such an identification fails.

The same applies for the restrictions $U{\restriction}z$ of systems U. □

14. Product almost disjoint extensions

Assume that $M \geq 1$ and U is an M-system in $\mathbf{L}$. Consider $\mathbf{P}[U] \in \mathbf{L}$ as a forcing notion. We'll study $\mathbf{P}[U]$-generic extensions $\mathbf{L}[G]$ of the ground universe $\mathbf{L}$. Define some elements of these extensions. Every set $G \subseteq {}^*\mathbf{P}_M$, generic over $\mathbf{L}$, can be viewed as an array $G = \{G(n,i)\}_{n,i<\omega}$, where $G(n,i) = \{p(n,i) : p \in G\} \subseteq {}^*P_M$ is ${}^*P_M$-generic over $\mathbf{L}$. We let

$$\mathbf{S}_G(n,i) = S_{G(n,i)} = \bigcup_{p \in G} \mathbf{S}_p(n,i) \;\text{ for any } n, i < \omega.$$

Thus $\mathbf{S}_G(n,i) \subseteq \mathbf{Seq}_M$. By the way, this defines a sequence $\vec{S}_G = \{\mathbf{S}_G(n,i)\}_{n,i<\omega}$ of subsets of $\mathbf{Seq}_M$.

If $z \subseteq \mathcal{I}$ then let $G{\restriction}z = \{p{\restriction}z : p \in G\}$.

If $z \in \mathbf{L}$ then $G{\restriction}z$ can be identified with $\{p \in G : |p| \subseteq z\}$.

Put $G{\restriction}_{\neq\langle n,i\rangle} = \{p \in G : \langle n,i\rangle \notin |p|\} = G{\restriction}(\mathcal{I} \smallsetminus \{\langle n,i\rangle\})$.

Lemma 14.1. *Let $M \geq 1$, U be an M-system in $\mathbf{L}$, and $G \subseteq \mathbf{P}[U]$ be a set $\mathbf{P}[U]$-generic over $\mathbf{L}$. Then $\mathbf{L}[G] = \mathbf{L}[\vec{S}_G]$, and if $n, i < \omega$ then the set $G(n,i) = \{p(n,i) : p \in G\} \in \mathbf{L}[G]$ is $P[U(n,i)]$-generic over $\mathbf{L}$, hence if $f \in \mathbf{Fun}_M$ then*

$$f \in U(n,i) \iff \mathbf{S}_G(n,i) \textit{ does not cover } f.$$

Proof. To prove $\mathbf{L}[G] = \mathbf{L}[\vec{S}_G]$ apply Lemma 11.4(ii). The genericity of $G(n,i)$ holds by the product forcing theorem, then use Lemma 11.4(i). □

Section IV. The forcing notion and the model

To get a model for Theorem 4.1, we define, in $\mathbf{L}$, a certain M-system $\mathbb{U}^M$ for each $M \geq 1$, such that its different layers $\{\mathbb{U}^M(n,i)\}_{i<\omega}$, $n < \omega$, satisfy different definability and completeness requirements that depend on M and n. Each system $\mathbb{U}^M$ is equal to the limit $\mathbb{U}^M = \bigvee_{\gamma<\Omega_{M+1}} \mathbb{U}^M_\gamma$ of an increasing sequence of *small* systems

$\mathbb{U}^M_\gamma$ given by Theorem 16.1 in Subsection 16. This key theorem was established in [34] on the basis of a diamond-style construction.

We define, in Subsection 17, forcing notions $\mathbb{P}^M = \mathbf{P}[\mathbb{U}^M]$, and consider the product forcing $\mathbb{C}^\Upsilon \times \prod_{M\in\Upsilon} \mathbb{P}^M$ (with the collapse forcing $\mathbb{C}^\Upsilon$ of Subsection 10) in $\mathbf{L}$. Any corresponding generic extension of $\mathbf{L}$ has the form $\mathbf{L}[\boldsymbol{\zeta}, \{G_M\}_{M\in\Upsilon}]$, where $\boldsymbol{\zeta} \subseteq \mathbb{C}^\Upsilon$ is $\mathbb{C}^\Upsilon$-generic and each $G_M \subseteq \mathbb{P}^M$ is $\mathbb{P}^M$-generic over $\mathbf{L}$, so that $G_M = \{G_M(n,i)\}_{n,i<\omega}$, where each factor $G_M(n,i) = \{p(n,i) : p \in G_M\}$ is $\mathbb{P}^M(n,i)$-generic over $\mathbf{L}$. Such an extension is a "basic product model", Subsection 18.

Further, using the sets $\boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}] \subseteq \omega\times\omega$, as in Definition 10.4, we define *the key submodel*

$$\mathbf{L}[\boldsymbol{\zeta}, \{G_M(n,i)\}_{M\in\Upsilon\wedge\langle n,i\rangle\in\boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]}] \tag{5}$$

in Subsection 19, which will be a model for Theorem 4.1. Theorem 20.2 communicates the main definability properties of the submodel (5), relevant to the proof of Theorem 4.1. *Those amount to the fact that, for any $M \in \Upsilon$, it is true in the submodel* (5) *that the set $\mathbf{D}_{1M}$ is equal to $\mathscr{P}(\omega)\cap\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]$ and belongs to $\mathbf{D}_{2m}$* (see Remark 20.4). *This basic fact is split into four separate claims of Theorem* 20.2. Of those, three claims are established in this section, while the fourth one takes more effort in the remainder. Lemmas 20.1, 22.1 provide important reductions of the definability claims involved to simpler models.

We'll take special care of the case $M \notin \Upsilon$ in Subsection 22. The proof of Theorem 4.2 (our second main result) follows in Subsection 23.

15. Systems, definability aspects

We **argue in L** and make use of notation of Definition 11.1, in particular

$$\Omega_M = \omega^{\mathbf{L}}_{\omega+M},$$
$$\mathbb{H}_M = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}} = \{x \in \mathbf{L} : \mathtt{card}\,(\mathrm{TC}\,(x)) < \Omega_{M+1} \text{ in } \mathbf{L}\}.$$

- An M-system U is *disjoint* iff $U(n,i)\cap U(k,j) = \varnothing$ whenever $\langle n,i\rangle \neq \langle k,j\rangle$. Let $\mathbf{DS}_M$ (disjoint systems) be the set of all disjoint M-systems.
- An M-system U is *small*, if each $U(n,i)$ has cardinality $\le \Omega_M$ in $\mathbf{L}$. Let $\mathbf{sDS}_M$ (*small disjoint* systems) be the set of all small M-systems $U \in \mathbf{DS}_M$.
- If U, V are M-systems and $U(n,i) \subseteq V(n,i)$ for all n,i, then V *extends* U, in symbol $U \preccurlyeq V$.
- If $\{U_\gamma\}_{\gamma<\lambda}$ is a sequence of M-systems then the limit M-system $U = \bigvee_{\gamma<\lambda} U_\gamma$ is defined by $U(n,i) = \bigcup_{\gamma<\lambda} U_\gamma(n,i)$, for all n,i.

Define $\mathbf{sDS}_M{\restriction}^{\ge m} = \{U{\restriction}^{\ge m} : U \in \mathbf{sDS}_M\}$, and similarly $\mathbf{sDS}_M{\restriction}^{<m}$ *etc.*

The sets $\mathbf{DS}_M$, $\mathbf{sDS}_M$, $\mathbf{sDS}_M{\restriction}^{\ge m}$, $\mathbf{DS}_M{\restriction}^{<m}$ *etc.*, and the order relation $\preccurlyeq$, **belong to L**, of course.

Lemma 15.1 (in **L**). *Let $M \ge 1$. The following sets belong to $\varDelta^{\mathbb{H}_M}_1(\{\Omega_M\})$ and to $\varDelta^{\mathbb{H}_M}_3$:*

$$\{\Omega_M\},\ \{\mathbf{Seq}_M\},\ \mathbf{Fun}_M,\ {}^*\mathbf{P}_M,\ \mathbf{sDS}_M,\ \mathbf{sDS}_M{\restriction}^{\ge m},\ \mathbf{sDS}_M{\restriction}^{<m},$$

as well as the set $\{\langle U,p\rangle : U \in \mathbf{sDS}_M \wedge p \in \mathbf{P}[U]\}$ and the relation $\preccurlyeq$ on $\mathbf{sDS}_M$.

Proof. All these sets have straightforward $\varDelta^{\mathbb{H}_M}_1(\{\Omega_M\})$ definitions, with $\Omega_M \in \mathbb{H}_M$ as the only parameter. To eliminate Ω_M, it suffices to prove that $\{\Omega_M\} \in \varDelta^{\mathbb{H}_M}_3$.

Note first of all that "ϑ is a cardinal (initial ordinal)" is a Π_1 formula

$$\vartheta \in \mathbf{Ord} \wedge \forall \alpha < \vartheta \, \forall f \, (f : \alpha \to \vartheta \implies \mathtt{ran}\, f \neq \vartheta).$$

On the other hand, Ω_M is the largest cardinal in $\mathbb{H}_M$, hence it holds in $\mathbb{H}_M$ that

$$\vartheta = \Omega_M \iff \forall \varkappa \, (\varkappa \text{ is a cardinal} \implies \varkappa \le \Omega_M).$$

We conclude that $\{\Omega_M\} \in \Pi_2^{\mathbb{H}_M} \subseteq \Delta_3^{\mathbb{H}_M}$. Finally the conversion $\Delta_1^{\mathbb{H}_M}(\{\Omega_M\}) \to \Delta_3^{\mathbb{H}_M}$ is routine. □

16. Complete sequences

The following major theorem was established in [34] (Theorem 6 in case $\Omega = \Omega_M$, $\Omega^{\oplus} = \Omega_{M+1}$, $\mathbf{sDS}_{\Omega} = \mathbf{sDS}_M$), and we take it here without a proof.

Theorem 16.1 (in $\mathbf{L}$)**.** *Let $M \ge 1$. There is a continuous* (that is, $\mathbb{U}_\lambda = \bigvee_{\gamma<\lambda} \mathbb{U}_\gamma$ for all limit ordinals $\lambda < \Omega_{M+1}$) *$\preccurlyeq$-increasing sequence $\{\mathbb{U}_\gamma^M\}_{\gamma<\Omega_{M+1}}$ of systems $\mathbb{U}_\gamma^M \in \mathbf{sDS}_M$, such that*

(i) *if $n < \omega$ then the "slice" $\{\mathbb{U}_\gamma^M \restriction^n\}_{\gamma<\Omega_{M+1}}$ is $\Delta_{n+4}^{\mathbb{H}_M}$;*

(ii) *if $m < \omega$ then the "tail" $\{\mathbb{U}_\gamma^M \restriction^{\ge m}\}_{\gamma<\Omega_{M+1}}$ is $(m+3)$-**complete**, in the sense that for any $\Sigma_{m+3}(\mathbb{H}_M)$ set $D \subseteq \mathbf{sDS}_M \restriction^{\ge m}$ there is $\gamma < \Omega_{M+1}$ such that the system $\mathbb{U}_\gamma^M \restriction^{\ge m}$ m-**solves** D, that is,*
   - *either $\mathbb{U}_\gamma^M \restriction^{\ge m} \in D$;*
   - *or there is no system $U \in D$ with $\mathbb{U}_\gamma^M \restriction^{\ge m} \preccurlyeq U$;*

(iii) *there is a recursive sequence of parameterfree $\in$-formulas $\chi_n(\gamma, x)$ such that if $\gamma < \Omega_{M+1}$ and $x \in \mathbb{H}_M$ then $\mathbb{H}_M \models \chi_n(\gamma, x)$ iff $x = \mathbb{U}_\gamma^M \restriction^n$.* □

Recall that $\Sigma_{m+3}(\mathbb{H}_M)$ in (ii) means Σ_{m+3} definability in $\mathbb{H}_M$ with arbitrary parameters in $\mathbb{H}_M$.

One of the difficulties in the proof of this theorem in [34] is that we have to account for different levels of completeness for different slices of the construction. This was handled via a diamond style lemma.[7] By the way, according to Lemma 16.2, the key completeness property of the sequence $\{\mathbb{U}_\gamma^M\}_{\gamma<\Omega_{M+1}}$, as in (ii) of Theorem 16.1, still holds, to some extent, in rather mild generic extensions of $\mathbf{L}$.

Lemma 16.2 (Lemma 20 in [34] for $\Omega = \Omega_M$)**.** *Let $M \ge 1$, and $Q \in \mathbf{L}$ be a forcing notion with $\mathtt{card}\, Q \le \Omega_M$ in $\mathbf{L}$. Let a set $F \subseteq Q$ be Q-generic over $\mathbf{L}$.*

*Assume that $m < \omega$, $\delta < \Omega_{M+1}$, and a set $D \in \mathbf{L}[F]$, $D \subseteq \mathbf{sDS}_M \restriction^{\ge m}$, belongs to $\Sigma_{m+3}(\mathbb{H}_M[F])$ and is **open** in $\mathbf{sDS}_M \restriction^{\ge m}$, so that any extension of a system $U \in D$ in $\mathbf{sDS}_M \restriction^{\ge m}$ belongs to D itself.*

*Then there is an ordinal γ, $\delta \le \gamma < \Omega_{M+1}$, such that $\mathbb{U}_\gamma^M \restriction^{\ge m}$ m-**solves** D.* □

We recall that $\mathbb{H}_M = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}}$ and $\mathbb{H}_M[F] = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[F]}$ by Definition 11.1, and Ω_{M+1} remains a cardinal in $\mathbf{L}[F]$ by the choice of Q in Lemma 16.2.

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[7] A similar construction was also applied in [37] to obtain a generic extension of $\mathbf{L}$ in which every non-empty analytically definable set of the reals contains an analytically definable real, but there are no analytically definable wellorderings of the reals.

17. Basic forcing notions

We make the following arrangements on the basis of Theorem 16.1.

Definition 17.1 (in $\mathbf{L}$)**.** If $M \geq 1$ then we fix a $\preccurlyeq$-increasing continuous sequence $\{\mathbb{U}^M_\gamma\}_{\gamma<\Omega_{M+1}}$ of systems in $\mathbf{sDS}_M$, satisfying conditions (i), (ii), (iii) of Theorem 16.1 for this M. We define:

- the limit system $\mathbb{U}^M = \bigvee_{\gamma<\Omega_{M+1}} \mathbb{U}^M_\gamma$,
- the forcing notion $\mathbb{P}^M = \mathbf{P}[\mathbb{U}^M]$ — thus $\mathbb{P}^M = \prod_{n,i<\omega} \mathbb{P}^M(n,i) \in \mathbf{L}$ is the full $\mathbf{L}$-product of sets $\mathbb{P}^M(n,i) = P[\mathbb{U}^M(n,i)]$, n, $i<\omega$,
- subforcings $\mathbb{P}^M_\gamma = \mathbf{P}[\mathbb{U}^M_\gamma]$, $\gamma < \Omega_{M+1}$,
- restrictions $\mathbb{P}^M\restriction z$ ($z \subseteq \mathcal{I}$), $\mathbb{P}^M\restriction_{\neq\langle n,i\rangle}$ *etc.* as in Subsection 13,
- the full $\mathbf{L}$-product $\mathbb{P}^\Upsilon = \prod_{M\in\Upsilon} \mathbb{P}^M$ (countable support),
- the subproducts $\mathbb{P}^\Upsilon_{\geq M} = \prod_{N\in\Upsilon, N\geq M} \mathbb{P}^N$, $\mathbb{P}^\Upsilon_{<M} = \prod_{N\in\Upsilon, N<M} \mathbb{P}^N$, and then $\mathbb{P}^\Upsilon_{\leq M}$, $\mathbb{P}^\Upsilon_{>M}$ similarly.

Then $\mathbb{P}^\Upsilon$ can be further factorized as $\mathbb{P}^\Upsilon = \prod_{M\in\Upsilon;\, n,i<\omega} \mathbb{P}^M(n,i)$. □

Lemma 13.2 immediately implies some cardinal characteristics of these sets.

Corollary 17.2. *Let $M \geq 1$. It holds in $\mathbf{L}$ that $\mathtt{card}\,(\mathbb{P}^M) = \Omega_M$ and*

(i) *the forcing notions $\mathbb{P}^M$ and $\mathbb{P}^\Upsilon_{\geq M}$ are Ω_{M-1}-closed and Ω_{M-1}-distributive, therefore $\mathbb{P}^M$ and $\mathbb{P}^\Upsilon_{\geq M}$ do not adjoin new subsets of Ω_{M-1} to $\mathbf{L}$;*

(ii) *the forcing notions $\mathbb{P}^M$, $\mathbb{P}^\Upsilon_{\leq M}$, and $\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{\leq M}$ satisfy Ω_{M+1}-CC, both in $\mathbf{L}$ and in any generic extension of $\mathbf{L}$ where Ω_{M+1} remains a cardinal.*

Proof. The cardinality claim and (i) follow from the corresponding claims of Lemma 13.2. To prove (ii) note that $\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{\leq M}$ can be identified with $\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{<M} \times \mathbb{P}^M$, where $\mathbb{P}^M$ is Ω_{M+1}-CC by Lemma 13.2 while $\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{<M}$ has cardinality Ω_M in $\mathbf{L}$. □

As for the definability properties, systems $\mathbb{U}^M$ of Theorem 16.1 are not necessarily parameterfree definable in the sets $\mathbb{H}_M = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}}$, yet their slices are:

Corollary 17.3 (in $\mathbf{L}$, by Definition 17.1 and (i), (iii) of Theorem 16.1)**.** *Let $M \geq 1$.*

(i) *If $n<\omega$ then the slice set $\mathbb{U}^M\restriction^n = \{\langle i,f\rangle : i<\omega \wedge f \in \mathbb{U}^M(n,i)\}$ belongs to $\Sigma^{\mathbb{H}_M}_\infty$, moreover there is a recursive sequence of parameterfree $\in$-formulas $\vartheta^M_n(\cdot,\cdot)$, $n<\omega$, such that $\mathbb{U}^M\restriction^n = \{\langle i,f\rangle : \mathbb{H}_M \models \vartheta^M_n(i,f)\}$.*

(ii) *It follows that each slice $\mathbb{P}^M\restriction^n$ belongs to $\Sigma^{\mathbb{H}_M}_\infty$, too, moreover there is a recursive sequence of parameterfree $\in$-formulas $\Theta^M_n(\cdot,\cdot)$, $n<\omega$, such that $\mathbb{P}^M(n,i) = \{p \in \mathbb{H}_M : \mathbb{H}_M \models \Theta^M_n(i,p)\}$ for all $n,i<\omega$.*

(iii) *Therefore if $M<M'$ then sets $\mathbb{P}^M, \mathbb{U}^M$ belong to $\mathbb{H}_{M'}$ and to $\Sigma^{\mathbb{H}_{M'}}_\infty$.* □

18. Basic product model

Theorem 18.2 studies the structure of $(\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon)$-generic extensions of $\mathbf{L}$.

Definition 18.1. Suppose that a pair $\langle \boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon)$-generic over $\mathbf{L}$. The extension $\mathbf{L}[\boldsymbol{\zeta}, G]$ is the *basic product model* of this subsection title. We have, by the product forcing theorem:

- $\boldsymbol{\zeta} = \langle \boldsymbol{\zeta}_{\omega+1}, \{\boldsymbol{\zeta}_M\}_{M\in\Upsilon}\rangle$ is $\mathbb{C}^\Upsilon$-generic over $\mathbf{L}$,
- $G = \{G_M\}_{M\in\Upsilon}$ is $\mathbb{P}^\Upsilon$-generic over $\mathbf{L}[\boldsymbol{\zeta}]$, and

- if $M \in \Upsilon$ then $G_M \subseteq \mathbb{P}^M$ is $\mathbb{P}^M$-generic over $\mathbf{L}[\boldsymbol{\zeta}, \{G_N\}_{N\in\Upsilon, N\neq M}]$.

Following Subsection 14, if $M \in \Upsilon$ then G_M can be viewed as an array $\{G_M(n,i)\}_{n,i<\omega}$, where each factor $G_M(n,i) = \{p(n,i) : p \in G_M\}$ is a set $\mathbb{P}^M(n,i)$-generic over $\mathbf{L}$.

We define sets $\mathbf{S}_{G_M}(n,i) = \bigcup_{p\in G_M} \mathbf{S}_p(n,i) \subseteq \mathbf{Seq}_M$ and a sequence $\vec{S}_{G_M} = \{\mathbf{S}_{G_M}(n,i)\}_{n,i<\omega}$. Note that $\mathbf{L}[G_M] = \mathbf{L}[S_{G_M}]$ and $\mathbf{L}[G_M(n,i)] = \mathbf{L}[\mathbf{S}_{G_M}(n,i)]$ by Lemma 14.1; thus $\mathbf{L}[G] = \mathbf{L}[\{G_M(n,i)\}_{M\in\Upsilon;\, n,i<\omega}] = \mathbf{L}[\{\mathbf{S}_{G_M}(n,i)\}_{M\in\Upsilon;\, n,i<\omega}]$.

We further define $G^{\Upsilon}_{<M} = \{G_N\}_{N\in\Upsilon, N<M}$, $G^{\Upsilon}_{\geq M} = \{G_N\}_{N\in\Upsilon, N\geq M}$, and $G^{\Upsilon}_{>M}$, $G^{\Upsilon}_{\leq M}$ the same way. □

Theorem 18.2. *Under the assumptions of Definition* 18.1*, let* $M \geq 1$*. Then*

(i) Ω_{M+1} *is a cardinal in* $\mathbf{L}[\boldsymbol{\zeta}, G]$ *but* Ω_1 *is a countable ordinal even in* $\mathbf{L}[\boldsymbol{\zeta}]$*;*

(ii) *any set* $X \in \mathbf{L}[\boldsymbol{\zeta}, G]$*,* $X \subseteq \Omega_{M+1}$*, bounded in* Ω_{M+1}*, belongs to* $\mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]$*;*

(iii) $\Omega_{M+1} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta},G]} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta}]}$ — recall that $\Omega_{M+1} = \omega^{\mathbf{L}}_{\omega+M+1}$;

(iv) *both the equality* $2^{\Omega_M} = \Omega_{M+1}$ *and* GCH *as a whole hold in* $\mathbf{L}[\boldsymbol{\zeta}, G]$*.*

If moreover $B \in \mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]$*,* $B \subseteq \mathbf{Ord}$*, and* $\boldsymbol{\zeta} \in \mathbf{L}[B]$*, then*

(v) *every set* $X \in \mathbf{L}[B, G^{\Upsilon}_{>M}]$*,* $X \subseteq \Omega_{M+1}$*, bounded in* Ω_{M+1}*, belongs to* $\mathbf{L}[B]$*, and hence we have* $(\mathbf{H}\Omega_{M+1})^{\mathbf{L}[B, G^{\Upsilon}_{>M}]} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[B]}$*;*

(vi) $\Omega_{M+1} = \omega_M^{\mathbf{L}[B, G^{\Upsilon}_{>M}]} = \omega_M^{\mathbf{L}[B]}$*;*

(vii) *both the equality* $2^{\Omega_M} = \Omega_{M+1}$ *and* GCH *as a whole hold in* $\mathbf{L}[B, G^{\Upsilon}_{>M}]$*.*

Proof. To prove (i) and (ii) apply Lemma 9.1 for $\vartheta = \Omega_M$, $\Omega = \Omega_{M+1}$, $Q = \mathbb{C}^{\Upsilon} \times \mathbb{P}^{\Upsilon}_{\leq M}$, $P = \mathbb{P}^{\Upsilon}_{>M}$, so that $\mathbf{L}[\boldsymbol{\zeta}, G] = \mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}][G^{\Upsilon}_{>M}]$ is a $(Q \times P)$-generic extension of $\mathbf{L}$. To make sure that Lemma 9.1 is applicable, note that P is Ω_M-closed in $\mathbf{L}$ and Q satisfies Ω_{M+1}-CC in $\mathbf{L}$ by Corollary 17.2.

As M is arbitrary in this argument, it follows that all Ω_{M+1} are cardinals in $\mathbf{L}[\boldsymbol{\zeta}, G]$.

Now Claim (iii) is a simple corollary.

To establish (iv), it suffices, by (ii), to prove $2^{\Omega_M} = \Omega_{M+1}$ for the model $\mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]$. But $\mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]$ is obtained essentially by adjoining countably many subsets of the ordinal $\Omega_{M+1} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]}$ to $\mathbf{L}$, and this implies the result required.

(v) Note that the forcing notion $P = \mathbb{P}^{\Upsilon}_{>M}$ is Ω_M-distributive in $\mathbf{L}[B]$, because it is such in $\mathbf{L}[\boldsymbol{\zeta}, G^{\Upsilon}_{\leq M}]$, a bigger model, by Lemma 9.1 and Corollary 17.2(i).

To prove (vi), (vii) apply (iv) via $\mathbf{L}[\boldsymbol{\zeta}] \subseteq \mathbf{L}[B] \subseteq \mathbf{L}[B, G^{\Upsilon}_{>M}] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G]$. □

19. The key submodel

The following definitions introduce our model for Theorem 4.1; it will be a submodel of the basic product model that include its *collapse core* $\mathbf{L}[\boldsymbol{\zeta}]$. We'll freely use the notation introduced by Definition 10.4 with respect to the latter.

Recall that a set $\Upsilon \subseteq \omega \smallsetminus \{0\}$, $\Upsilon \in \mathbf{L}$, is fixed by Theorem 4.1.

Definition 19.1. Assume that a pair $\langle \boldsymbol{\zeta}, G\rangle$ is $\mathbb{C}^{\Upsilon} \times \mathbb{P}^{\Upsilon}$-generic over $\mathbf{L}$. Then $\boldsymbol{\zeta}$ is $\mathbb{C}^{\Upsilon}$-generic, hence Definition 10.4 applies and sets $\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}] \subseteq \omega \times \omega$, $\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}] \in \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\leq m}]$ are defined for $m \geq 1$, as well as the sequence $\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}] = \{\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}]\}_{m\in\Upsilon}$. Moreover, $G = \{G_M\}_{M\in\Upsilon}$, where each set $G_m = \{G_m(n,i)\}_{n,i<\omega} \subseteq \mathbb{P}^m$ is $\mathbb{P}^m$-generic, and hence the restriction

$$G_m{\restriction}\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}] = \{p{\restriction}\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}] : p \in G_m\}$$

is well-defined. With some abuse of notation, we define indexed sets

$$G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}] = \{G_m{\restriction}\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}]\}_{m\in\Upsilon},$$
$$G{\restriction}\boldsymbol{w}^{\Upsilon}_{<M}[\boldsymbol{\zeta}] = \{G_m{\restriction}\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}]\}_{m\in\Upsilon\wedge m<M},$$
$$G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}] = \{G_m{\restriction}\boldsymbol{w}^{\Upsilon}_m[\boldsymbol{\zeta}]\}_{m\in\Upsilon\wedge m\le M},$$

and consider the models

$$\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]].$$

□

We may note that $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$ is a submodel of $\mathbf{L}[\boldsymbol{\zeta}, G]$, generated by $\boldsymbol{\zeta}$ and those factors $G_M(n,i)$ of the product generic set $G = \{G_M(n,i)\}_{M\in\Upsilon\wedge n,i<\omega}$, which satisfy $\langle n,i\rangle \in \boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]$.

We claim that the model $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G]$ satisfies Theorem 4.1.

The proof of the claim will go in the remainder essentially by means of three successive **reductions** of $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$ to simpler models $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ (Subsection 20), $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]$ (Subsection 22), and $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}, G]$ (Subsection 24).

20. First reduction

Lemma 20.1 provides an essential reduction of $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$ to the submodel $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$, based on rather standard forcing results by Theorem 18.2.

Lemma 20.1. *Under the assumptions of Definition* 19.1*, let $M \ge 1$. Then*

(i) *if a set $X \in \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$, $X \subseteq \Omega_{M+1}$, is bounded in Ω_{M+1} then X belongs to $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$, therefore*

$$(\mathbf{H}\Omega_{M+1})^{\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]};$$

(ii) $\Omega_{M+1} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]}$;

(iii) *both $2^{\Omega_M} = \Omega_{M+1}$ and even* GCH *as a whole hold in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$.*

Proof. Apply Theorem 18.2(v),(vi),(vii) with $B = \langle \boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]\rangle$. It is clear that $\mathbf{L}[B] = \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[B, G^{\Upsilon}_{>M}]$. □

Lemma 20.1 allows us to focus on Theorem 20.2.

Theorem 20.2. *Assume that a pair $\langle \boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^{\Upsilon} \times \mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$, and $M \ge 1$. Then it holds in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$, hence in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$, that*

(A) *if $M \in \Upsilon$ then $\mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}] \subseteq \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}$;*

(B) *if $M \in \Upsilon$ then $\mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}] \in \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}$;*

(C) *if $M \in \Upsilon$ then $\mathscr{P}(\omega) \cap \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty} \subseteq \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}F]$;*

(D) *but if $M \notin \Upsilon$ then $\mathscr{P}(\omega) \cap \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty} \notin \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}$.*

Remark 20.3. The "hence" claim of the theorem follows from Lemma 20.1(i) which asserts that the set $\mathbf{H}\Omega_{M+1}$ is the same in both models. □

Remark 20.4. Theorem 20.2 implies Theorem 4.1. Indeed let $M \in \Upsilon$. We have to prove $\mathbf{D}_{1M} \in \mathbf{D}_{2M}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}[\boldsymbol{\zeta}]]$. By Theorem 6.1, this is equivalent to

$$\Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty} \cap \mathscr{P}(\omega) \in \Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}, \tag{6}$$

since $\Omega_{M+1} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta}, G\restriction \boldsymbol{w}^\Upsilon[\boldsymbol{\zeta}]]}$ and GCH holds by Lemma 20.1. But (6) is an immediate corollary of Theorem 20.2.

On the other hand, if $M \notin \Upsilon$ then $\mathbf{D}_{1M} \notin \mathbf{D}_{2M}$ in $\mathbf{L}[\boldsymbol{\zeta}, G\restriction \boldsymbol{w}^\Upsilon[\boldsymbol{\zeta}]]$ follows from Theorem 20.2(D) via Theorem 6.1. □

The proof of Theorem 20.2 follows in the remainder. Claims (A), (B) will be established in Subsection 21, Claim (D) in Subsection 22, Claim (C) will be accomplished in Subsection 33, based on the substantial work in Sections V and VI.

21. Definability in subextensions

Here we prove Claims (A), (B) of Theorem 20.2.

Arguing under the assumptions and notation of Definition 17.1, we let formulas $\Gamma_n^M(\cdot,\cdot)$ ($n < \omega$, $M \ge 1$) be defined as follows:

$$\Gamma_n^M(i, S) := i < \omega \wedge S \subseteq \mathbf{Seq}_M \wedge \forall f \in \mathbf{Fun}_M \big(f \in \mathbb{U}^M(n, i) \Longleftrightarrow S \text{ doesn't cover } f\big).$$

Theorem 21.1 shows that any real in $\mathbf{L}$ and even in some generic extensions of $\mathbf{L}$ can be made parameterfree-definable in appropriate subextensions of $\mathbb{P}^M$-generic extensions, basically by means of the formulas Γ_n^M. This is Theorem 7 in [34] with

$$\Omega = \Omega_M, \ \ \Omega^\oplus = \Omega_{M+1}, \ \ \mathbb{P}^\Omega = \mathbb{P}^M, \ \ Y = B,$$

of which we only present here Claim (v), actually used below, and Claim (iv), which shows how definability properties are obtained in suitable subextensions.

Theorem 21.1 (part of Theorem 7 in [34]). *Assume that $M \ge 1$, $Q \in \mathbf{L}$ is a forcing notion,* $\mathtt{card}\, Q \le \Omega_M$ *in $\mathbf{L}$, a pair $\langle W, G_M\rangle$ is $(Q \times \mathbb{P}^M)$-generic over $\mathbf{L}$, $B \in \mathbf{L}[W]$, and $z \in \mathbf{L}[B]$, $z \subseteq \mathcal{I} = \omega \times \omega$. Then*

(iv) *it is true in $\mathbf{L}[B, G_M\restriction z]$ that $z = \{\langle n, i\rangle : \exists\, S \subseteq \mathbf{Seq}_M\, \Gamma_n^M(i, S)\}$;*

(v) *if $n < \omega$ then the n-th slice $(z)_n = \{i : \langle n, i\rangle \in z\}$ belongs to $\Sigma_\infty^{\mathbb{T}}$, where $\mathbb{T} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[B, G_M\restriction z]}$.*

Proof (outline). Regarding the proof of [34, Theorem 7], we may note that the implication $\langle n, i\rangle \in z \Longrightarrow \exists\, S \subseteq \mathbf{Seq}_M\, \Gamma_n^M(i, S)$ in (iv) is a consequence of Lemma 14.1, for take $S = \mathbf{S}_{G_M}(n, i)$. The inverse implication was established in [34] on the basis of the choice of the sequence $\{\mathbb{U}_\gamma^M\}_{\gamma < \Omega_{M+1}}$ of systems in $\mathbf{sDS}_M$ by Definition 17.1, so that in particular condition (ii) of Theorem 16.1 is satisfied.

As for Claim (v), in follows from (iv) that the equality

$$(z)_n = \{i < \omega : \exists\, S \subseteq \mathbf{Seq}_M\, \Gamma_n^M(i, S)\}$$

holds both in $\mathbf{L}[B, G_M\restriction z]$ and in $\mathbb{T} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[B, G_M\restriction z]}$. Now, using the formulas ϑ_n^M as in Corollary 17.3(i), another recursive sequence of $\in$-formulas τ_n^M can be defined with the help of Lemma 8.1, such that $(z)_n = \{i < \omega : \tau_n^M(i)\}$ holds in $\mathbb{T}$. This immediately implies (v), and **moreover**, implies that if $M' > M$ then z as a whole belongs to $\Sigma_\infty^{\mathbb{T}'}$, where $\mathbb{T}' = (\mathbf{H}\Omega_{M'+1})^{\mathbf{L}[B, G_M\restriction z]}$, because the truth in $\mathbb{T}$ is uniformly expressible in a much bigger set $\mathbb{T}'$. □

Now we demonstrate how Theorem 21.1 implies two first claims of Theorem 20.2.

Proof of Claims (A), (B) *of Theorem* 20.2. Let $M \in \Upsilon$. Arguing under the assumptions of Theorem 20.2, we show first of all how Theorem 21.1 will be applied. We let $Q = \mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{<M}$, so that $Q \times \mathbb{P}^M = \mathbb{C}^\Upsilon \times \mathbb{P}^\Upsilon_{\le M}$ and $\mathtt{card}\, Q \le \Omega_M$ by Corollary 17.2. We accordingly let $B = \langle \boldsymbol{\zeta}, G\restriction \boldsymbol{w}^\Upsilon_{<M}[\boldsymbol{\zeta}]\rangle$ and $W = \langle \boldsymbol{\zeta}, G^\Upsilon_{<M}\rangle$, so

that $B \in \mathbf{L}[W]$ and the pair $\langle W, G_M \rangle$ is $(Q \times \mathbb{P}^M)$-generic over $\mathbf{L}$ by the product forcing theorem. We finally let $z = \boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]$, so that $z \in \mathbf{L}[\boldsymbol{\zeta}] \subseteq \mathbf{L}[B]$. Thus general assumptions of Theorem 21.1 are verified, and by the way

$$\mathbf{L}[B, G_M{\restriction}z] = \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]] = \mathbf{L}[\boldsymbol{\zeta}, \{G_N{\restriction}\boldsymbol{w}^{\Upsilon}_N[\boldsymbol{\zeta}]\}_{N\in\Upsilon, N\le M}].$$

Now to prove Claim (A) of Theorem 20.2, assume that $x \in \mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}]$. Then $x = (\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}])_n = (z)_n$ for some n by Lemma 10.5. Therefore $x = (z)_n \in \Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[B, G_M{\restriction}z] = \mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ by Theorem 21.1(v).

To prove Claim (B) of Theorem 20.2, we note first of all that each set $\boldsymbol{w}^{\Upsilon}_n[\boldsymbol{\zeta}]$ with $n \in \Upsilon$, $n < M$, belongs to $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}]$, and hence to $\Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ by Claim (A). It easily follows by construction that the string $\boldsymbol{\zeta}^{\Upsilon}_{<M}$ itself belongs to $\Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$. This immediately implies Claim (B). □

22. Second reduction

This subsection contains another vital step in the proof of Claims (C) and (D) of Theorem 20.2. This step provides a reduction of $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ to the smaller model $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]$ and to the smaller structure $\langle \mathscr{P}(\Omega_M); \mathbb{p}, \in\rangle$ (see Subsection 7) in that model.

Lemma 22.1. *Assume that $M \ge 1$, a pair $\langle \boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^{\Upsilon} \times \mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$,*

$$\mathbb{T} = \big(\mathbf{H}\Omega_{M+1}\big)^{\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]}, \quad \mathbb{S} = \big(\mathbf{H}\Omega_{M+1}\big)^{\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]},$$

and $a \in \mathbb{T}$, $a \subseteq \omega$, $a \in \Sigma^{\mathbb{T}}_\infty$. Then $a \in \mathbb{S}$ and $a \in \Sigma^{\mathbb{S}}_\infty(\boldsymbol{\zeta}^{\Upsilon}_{<M})$, that is, a belongs to $\mathbb{S}$ and a is $\in$-definable in $\mathbb{S}$ with $\boldsymbol{\zeta}^{\Upsilon}_{<M}$ as the only parameter.

In addition, if $M \in \Upsilon$, then $a \in \Sigma_\infty^{\mathscr{P}(\Omega_M), \mathbb{p}}$ in $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]$.

Recall that $\mathbb{p}$ in the upper index means that the defining formula can contain occurrences of $\mathbb{p}(\cdot,\cdot)$, where $\mathbb{p}$ is interpreted as $\mathbb{p}{\restriction}(\Omega_M \times \Omega_M)$ (a bijection $\Omega_M \times \Omega_M \xrightarrow{\text{onto}} \Omega_M$, see Subsection 7).

Proof. By the product forcing theory the model $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ is a Q-generic extension of $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\le M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]$, where Q is the finite support product

$$Q = \underbrace{\mathbb{C}_{\omega+1} \times \prod_{n\in\Upsilon, n>M} \mathbb{C}_n}_{Q'} \times \underbrace{\prod_{n\in\Upsilon, n<M} (\mathbb{P}^n{\restriction}\boldsymbol{w}^{\Upsilon}_n[\boldsymbol{\zeta}])}_{P} \in \mathbb{S}. \tag{7}$$

The first factor Q' here is order-isomorphic to $\mathbb{C}_{\omega+1}$, so that $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\le M}[\boldsymbol{\zeta}]]$ is a $(\mathbb{C}_{\omega+1} \times P)$-generic extension of $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}]$, and accordingly $\mathbb{T}$ is a $(\mathbb{C}_{\omega+1} \times P)$-generic extension of $\mathbb{S}$. Note that $\mathbb{C}_{\omega+1}$ is a cone homogeneous forcing notion in the sense of Definition 9.2 (in fact even fully homogeneous), and the second factor P is cone homogeneous as well by Corollary 12.2.

Moreover each $\mathbb{P}^n$, $n < M$, belongs to $\mathbb{H} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}}$ and is parameterfree $\in$-definable in $\mathbb{H}$ by Corollary 17.3(iii), and hence $\mathbb{P}^n \in \mathbb{S}$ and $\mathbb{P}^n \in \Sigma^{\mathbb{S}}_\infty$ by Lemma 8.1. Further, if $n \in \Upsilon$ and $n < M$ then the set $\boldsymbol{w}^{\Upsilon}_n[\boldsymbol{\zeta}] \subseteq \omega \times \omega$ belongs to $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}]$ by Lemma 10.5, hence to $\mathbb{S}$, and $\boldsymbol{w}^{\Upsilon}_n[\boldsymbol{\zeta}]$ is $\in$-definable in $\mathbb{S}$ with $\boldsymbol{\zeta}^{\Upsilon}_{<M} \in \mathbb{S}$ as the only parameter. To conclude, P as a whole belongs to $\Sigma^{\mathbb{S}}_\infty(\boldsymbol{\zeta}^{\Upsilon}_{<M})$, so obviously does $\mathbb{C}_{\omega+1}$, and both $\mathbb{C}_{\omega+1}, P$ are cone homogeneous by the above.

It immediately follows by Lemma 9.3 that $a \in \Sigma^{\mathbb{S}}_\infty(\boldsymbol{\zeta}^\Upsilon_{\le N})$, as required.

To prove the additional claim of the lemma, suppose that $M \in \Upsilon$. Then we have $\mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}] \subseteq \Sigma^{\mathbb{S}}_\infty$ — this is essentially a clone of Claim (A) of Theorem 20.2 with the same proof as in Subsection 21 based on Theorem 21.1(v) with

$$Q = \mathbb{C}^\Upsilon_{\le M}, \quad B = W = \boldsymbol{\zeta}^\Upsilon_{<M}, \quad z = \boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}],$$

and we leave this to the reader. It follows that in fact $P \in \Sigma^{\mathbb{S}}_\infty$.

To conclude, the sets $\mathbb{C}_{\omega+1}$ and P belong to $\mathbb{S}$, are parameterfree $\in$-definable in $\mathbb{S}$, and are cone homogeneous by the above. Now Lemma 9.3 implies $a \in \Sigma^{\mathbb{S}}_\infty$. To get the final result $a \in \Sigma^{\mathscr{P}(\Omega_M),\mathbb{P}}_\infty$ in $\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{\le M}, G_M \restriction \boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]]$, apply Theorem 7.3. $\square$

Proof of Claim (D) *of Theorem* 20.2. Suppose that $M \ge 1$, $M \notin \Upsilon$. Let

$$\mathbb{T} = \big(\mathbf{H}\Omega_{M+1}\big)^{\mathbf{L}[\boldsymbol{\zeta}, G\restriction \boldsymbol{w}^\Upsilon_{\le M}[\boldsymbol{\zeta}]]}, \quad \mathbb{S} = \big(\mathbf{H}\Omega_{M+1}\big)^{\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{\le M}, G_M\restriction \boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]]},$$

as above, but in this case $\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{\le M}, G_M \restriction \boldsymbol{w}^\Upsilon_M[\boldsymbol{\zeta}]] = \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]$ and $\mathbb{S} = \big(\mathbf{H}\Omega_{M+1}\big)^{\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]}$, of course. We have to prove that $\mathscr{P}(\omega) \cap \Sigma^{\mathbb{T}}_\infty \notin \Sigma^{\mathbb{T}}_\infty$.

Suppose to the contrary that $\mathscr{P}(\omega) \cap \Sigma^{\mathbb{T}}_\infty \in \Sigma^{\mathbb{T}}_\infty$. We have $\mathscr{P}(\omega) \cap \Sigma^{\mathbb{T}}_\infty \subseteq \Sigma^{\mathbb{S}}_\infty(\boldsymbol{\zeta}^\Upsilon_{<M})$ by Lemma 22.1, therefore the set $X = \big(\mathscr{P}(\omega) \cap \mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]\big) \smallsetminus \Sigma^{\mathbb{T}}_\infty$ is non-empty. Let $\leqslant^*$ be the canonical Gödel wellordering of $\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]$, definable in $\mathbf{L}[\boldsymbol{\zeta}^\Upsilon_{<M}]$ with $\boldsymbol{\zeta}^\Upsilon_{<M}$ as the only parameter. Let x be the $\leqslant^*$-least element in X. Note that the string $\boldsymbol{\zeta}^\Upsilon_{<M}$ belongs to $\mathbb{T}$ and to $\Sigma^{\mathbb{T}}_\infty$ by Theorem 20.2(A). Therefore x belongs to $\Sigma^{\mathbb{T}}_\infty$ as well by the contrary assumption. $\square$

23. Second main theorem

This subsection is devoted to Theorem 4.2, our second main result. The proof follows more or less the same plan with some changes *mutatis mutandis*. In particular, the whole major part of of the proof of Theorem 4.1 presented in Sections V and VI will not be involved in the proof of Theorem 4.2. The first change begins with the definition of some key sets in Subsection 10.

Definition 23.1 (modification of Definition 10.4)**.** We put $\Upsilon = \omega \smallsetminus \{0\}$, accordingly $\mathbb{C}^\Upsilon = \mathbb{C}_{\omega+1} \times \prod_{n\ge 1} \mathbb{C}_n$ (*finite-support* product). $\square$

Each generic set $\boldsymbol{\zeta} \subseteq \mathbb{C}^\Upsilon$ naturally adjoins a $\mathbb{C}_{\omega+1}$-generic map $\boldsymbol{\zeta}_{\omega+1} : \omega \xrightarrow{\text{onto}} \Omega_1$ and, for each $n \ge 1$, a $\mathbb{C}_n$-generic map $\boldsymbol{\zeta}_n : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_n$.

Definition 23.2 (modification of Definition 10.4)**.** Let

$$\mathscr{P}(\omega) \cap \mathbf{L} = \{A_\nu : \nu < \omega^{\mathbf{L}}_1\} \tag{8}$$

be a canonical Gödel enumeration parameterfree definable in $\mathbf{L}_{\omega^{\mathbf{L}}_1}$. Assume that a set $\boldsymbol{\zeta} \subseteq \mathbb{C}^\Upsilon$ is $\mathbb{C}^\Upsilon$-generic over $\mathbf{L}$, hence $\boldsymbol{\zeta}_1 : \omega \xrightarrow{\text{onto}} \omega^{\mathbf{L}}_1$ is defined. If $M \ge 1$ then put

$$\mathbb{w}_M[\boldsymbol{\zeta}] := \{\langle n, i\rangle : n, i < \omega \wedge i \in A_{\boldsymbol{\zeta}_1(M)}\},$$

so $(\mathbb{w}_M[\boldsymbol{\zeta}])_n = A_{\boldsymbol{\zeta}_1(M)}$ for each n, and, by the genericity and (8),

$$\mathscr{P}(\omega) \cap \mathbf{L} = \{A_{\boldsymbol{\zeta}_1(M)} : M \ge 1\} = \{(\mathbb{w}_M[\boldsymbol{\zeta}])_0 : M \ge 1\}.$$

$\square$

Remark 23.3. In the context of Definition 23.2, the sets $\mathbb{w}_M[F]$ are constructible, unlike, generally speaking, the sets $\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]$ in Definition 23.2. This implies essential simplifications in the proof of Theorem 4.2. □

Following Definition 17.1, put $\mathbb{P}^{\Upsilon} = \prod_{M\ge 1}\mathbb{P}^M$ (countable support), and define $\mathbb{P}^{\Upsilon}_{\ge M} = \prod_{n\ge M}\mathbb{P}^M$, $\mathbb{P}^{\Upsilon}_{<M} = \prod_{1\le n<M}\mathbb{P}^M$, and $\mathbb{P}^{\Upsilon}_{\le M}$, $\mathbb{P}^{\Upsilon}_{>M}$ similarly.

Following Definition 18.1, suppose that a pair $\langle\boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$. With some abuse of notation, we let

$$G\restriction\mathbb{w}[\boldsymbol{\zeta}] = \{G_m\restriction\mathbb{w}_m[\boldsymbol{\zeta}]\}_{m\ge 1} \quad\text{and}\quad G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}] = \{G_m\restriction\mathbb{w}_m[F]\}_{1\le m\le M},$$

and consider the models

$$\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}[\boldsymbol{\zeta}]] \subseteq \mathbf{L}[\boldsymbol{\zeta}, G].$$

The model $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}[\boldsymbol{\zeta}]]$ is our **model for Theorem 4.2**.

Lemma 23.4 provides a useful reduction similar to Lemma 20.1.

Lemma 23.4. *Let a pair $\langle\boldsymbol{\zeta}, G\rangle$ be $(\mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$, and $M\ge 1$. Then*

(i) *if a set $X\in\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}[\boldsymbol{\zeta}]]$, $X\subseteq\Omega_{M+1}$, is bounded in Ω_{M+1} then X belongs to $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$, thus*

$$(\mathbf{H}\Omega_{M+1})^{\mathbf{L}[\boldsymbol{\zeta},G\restriction\mathbb{w}[\boldsymbol{\zeta}]]} = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}[\boldsymbol{\zeta},G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]};$$

(ii) $\Omega_{M+1} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta},G\restriction\mathbb{w}[\boldsymbol{\zeta}]]} = \omega_M^{\mathbf{L}[\boldsymbol{\zeta},G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]}$;

(iii) *both* $\exp(\Omega_M) = \Omega_{M+1}$ *and* GCH *hold in* $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}[\boldsymbol{\zeta}]]$. □

Now we prove the following result pretty analogous to Theorem 20.2.

Theorem 23.5. *Assume that a pair $\langle\boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$, and $M\ge 1$. Then it is true in $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$, and hence in $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}[\boldsymbol{\zeta}]]$, that*

(A) *the set $A_{\boldsymbol{\zeta}_1(M)}$ (see Definition 23.2) belongs to $\Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}$;*

(B) $\mathscr{P}(\omega)\cap\Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}\subseteq\mathbf{L}$.

Proof. (A) We follow the proof of Claim (A) of Theorem 20.2 in Subsection 21. To apply Theorem 21.1, we let $Q = \mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon}_{<M}$, so that $\mathtt{card}\,Q\le\Omega_M$ in $\mathbf{L}$ by Corollary 17.2, and $Q\times\mathbb{P}^M = \mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon}_{\le M}$. We accordingly let $B = \langle\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]\rangle$ and $W = \langle\boldsymbol{\zeta}, G^{\Upsilon}_{<M}\rangle$, so that $B\in\mathbf{L}[W]$ and the pair $\langle W, G_M\rangle$ is $(Q\times\mathbb{P}^M)$-generic over $\mathbf{L}$ by the product forcing theorem. We finally let $z = \mathbb{w}_M[\boldsymbol{\zeta}]$, so that $z\in\mathbf{L}$. Thus general assumptions of Theorem 21.1 are verified, and we have $\mathbf{L}[B, G_M\restriction z] = \mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$.

Now to prove Claim (A) of Theorem 23.5, let $x = A_{\boldsymbol{\zeta}_1(M)}$. Then $x = (z)_n$, $\forall n$, by construction. It follows that $x\in\Sigma^{\mathbf{H}\Omega_{M+1}}_{\infty}$ in $\mathbf{L}[B, G_M\restriction z] = \mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$ by Theorem 21.1(v), as required.

(B) By the product forcing theory the model $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$ is a Q-generic extension of $\mathbf{L}$, where $Q = \mathbb{C}^{\Upsilon}\times P$, and $P = \prod_{1\le n\le M}(\mathbb{P}^n\restriction\mathbb{w}_n[\boldsymbol{\zeta}])$. Here $\mathbb{C}^{\Upsilon}\in\mathbf{L}$ is a cone homogeneous forcing notion in the sense of Definition 9.2. The second factor P is cone homogeneous as well by Corollary 12.2. Moreover each $\mathbb{P}^n$ belongs to $\mathbf{L}$, and each set $\mathbb{w}_n[F]\subseteq\omega\times\omega$ belongs to $\mathbf{L}$, too (see Remark 23.3). To conclude, P as a whole belongs to $\mathbf{L}$. So obviously does $\mathbb{C}^{\Upsilon}$, and both $\mathbb{C}^{\Upsilon}, P$ are cone homogeneous by the above. It follows then (see Lemma 9.3) that if a set $a\subseteq\omega$ is parameterfree $\in$-definable in $\mathbf{L}[\boldsymbol{\zeta}, G\restriction\mathbb{w}_{\le M}[\boldsymbol{\zeta}]]$ then $a\in\mathbf{L}$. This ends the proof of Theorem 23.5. □

Proof of Theorem 4.2. Assume that a pair $\langle \boldsymbol{\zeta}, G\rangle$ is $(\mathbb{C}^{\Upsilon}\times\mathbb{P}^{\Upsilon})$-generic over $\mathbf{L}$. Prove the equality $\mathbf{D}_1=\mathbf{L}\cap\mathscr{P}(\omega)$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$.

Let $x\in\mathbf{L}$, $x\subseteq\omega$. Then $x=A_{\boldsymbol{\zeta}_1(M)}$ for some $M\geq 1$ by the genericity. It follows that $x\in\Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}_{\leq M}[\boldsymbol{\zeta}]]$ — and hence in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$, by Theorem 23.5(A). But $\Omega_{M+1}=\omega_M$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$, and GCH holds, by Lemma 23.4. Therefore we have $x\in\mathbf{D}_{1M}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$ by Theorem 6.1.

Let conversely $x\subseteq\omega$, $x\in\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$, and $x\in\mathbf{D}_1$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$. Then $x\in\mathbf{D}_{1M}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$ for some $M\geq 1$, therefore $x\in\Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\mathbb{w}[\boldsymbol{\zeta}]]$ by Theorem 6.1, as above. Then $x\in\mathbf{L}$ by Theorem 23.5(B). This completes the proof of Theorem 4.2. □

Section V. Forcing approximation

Beginning here is a lengthy **proof of Claim (C) of Theorem 20.2**. Our plan will be to establish the following somewhat unexpected result of Theorem 24.1. It asserts that under suitable definability conditions reals in a $(\mathbb{C}^{\Upsilon}_{<M}\times\mathbb{C}_M\times\mathbb{P}^M)$-generic extension of $\mathbf{L}$ belong to an associated $(\mathbb{C}^{\Upsilon}_{<M}\times\mathbb{P}^M)$-generic extension, eliminating the M-th forcing component $\mathbb{C}_M$. Thus we call it *the elimination theorem.*

24. Third reduction

Theorem 24.1 (elimination theorem). *Assume that*

(∗) $M\geq 1$, *a triple* $\langle\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}\rangle$ *is* $(\mathbb{C}^{\Upsilon}_{<M}\times\mathbb{C}_M\times\mathbb{P}^M)$*-generic over* $\mathbf{L}$, *and* $\boldsymbol{w}=\boldsymbol{w}[\vec{\mathbf{s}},\boldsymbol{\theta}]$,[8]

and $a\in\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}{\restriction}\boldsymbol{w}]$, $a\subseteq\omega$, $a\in\Sigma_\infty^{\mathscr{P}(\Omega_M),\mathbb{P}}$ *in* $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}{\restriction}\boldsymbol{w}]$. *Then* $a\in\mathbf{L}[\vec{\mathbf{s}},\mathbf{G}]$.

Remark 24.2. Theorem 24.1 implies Claim (C) of Theorem 20.2 — and hence completes the proof of Theorem 20.2 as a whole because the other claims (A), (B), (D) of Theorem 20.2 have been already established, see Subsections 21 and 22.

To prove the implication, assume, under the conditions of Theorem 20.2, that $a\subseteq\omega$, $a\in\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\leq M}[\boldsymbol{\zeta}]]$, and $a\in\Sigma_\infty^{\mathbf{H}\Omega_{M+1}}$ in $\mathbf{L}[\boldsymbol{\zeta}, G{\restriction}\boldsymbol{w}^{\Upsilon}_{\leq M}[\boldsymbol{\zeta}]]$. Then, by Lemma 22.1, $a\in\Sigma_\infty^{\mathscr{P}(\Omega_M),\mathbb{P}}$ holds in the submodel $\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\leq M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]$. Let

$$\vec{\mathbf{s}}=\boldsymbol{\zeta}^{\Upsilon}_{<M},\quad \boldsymbol{\theta}=\boldsymbol{\zeta}_M,\quad \mathbf{G}=G_M,\quad \boldsymbol{w}=\boldsymbol{w}[\vec{\mathbf{s}},\boldsymbol{\theta}]. \tag{9}$$

Then $\boldsymbol{w}=\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]$ and the triple $\langle\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}\rangle$ is $(\mathbb{C}^{\Upsilon}_{<M}\times\mathbb{C}_M\times\mathbb{P}^M)$-generic over $\mathbf{L}$ by the product forcing theorem. In addition, we have

$$\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{\leq M}, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]=\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M},\boldsymbol{\zeta}_M, G_M{\restriction}\boldsymbol{w}^{\Upsilon}_M[\boldsymbol{\zeta}]]=\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}{\restriction}\boldsymbol{w}].$$

Therefore $a\in\mathbf{L}[\vec{\mathbf{s}},\mathbf{G}]$ by Theorem 24.1.

However $\mathbf{L}[\vec{\mathbf{s}},\mathbf{G}]$ is a $(\mathbb{C}^{\Upsilon}_{<M}\times\mathbb{P}^M)$-generic extension of $\mathbf{L}$ by (9), where the forcing notion $\mathbb{C}^{\Upsilon}_{<M}=\prod_{n<M}\mathbb{C}_n\in\mathbf{L}$ satisfies $\mathtt{card}\,\mathbb{C}^{\Upsilon}_{<M}=\omega^{\mathbf{L}}_{M-1}$ in $\mathbf{L}$ (see Subsection 10) whereas $\mathbb{P}^M$ is Ω_{M-1}-closed in $\mathbf{L}$ by Corollary 17.2. It follows by Lemma 9.1(ii) that any real in $\mathbf{L}[\vec{\mathbf{s}},\mathbf{G}]$ belongs to the model $\mathbf{L}[\vec{\mathbf{s}}]=\mathbf{L}[\boldsymbol{\zeta}^{\Upsilon}_{<M}]$, as required. □

To prove Theorem 24.1, we are going to define a forcing-like relation **forc** rather analogous to approximate relations considered in our recent works [34, 35], and also in [30, 32, 33, 36] on the basis of forcing notions not of the almost-disjoint type.

[8] See Definition 10.4 on $\boldsymbol{w}[\vec{\mathbf{s}},\boldsymbol{\theta}]$.

Then we exploit certain symmetries of objects related to **forc**. Generally these arguments will go pretty similar to the proof of Theorem 9 in sections 5, 6 in [34].

25. Names and formulas

We argue under the assumptions and notation of $(*)$ of Theorem 24.1.

Assume that $z \in \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, $z \subseteq \mathcal{I} = \omega \times \omega$. Then let $\mathbf{Nam}_z \in \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$ be the set of all sets $\tau \in \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, $\tau \subseteq ({}^*\mathbf{P}_M{\restriction}z) \times \Omega_M$, with $\mathtt{card}\,\tau \le \Omega_M$ in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$.

Note that ${}^*\mathbf{P}_M$, a bigger forcing notion, is used instead of $\mathbb{P}^M$ in this definition. An advantage is that ${}^*\mathbf{P}_M$ is $\in$-definable in $\mathbb{H}_M = (\mathbf{H}\Omega_{M+1})^{\mathbf{L}}$ by Lemma 15.1.

If $\tau \in \mathbf{Nam}_z$ and $G \subseteq {}^*\mathbf{P}_M{\restriction}z$ then put $\tau[G] = \{\alpha < \Omega_M : \exists\, p \in G\,(\langle p,\alpha\rangle \in \tau)\}$.

Lemma 25.1. $\mathscr{P}(\Omega_M) \cap \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}{\restriction}\boldsymbol{w}] = \{\tau[\mathbf{G}{\restriction}\boldsymbol{w}] : \tau \in \mathbf{Nam}_{\boldsymbol{w}}\}$.

Proof. Let $X \in \mathscr{P}(\Omega_M) \cap \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}{\restriction}\boldsymbol{w}]$. By the product forcing theory, $\mathbf{G}{\restriction}\boldsymbol{w}$ is $(\mathbb{P}^M{\restriction}\boldsymbol{w})$-generic over $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$. Therefore, by a well-known property of generic extensions (see, *e.g.*, [26]), there is a name $t \in \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, $t \subseteq (\mathbb{P}^M{\restriction}\boldsymbol{w}) \times \Omega_M$, such that $X = t[\mathbf{G}{\restriction}\boldsymbol{w}]$. To reduce t to a name τ with the same property, satisfying $\mathtt{card}\,\tau \le \Omega_M$, apply Lemma 13.2. □

Now, arguing in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, we introduce a language that will help us to study analytic definability in the generic extensions considered.

Let $\mathcal{L}$ be the 2nd order language, with variables α, β, …, assumed to vary over ordinals $< \Omega_M$, and separately X, Y, …, assumed to vary over the subsets of Ω_M. Atomic formulas of the following types are allowed:

$$\alpha < \beta, \quad \alpha = \beta, \quad \alpha \in X, \quad \mathbb{p}(\alpha,\beta) = \nu \quad \text{(see Subsection 7 on } \mathbb{p}).$$

Only the connectives $\wedge$ and $\neg$ and quantifiers $\exists\,\alpha$ and $\exists\,X$ are allowed, the other connectives and $\forall$ are treated as shortcuts, and, to reduce the number of cases, the equality $X = Y$ will be treated as a shortcut for $\forall\,\alpha(\alpha \in X \iff \alpha \in Y)$.

The *complexity* $\#(\varphi)$ of an $\mathcal{L}$-formula φ is defined by induction so that

- $\#(\varphi) = 0$ for all atomic formulas,
- $\#(\varphi \wedge \psi) = \max\{\#(\varphi), \#(\psi)\}$,
- $\#(\exists\,\alpha\,\varphi(\alpha)) = \#(\varphi(\alpha))$ and $\#(\exists\,X\,\varphi(X)) = \#(\varphi(X))$,
- finally, $\#(\neg\,\varphi) = \#(\varphi) + 1$.

For any n there is a true quantifier-free formula $\mathbf{tr}_n$ with $\#(\mathbf{tr}_n) = n$, namely

$$\text{(10)} \qquad \mathbf{tr}_n := \begin{cases} \neg\neg\ldots\neg(0=0) & (n \text{ negations}), \quad \text{in case} \quad n = 2k; \\ \neg\neg\ldots\neg(0=1) & (n \text{ negations}), \quad \text{in case} \quad n = 2k+1. \end{cases}$$

If $z \in \mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, $z \subseteq \omega \times \omega$, then let $\mathcal{L}(z)$ be the extension of $\mathcal{L}$ by:

– ordinals $\alpha < \Omega_M$ to substitute variables over Ω_M,
– names in $\mathbf{Nam}_z$ to substitute variables X, Y, … over $\mathscr{P}(\Omega_M)$.

If $G \subseteq {}^*\mathbf{P}_M{\restriction}z$, then the *valuation* $\varphi[G]$ of such a formula φ is defined by substitution of $\tau[G]$ for any name $\tau \in \mathbf{Nam}_z$ that occurs in φ, and relativization of each quantifier $\exists\,\alpha$ or $\exists\,X$ to resp. Ω_M, $\mathscr{P}(\Omega_M)$. Thus $\varphi[G]$ is a formula of $\mathcal{L}$ with parameters in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},G] \cap (\Omega_M \cup \mathscr{P}(\Omega_M))$ and quantifiers relativized as above, so it can be Ω_M-true or Ω_M-false in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},G]$ in the sense of Subsection 7.

26. Forcing approximation

We still argue under the assumptions and notation of (∗) of Theorem 24.1.

Our next goal is to define, **in** $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, a forcing-style relation $p\,\mathbf{forc}^z_U\,\varphi$. In case $z=\boldsymbol{w}$ and $U=\mathbb{U}^M_\gamma$, $\gamma<\Omega_{M+1}$, the relation $\mathbf{forc}^z_U$ will be compatible with the truth in the model $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta},\mathbf{G}\restriction\boldsymbol{w}]=\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}][\mathbf{G}\restriction\boldsymbol{w}]$, viewed as a $(\mathbb{P}^M\restriction\boldsymbol{w})$-generic extension of $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$. But, unlike the true forcing relation associated with $\mathbb{P}^M\restriction\boldsymbol{w}$, the relation $\mathbf{forc}^z_U$ will be invariant under certain transformations.

The definition goes on in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$ by induction on the complexity of φ.

(F1) When writing $p\,\mathbf{forc}^z_U\,\varphi$, it will always be assumed that $U\in\mathbf{sDS}_M$, $z\in\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, $z\subseteq\omega\times\omega$, $p\in\mathbf{P}[U]\restriction z$, φ is a closed formula in $\mathcal{L}(z)$.

(F2) If U,p,z are as in (F1), and $\alpha,\beta,\nu<\Omega_M$, then: $p\,\mathbf{forc}^z_U\,\alpha<\beta$ iff in fact $\alpha<\beta$, and the same for the formulas $\alpha=\beta$ and $\mathbb{p}(\alpha,\beta)=\nu$.

(F3) If U,p,z are as above, $\alpha<\Omega_M$, $Y\in\mathbf{Nam}_z$, then: $p\,\mathbf{forc}^z_U\,\alpha\in Y$ iff there exists a condition $q\in\mathbf{P}[U]\restriction z$ such that $\langle q,\alpha\rangle\in Y$ and $p\leqslant q$.

(F4) If U,p,z are as above, then: $p\,\mathbf{forc}^z_U\,(\varphi\wedge\psi)$ iff $p\,\mathbf{forc}^z_U\,\varphi$ and $p\,\mathbf{forc}^z_U\,\psi$.

(F5) If U,p,z are as above, then $p\,\mathbf{forc}^z_U\,\exists\,\alpha\,\varphi(\alpha)$ iff there is $\alpha<\Omega_M$ such that $p\,\mathbf{forc}^z_U\,\varphi(\alpha)$.

(F6) If U,p,z are as above, then $p\,\mathbf{forc}^z_U\,\exists X\,\varphi(X)$ iff there exists a name $\tau\in\mathbf{Nam}_z$ such that $p\,\mathbf{forc}^z_U\,\varphi(\tau)$.

We precede the last item (F7) with another definition. If $n<\omega$ then let $\mathbf{sDS}_M[n]$ be the set of all systems $U\in\mathbf{sDS}_M$ such that $U\restriction^{<n}=\mathbb{U}^M_\gamma\restriction^{<n}$ for some $\gamma<\Omega_{M+1}$. (See Definition 17.1 on $\mathbb{U}^M_\gamma$.) Thus $\mathbb{U}^M_\gamma\in\bigcap_n\mathbf{sDS}_M[n]$ for all γ, and

$$\mathbf{sDS}_M=\mathbf{sDS}_M[0]\supseteq\mathbf{sDS}_M[1]\supseteq\mathbf{sDS}_M[2]\supseteq\dots.$$

(F7) If U,p,z are as in (F1), φ is a closed $\mathcal{L}(z)$ formula, $n=\#(\varphi)$, then we define $p\,\mathbf{forc}^z_U\,\neg\,\varphi$ iff there is no system $U'\in\mathbf{sDS}_M[n]$ extending U, and no condition $q\in\mathbf{P}[U']\restriction z$, $q\leqslant p$, such that $q\,\mathbf{forc}^z_{U'}\,\varphi$.

Exercise 26.1. Assume that U,p,z are as in (F1), $n<\omega$, and $\mathbf{tr}_n$ is defined by (10). Prove using (F2) and (F7) that $p\,\mathbf{forc}^z_U\,\mathbf{tr}_n$ but $\neg(p\,\mathbf{forc}^z_U\,\neg\,\mathbf{tr}_n)$. □

Lemma 26.2 (in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$)**.** *Assume that U,p,z,φ satisfy* (F1)*. Then:*

(i) *if $p\,\mathbf{forc}^z_U\,\varphi$, a system $U'\in\mathbf{sDS}_M$ extends U, and $q\in\mathbf{P}[U']\restriction z$, $q\le p$, then $q\,\mathbf{forc}^z_{U'}\,\varphi$.*

(ii) *if $U\in\mathbf{sDS}_M[n]$, $\#(\varphi)\le n$, and $p\,\mathbf{forc}^z_U\,\varphi$, then $p\,\mathbf{forc}^z_U\,\neg\,\varphi$ fails.*

Proof. The proof of (i) by straightforward induction is elementary. As for (ii), make use of (F7). □

Now let's evaluate the complexity of the relation **forc**. Recall that

$$\mathbb{H}_M=(\mathbf{H}\Omega_{M+1})^{\mathbf{L}}\quad\text{and}\quad\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]=(\mathbf{H}\Omega_{M+1})^{\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]}.$$

Lemma 26.3 (in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$)**.** *Assume that $\varphi(\alpha,\beta,\dots,X,Y,\dots)$ is a parameterfree $\mathcal{L}$-formula and $n=\#(\varphi)$. Then the following set is $\Sigma^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}_{n+3}$:*

$$\begin{aligned}\mathbf{Forc}(\varphi)=\{\langle z,U,p,\alpha,\beta,\dots,\tau_X,\tau_Y,\dots\rangle:\ &U\in\mathbf{sDS}_M\wedge z\subseteq\omega\times\omega\\ &\wedge\ p\in\mathbf{P}[U]\restriction z\wedge\alpha,\beta,\dots<\Omega_M\wedge\tau_X,\tau_Y,\dots\in\mathbf{Nam}_z\\ &\wedge\ p\,\mathbf{forc}^z_U\,\varphi(\alpha,\beta,\dots,\tau_X,\tau_Y,\dots)\ \}.\end{aligned}$$

Proof. The set $\mathbf{sDS}_M$ is $\Delta_3^{\mathbb{H}_M}$ by Lemma 15.1, hence is $\Delta_3^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$ as well by Lemma 8.1. The relations $p \in \mathbf{P}[U]\restriction z$ and $\tau \in \mathbf{Nam}_z$, with arguments resp. p, U, z and τ, z, are routinely checked to be $\Delta_3^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$, too. (Bounded quantifiers preserve $\Delta_3^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$.) After this remark, prove the lemma by induction on the structure of φ.

The case of atomic formulas (F2) is clear: the pairing function $\mathbb{p}$ in (F2) is $\Delta_1^{\mathbb{H}_M}$ by Lemma 7.1. The result for atomic formulas (F3) amounts to the formula $\exists\, q \in {}^*\mathbf{P}_M\restriction z\,(\langle q, \alpha\rangle \in Y \wedge p \leqslant q)$, which is $\Sigma_3^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$ by the above. The step (F4) amounts to the intersection of two sets, and hence is obvious. And so are steps (F5) and (F6) (a $\exists$-quantification on the top of $\Sigma_{\#(\varphi)+3}^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$).

To carry out the step (F7), note that $\mathbf{sDS}_M[n]$ is $\Sigma_{n+3}^{\mathbb{H}_M}$: indeed by Definition 17.1 the systems $\mathbb{U}_\gamma^M$ satisfy condition (i) of Theorem 16.1. It follows that $\mathbf{sDS}_M[n]$ is $\Sigma_{n+3}^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$ as well by Lemma 8.1. Thus if $\mathbf{Forc}(\varphi)$ is $\Sigma_{n+3}^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$ then $\mathbf{Forc}(\neg\,\varphi)$ is $\Pi_{n+3}^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$, therefore $\Sigma_{n+4}^{\mathbb{H}_M[\vec{\mathbf{s}},\boldsymbol{\theta}]}$, as required. □

27. Consequences for the complete forcing notions

We continue to argue under the assumptions and notation of $(*)$ of Theorem 24.1. Coming back to the sequence of systems $\mathbb{U}_\gamma^M \in \mathbf{sDS}_M$ given by Definition 17.1, we observe that every system $\mathbb{U}_\gamma^M$ belongs to $\bigcap_n \mathbf{sDS}_M[n]$.

Let $\mathbf{forc}_\gamma^z$ be $\mathbf{forc}_{\mathbb{U}_\gamma^M}^z$, and let $p\,\mathbf{forc}_\infty^z\,\varphi$ mean: $\exists\,\gamma < \Omega_{M+1}\,(p\,\mathbf{forc}_\gamma^z\,\varphi)$.

Note that $p\,\mathbf{forc}_\gamma^z\,\varphi$ implies $p \in \mathbb{P}_\gamma^M\restriction z$, whereas $p\,\mathbf{forc}_\infty^z\,\varphi$ implies $p \in \mathbb{P}^M\restriction z$. Lemma 26.2 takes the following form:

Lemma 27.1 (in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$)**.** *Assume that $z \subseteq \omega\times\omega$, φ is a closed $\mathcal{L}(z)$ formula, and $p \in \mathbb{P}^M\restriction z$. Then:*

(i) *if $p\,\mathbf{forc}_\gamma^z\,\varphi$ and $\gamma \le \eta < \Omega_{M+1}$, $q \in \mathbb{P}_\eta^M\restriction z$, $q \leqslant p$, then $q\,\mathbf{forc}_\eta^z\,\varphi$, and accordingly, if $p\,\mathbf{forc}_\infty^z\,\varphi$ and $q \in \mathbb{P}^M\restriction z$, $q \leqslant p$, then $q\,\mathbf{forc}_\infty^z\,\varphi$;*

(ii) *$p\,\mathbf{forc}_\infty^z\,\varphi$ and $p\,\mathbf{forc}_\infty^z\,\neg\,\varphi$ contradict each other.* □

The following "completeness lemma" will be very important.

Lemma 27.2 (in $\mathbf{L}[\vec{\mathbf{s}},\boldsymbol{\theta}]$)**.** *If $z \subseteq \omega\times\omega$, φ is a closed $\mathcal{L}(z)$ formula, $p \in \mathbb{P}^M\restriction z$, then there is a condition $q \in \mathbb{P}^M\restriction z$, $q \leqslant p$, such that either $q\,\mathbf{forc}_\infty^z\,\varphi$, or $q\,\mathbf{forc}_\infty^z\,\neg\,\varphi$.*

Proof. Let $n = \#(\varphi)$. There is an ordinal $\eta < \Omega_{M+1}$ such that $p \in \mathbb{P}_\eta^M\restriction z$. (See Definition 17.1 on forcing notions $\mathbb{P}_\gamma^M$.) Consider the set D of all systems $U' \in \mathbf{sDS}_M\restriction^{\ge n}$ such that there is a system $U \in \mathbf{sDS}_M[n]$ that extends $\mathbb{U}_\eta^M$ and satisfies $U\restriction^{\ge n} = U'$, and also there is a condition $q \in \mathbf{P}[U]\restriction z$, $q \leqslant p$, satisfying $q\,\mathbf{forc}_U^z\,\varphi$. The set D belongs to $\Sigma_{n+3}(\mathbb{H}[\vec{\mathbf{s}},\boldsymbol{\theta}])$ (with $\mathbb{U}_\eta^M, p$ as definability parameters) by Lemma 26.3. Therefore by Lemma 16.2 there is an ordinal γ, $\eta \leqslant \gamma < \Omega_{M+1}$, such that the system $\mathbb{U}_\gamma^M\restriction^{\ge n}$ n-solves D. We have two cases.

Case 1. $\mathbb{U}_\gamma^M\restriction^{\ge n} \in D$. Then there exist: a system $U \in \mathbf{sDS}_M[n]$ extending $\mathbb{U}_\eta^M$ and satisfying $U\restriction^{\ge n} = \mathbb{U}_\gamma^M\restriction^{\ge n}$, and a condition $q \in \mathbf{P}[U]\restriction z$, $q \leqslant p$, with $q\,\mathbf{forc}_U^z\,\varphi$. By definition there is an ordinal $\vartheta < \Omega_{M+1}$ such that $U\restriction^{<n} = \mathbb{U}_\vartheta^M\restriction^{<n}$. Now let $\mu = \mathtt{max}\{\gamma, \vartheta\}$. Then $\mathbb{U}_\mu^M$ extends U, hence $q\,\mathbf{forc}_\mu^z\,\varphi$ and $q\,\mathbf{forc}_\infty^z\,\varphi$.

Case 2. There is no system $U \in D$ that extends $\mathbb{U}_\gamma^M\restriction^{\ge n}$. Prove that $p\,\mathbf{forc}_\gamma^z\,\neg\,\varphi$. Suppose towards the contrary that this fails. Then, by (F7) in Subsection 26, there exists a system $U \in \mathbf{sDS}_M[n]$ extending $\mathbb{U}^M_\gamma$, and a condition $q \in \mathbf{P}[U]$, $q \le p$, such that $q\,\mathbf{forc}^z_U\,\varphi$. By definition the system $U' = U{\restriction}^{\ge n}$ belongs to $\mathbf{sDS}_M{\restriction}^{\ge n}$, and moreover U witnesses that $U' \in D$. But this contradicts the Case 2 assumption. □

28. Truth lemma

We continue to argue under the assumptions and notation of $(*)$ of Theorem 24.1.

According to Theorem 28.1 ("the truth lemma"), the truth in the generic extensions considered is connected usual way with the relation $\mathbf{forc}_\infty$.

Theorem 28.1. *Assume that φ is a $\mathcal{L}(\vec{\mathbf{s}}, \boldsymbol{\theta})$-formula. Then $\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-true in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$ iff there is a condition $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ such that $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\varphi$.*

Proof. We proceed by induction. Suppose that φ is an atomic formula $\alpha \in \tau$ of type (F3) of Subsection 26, where $\alpha < \Omega_M$ and $\tau \in \mathbf{Nam}_{\boldsymbol{w}}$. (The case of formulas as in (F2) is pretty elementary.) To prove the implication $\Longleftarrow$, assume that $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ and $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\alpha \in \tau$. Then by definition ((F3)) in Subsection 26 there exists a condition $q \in \mathbb{P}^M{\restriction}\boldsymbol{w}$ satisfying $p \leqslant q$ and $\langle q, \alpha\rangle \in \tau$. There are conditions $p', q' \in \mathbb{P}^M$ such that $p = p'{\restriction}\boldsymbol{w}$ and $q = q'{\restriction}\boldsymbol{w}$, but not necessarily $p' \leqslant q'$. We only know that $p'(n,i) \leqslant q'(n,i)$ for all $\langle n,i\rangle \in \boldsymbol{w}$. Therefore $\boldsymbol{w} \subseteq Z = \{\langle n,i\rangle : p'(n,i) \leqslant q'(n,i)\}$. The set Z belongs to $\mathbf{L}$ since so do p', q' as elements of $\mathbb{P}^M \in \mathbf{L}$ (whereas about $\boldsymbol{w}$ we only assert that $\boldsymbol{w} \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$). Therefore a condition $q'' \in \mathbb{P}^M$ can be defined by

$$q''(n,i) = \begin{cases} q(n,i), & \text{in case} \quad \langle n,i\rangle \in Z, \\ p(n,i), & \text{in case} \quad \langle n,i\rangle \notin Z, \end{cases}$$

and still $q''{\restriction}\boldsymbol{w} = q'{\restriction}\boldsymbol{w}$ and $p' \leqslant q''$. It follows that $q'' \in \mathbf{G}$ by genericity, hence $q''{\restriction}\boldsymbol{w} = q'{\restriction}\boldsymbol{w} \in \mathbf{G}{\restriction}\boldsymbol{w}$. Then $\alpha \in \tau[\mathbf{G}{\restriction}\boldsymbol{w}]$, *i.e.*, $(\alpha \in \tau)[\mathbf{G}{\restriction}\boldsymbol{w}]$ is true, as required.

To prove the converse, assume that $\alpha \in \tau[\mathbf{G}{\restriction}\boldsymbol{w}]$. There exists a condition $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ such that $\langle p, \alpha\rangle \in \tau$, and we have $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\alpha \in \tau$, as required.

The rather simple inductive steps (F4), (F5) of Subsection 26 are left for the reader.

Let's carry out **the step (F6)**. Let φ be $\exists X\,\psi(X)$. Suppose that $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ and $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\varphi$. By definition there is a name $\tau \in \mathbf{Nam}_{\boldsymbol{w}}$ such that $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\psi(\tau)$. The formula $\psi(\tau)[\mathbf{G}{\restriction}\boldsymbol{w}]$ is then true in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$ by the inductive hypothesis. But $\psi(\tau)[\mathbf{G}{\restriction}\boldsymbol{w}]$ coincides with $\psi[\mathbf{G}{\restriction}\boldsymbol{w}](X)$, where $X = \tau[\mathbf{G}{\restriction}\boldsymbol{w}] \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$. We conclude that $\exists X\,\psi(X)[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-true in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$, as required.

To prove the converse, assume that $\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$, that is, $\exists X\,\psi(X)[\mathbf{G}{\restriction}\boldsymbol{w}]$, is Ω_M-true in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$, hence there exists a set $X \in \mathscr{P}(\Omega_M) \cap \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$ such that $\psi[\mathbf{G}{\restriction}\boldsymbol{w}](X)$ is Ω_M-true. By Lemma 25.1, there is a name $\tau \in \mathbf{Nam}_{\boldsymbol{w}}$ with $X = \tau[\mathbf{G}{\restriction}\boldsymbol{w}]$. Then $\psi(\tau)[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-true in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$. The inductive hypothesis implies that some $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ satisfies $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\psi(\tau)$, hence $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\varphi$, as required.

Finally, let's carry out **the step (F7)**, which is somewhat less trivial. Prove the lemma for an $\mathcal{L}(\boldsymbol{w})$ formula $\neg\varphi$, assuming that the result holds for φ. If $\neg\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-false in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$ then $\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-true. Thus by the inductive hypothesis, there is a condition $p \in \mathbf{G}{\restriction}\boldsymbol{w}$ such that $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\varphi$. Then $q\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\neg\varphi$ for any $q \in \mathbf{G}{\restriction}\boldsymbol{w}$ is impossible by Lemma 27.1.

Conversely suppose that $p\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\neg\varphi$ holds for no condition $p \in \mathbf{G}{\restriction}\boldsymbol{w}$. Then by Lemma 27.2 and the genericity there is a condition $q \in \mathbf{G}{\restriction}\boldsymbol{w}$ satisfying $q\,\mathbf{forc}^{\boldsymbol{w}}_\infty\,\varphi$. It follows that $\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-true by the inductive hypothesis, therefore $\neg\varphi[\mathbf{G}{\restriction}\boldsymbol{w}]$ is Ω_M-false. □

Section VI. Invariance

The goal of this section is **to prove Theorem 24.1**, and thereby accomplish the proof of Claim (C) of Theorem 20.2, and the proof of Theorem 4.1 (the first main theorem) itself. The proof makes use of the relation **forc** introduced in Section V, and exploits certain symmetries in **forc**, investigated in Subsection 32.

29. Hidden invariance

Theorem 24.1 belongs to a wide group of results on the structure of generic models which assert that such-and-such elements of a given generic extension belong to a smaller and/or better shaped subextension. One of possible methods to prove such results is to exploit the homogeneity of the forcing notion considered, or in different words, its invariance w.r.t. a sufficiently large system of order-preserving transformations. In particular, for a straightforward proof of Theorem 30.1 (the key technical step in the proof of Theorem 24.1) the invariance of the forcing notion $\mathbb{P}^M$ under permutations of indices in $\mathcal{I} = \omega \times \omega$ would be naturally required — but $\mathbb{P}^M$ is definitely not invariant w.r.t. permutations.

On the other hand, the auxiliary forcing relation **forc** is invariant w.r.t. permutations. Theorem 28.1 conveniently connects the relation **forc** with the truth in $\mathbb{P}^M$-generic extensions by means of a forcing-style association. This principal association was based on the M-completeness property (Theorem 16.1). Basically it occurs that some transformations, that is, permutations, are *hidden* in construction of $\mathbb{P}^M$, so that they do not act explicitly, but their influence is preserved and can be recovered to some extent via the relation **forc**.

This method of *hidden invariance*, that is, invariance properties (of an auxiliary forcing-type relation like **forc**) hidden in $\mathbb{P}^M$ by a suitable generic-style construction of $\mathbb{P}^M$, was introduced in Harrington's notes [24] in a different terminology.

30. The invariance theorem

Let $\mathbb{H}$ be the group of all *finite idempotent permutations* of ω, that is, all bijections $\mathbb{h} : \omega \xrightarrow{\text{onto}} \omega$ sich that $\mathbb{h} = \mathbb{h}^{-1}$ and the set $|\mathbb{h}| = \{k : \mathbb{h}(k) \neq k\}$ is finite. If $n < \omega$ then the subgroup $\mathbb{H}_n$ consists of all $\mathbb{h} \in \mathbb{H}$ satisfying $\mathbb{h}(k) = k$ for all $k < n$.

If $\mathbb{h} \in \mathbb{H}$, and $z \subseteq \omega \times \omega$ then put $\mathbb{h}z = \{\langle \mathbb{h}(n), i\rangle : \langle n, i\rangle \in z\}$.

If g is a function defined on $\mathtt{dom}\, g \subseteq \omega$ then define a function $\mathbb{h}g$ on the set $\mathbb{h}[\mathtt{dom}\, g] = \{\mathbb{h}(k) : k \in \mathtt{dom}\, g\}$ by $(\mathbb{h}g)(\mathbb{h}(k)) = g(k)$, for all $k \in \mathtt{dom}\, g$.

The following is the invariance theorem. Note that $\mathbb{h}$ does **not** act on p_0!

Theorem 30.1 (in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$)**.** *Under the assumptions of* $(*)$ *of Theorem* 24.1*, suppose that φ is a closed parameter-free formula of $\mathcal{L}(\boldsymbol{w})$, $n = \#(\varphi)$, $\mathbb{h} \in \mathbb{H}_n$, $\boldsymbol{w}' = \mathbb{h}\boldsymbol{w}$, and $p_0 \in \mathbb{P}^M$. Then $p_0 {\restriction} \boldsymbol{w}\, \mathbf{forc}^{\boldsymbol{w}}_{\infty} \neg\varphi$ iff $p_0 {\restriction} \boldsymbol{w}'\, \mathbf{forc}^{\boldsymbol{w}'}_{\infty} \neg\varphi$.*

A lengthy **proof of Theorem 30.1** follows below in Subsection 31–33.

Now we show how the theorem *works in the proof of Theorem* 24.1.

Proof of Theorem 24.1 *from Theorem* 30.1. Arguing under the assumptions of Theorem 24.1, consider an arbitrary set $a \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$, $a \subseteq \omega$, such that $a \in \Sigma^{\mathscr{P}(\Omega_M), \mathbb{P}}_{\infty}$ in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$. Then it holds in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}, \mathbf{G}{\restriction}\boldsymbol{w}]$ that

$$a = \{j < \omega : \neg\varphi(j) \text{ is } \Omega_M\text{-true}\},$$

where $\varphi(\cdot)$ is a parameter-free $\mathcal{L}$-formula, hence, $\mathbb{p}$-formula. Let $n = \#(\varphi)$. The goal is to prove that $a \in \mathbf{L}[\vec{\mathbf{s}}, \mathbf{G}]$. This is based on Lemma 30.2.

Lemma 30.2. *The set* $T = \{\langle p, j\rangle : p \in \mathbb{P}^{\Omega} \wedge p{\restriction}\boldsymbol{w}\,\mathbf{forc}^{\boldsymbol{w}}_{\infty} \neg\varphi(j)\}$ *belongs to* $\mathbf{L}[\vec{\mathbf{s}}]$.

(Note that $\boldsymbol{w}$ does not belong to $\mathbf{L}[\vec{\mathbf{s}}]$!)

Proof. We observe that, by Lemma 26.3, the set

$$K = \{\langle z, p, j\rangle : p \in \mathbb{P}^{\Omega} \wedge z \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}] \wedge z \subseteq \omega \times \omega \wedge j < \omega \wedge p{\restriction}z\,\mathbf{forc}^{z}_{\infty} \neg\varphi(j)\}$$

is definable in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$ by a formula with sets in $\mathbf{L}$ as parameters, say

$$K = \{\langle z, p, j\rangle : \chi(z, p, j, Y)\} \quad \text{in} \quad \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}],$$

with $Y \in \mathbf{L}$ as a sole parameter. Recall that $\boldsymbol{\theta} : \omega \to \omega^{\mathbf{L}}_{M}$ is $\mathbb{C}_M$-generic over $\mathbf{L}[\vec{\mathbf{s}}]$ by $(*)$ of Theorem 24.1, and $\boldsymbol{w} = \boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}] = \{\langle k, j\rangle : j \in A_{\boldsymbol{\theta}(k)}[\vec{\mathbf{s}}]\}$, as in Definition 10.4.

We let $\breve{\boldsymbol{\theta}} \in \mathbf{L}$ be a canonical $\mathbb{C}_M$-name for $\boldsymbol{\theta}$, and let $\Vdash$ be the $\mathbb{C}_M$-forcing relation over $\mathbf{L}[\vec{\mathbf{s}}]$, definable in $\mathbf{L}[\vec{\mathbf{s}}]$, as usual. We claim that

$$\chi(\boldsymbol{w}, p, j, Y) \text{ holds in } \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}] \quad \text{iff} \quad \boldsymbol{\theta}{\restriction}n \Vdash \chi(\boldsymbol{w}[\vec{\mathbf{s}}, \breve{\boldsymbol{\theta}}], p, j, Y), \tag{11}$$

where $n = \#(\varphi)$ as above; $\boldsymbol{\theta}{\restriction}n$ belongs to $\mathbb{C}_M$, of course.

The direction $\Longleftarrow$ is obvious.

To establish $\Longrightarrow$, assume that the right-hand side of (11) fails. There is a condition $e_0 \in \mathbb{C}_M$ such that $\boldsymbol{\theta}{\restriction}n \subseteq e_0$ and $e_0 \Vdash \neg\chi(\boldsymbol{w}[\vec{\mathbf{s}}, \breve{\boldsymbol{\theta}}], p, j, Y)$. We note that

$$D = \{e \in \mathbb{C}_M : \boldsymbol{\theta}{\restriction}n \subseteq e \wedge \exists\, \mathbb{h} \in \mathbb{H}_n\, (|\mathbb{h}| \subseteq \mathtt{dom}\, e \wedge e_0 \subseteq \mathbb{h}e)\}$$

is a set open dense in $\mathbb{C}_M$ over $\boldsymbol{\theta}{\restriction}n$. Therefore, by the genericity of $\boldsymbol{\theta}$, there exists a number $k > n$ such that $e = \boldsymbol{\theta}{\restriction}k \in D$. Accordingly, there is a permutation $\mathbb{h} \in \mathbb{H}_n$ satisfying $|\mathbb{h}| \subseteq k$ and $e_0 \subseteq \mathbb{h}e$.

We put $\boldsymbol{\theta}' = \mathbb{h}\boldsymbol{\theta}$; this is still a $\mathbb{C}_M$-generic map $\omega \to \omega^{\mathbf{L}}_M$, with $\mathbf{L}[\boldsymbol{\theta}'] = \mathbf{L}[\boldsymbol{\theta}]$ since $\mathbb{h} \in \mathbf{L}$, and we have $e_0 \subseteq \mathbb{h}e \subset \boldsymbol{\theta}'$. It follows, by the choice of e_0, that $\chi(\boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}'], p, j, Y)$ *fails* in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}'] = \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$, and hence $\langle \boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}'], p, j\rangle \notin K$ by the choice of χ. However $\boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}'] = \mathbb{h} \cdot \boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}] = \mathbb{h}\boldsymbol{w}$, thus we have $\langle \mathbb{h}\boldsymbol{w}, p, j\rangle \notin K$.

We conclude that $p{\restriction}\mathbb{h}\boldsymbol{w}\,\mathbf{forc}^{\mathbb{h}\boldsymbol{w}}_{\infty} \neg\varphi(j)$ *fails* by the definition of K. Therefore $p{\restriction}\boldsymbol{w}\,\mathbf{forc}^{\boldsymbol{w}}_{\infty} \neg\varphi(j)$ *fails* as well by Theorem 30.1, so we have $\langle \boldsymbol{w}, p, j\rangle \notin K$, and hence $\chi(\boldsymbol{w}, p, j, Y)$ fails in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}'] = \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$. This completes the proof of (11).

And coming back to the lemma, (11) implies the equality

$$T = \{\langle p, j\rangle : p \in \mathbb{P}^{\Omega} \wedge j < \omega \wedge \boldsymbol{\theta}{\restriction}n \Vdash \chi(\boldsymbol{w}[\vec{\mathbf{s}}, \breve{\boldsymbol{\theta}}], p, j, Y)\}.$$

This implies $T \in \mathbf{L}[\vec{\mathbf{s}}]$ since $\Vdash$ is definable in $\mathbf{L}[\vec{\mathbf{s}}]$. Lemma 30.2 is established. □

It remains to notice that, by Theorem 28.1,

$$\begin{aligned} j \in a &\iff \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}] \models \neg\varphi(j) \text{ is } \Omega_M\text{-true} \\ &\iff \exists\, p \in \mathbf{G}{\restriction}\boldsymbol{w}\, (p\,\mathbf{forc}^{\boldsymbol{w}}_{\infty} \neg\varphi(j)) \\ &\iff \exists\, p \in \mathbf{G}\, (p{\restriction}\boldsymbol{w}\,\mathbf{forc}^{\boldsymbol{w}}_{\infty} \neg\varphi(j)). \end{aligned}$$

Therefore $j \in a \iff \exists\, p \in \mathbf{G}\, (\langle p, j\rangle \in T)$. But $T \in \mathbf{L}[\vec{\mathbf{s}}]$ by Lemma 30.2.

We conclude that $a \in \mathbf{L}[\vec{\mathbf{s}}, \mathbf{G}]$, as required. This completes the proof of Theorem 24.1 from Theorem 30.1. □

31. The invariance theorem: setup

We continue to argue under the assumptions and notation of $(*)$ of Theorem 24.1.

Proof of Theorem 30.1 *(will be completed in Subsection* 33*).* We fix p_0, $\boldsymbol{w}=\boldsymbol{w}[\vec{\mathbf{s}},\boldsymbol{\theta}]$, φ, $n=\#(\varphi)$, $\mathbb{h}\in\mathbb{H}_n$, $\widehat{\boldsymbol{w}}=\mathbb{h}\boldsymbol{w}$, as in Theorem 30.1.

Suppose to the contrary that $p_0\restriction\widehat{\boldsymbol{w}}\ \mathbf{forc}_\infty^{\widehat{\boldsymbol{w}}}\ \neg\varphi$ but $\neg\,(p_0\restriction\boldsymbol{w}\ \mathbf{forc}_\infty^{\boldsymbol{w}}\ \neg\varphi)$. There is an ordinal $\gamma<\Omega_{M+1}$ with $p_0\restriction\widehat{\boldsymbol{w}}\ \mathbf{forc}_\gamma^{\widehat{\boldsymbol{w}}}\ \neg\varphi$ but $\neg\,(p_0\restriction\boldsymbol{w}\ \mathbf{forc}_\gamma^{\boldsymbol{w}}\ \neg\varphi)$. Then there is:

(A) a system $U^1\in\mathbf{sDS}_M[n]$ extending $\mathbb{U}^M_\gamma$, and a condition $p_1\in\mathbf{P}[U^1]$, $p_1\leqslant p_0$, such that $p_1\restriction\boldsymbol{w}\ \mathbf{forc}_{U^1}^{\boldsymbol{w}}\ \varphi$, but still $p_1\restriction\widehat{\boldsymbol{w}}\ \mathbf{forc}_{U^1}^{\widehat{\boldsymbol{w}}}\ \neg\varphi$ by Lemma 26.2.

Recall that any condition $p\in{}^*\mathbf{P}_M$ is a map $p\in\mathbf{L}$, defined on $\omega\times\omega$, and each value $p(k,i)=\langle\mathbf{S}_p(n,i);\mathbf{F}_p(k,i)\rangle$ is a pair of a set $\mathbf{S}_p(k,i)\subseteq\mathbf{Seq}_M$ and $\mathbf{F}_p(k,i)\subseteq\mathbf{Fun}_M$, with $\mathtt{card}\,(\mathbf{S}_p(k,i)\cup\mathbf{F}_p(k,i))<\Omega_M$ strictly, in $\mathbf{L}$. We define the *support* $\|p\|=\bigcup_{k,i<\omega}\|p\|_{ki}$, where $\|p\|_{ki}=\{s(0):s\in\mathbf{S}_p(k,i)\}\cup\{f(0):f\in\mathbf{F}_p(k,i)\}$; then $\|p\|\in\mathbf{L}$, $\|p\|$ is a bounded subset of Ω_M. We conclude that there exists:

(B) a bijection $\mathbb{b}\in\mathbf{L}$, $\mathbb{b}:\Omega_M\xrightarrow{\text{onto}}\Omega_M$ with $\|p_1\|\cap(\mathbb{b}``\|p_1\|)=\varnothing$ and $\mathbb{b}=\mathbb{b}^{-1}$.

Further, the system $U^1\in\mathbf{sDS}_M[n]$ is small, hence the set $J=\bigcup_{k,i<\omega}U^1(k,i)\in\mathbf{L}$ satisfies $\mathtt{card}\,J\le\Omega_M$ in $\mathbf{L}$. It follows that there is:

(C) a sequence $\{\mathbb{B}_\alpha\}_{\alpha<\Omega_M}\in\mathbf{L}$ of bijections $\mathbb{B}_\alpha:\Omega_M\xrightarrow{\text{onto}}\Omega_M$, such that $\mathbb{B}_0=\mathbb{b}$, $\mathbb{B}_\alpha={\mathbb{B}_\alpha}^{-1}$, and if $f,g\in J$ then $f(\alpha)\ne\mathbb{B}_\alpha(g(\alpha))$ for some ordinal $\alpha<\Omega_M$.

32. Transformation

In continuation of the proof of Theorem 30.1, we now define an automorphism acting on several different domains in $\mathbf{L}$. It is based on $\mathbb{h}$, $\mathbb{b}$, and $\mathbb{B}_\alpha$ of Subsection 31 and its action will be denoted by $\widehat{\ }$. Along the way we'll formulate properties (D)–(H) of the automorphism, a routine check of which is left to the reader.

We argue under the assumptions and notation of $(*)$ of Theorem 24.1.

If $\alpha\le\Omega_M$ and $f:\alpha\to\Omega_M$ then $\widehat{f}:\alpha\to\Omega_M$ is defined by $\widehat{f}(\nu)=\mathbb{B}_\nu(f(\nu))$ for all $\nu<\alpha$. In particular, $\widehat{f}(0)=\mathbb{B}_0(f(0))=b(f(0))$. This defines $\widehat{s}\in\mathbf{Seq}_M$ and $\widehat{f}\in\mathbf{Fun}_M$ for all $s\in\mathbf{Seq}_M$ and $f\in\mathbf{Fun}_M$.

(D) $f\longmapsto\widehat{f}$ is a bijection $\mathbf{Seq}_M\xrightarrow{\text{onto}}\mathbf{Seq}_M$ and $\mathbf{Fun}_M\xrightarrow{\text{onto}}\mathbf{Fun}_M$, and if $f,g\in J=\bigcup_{k,i<\omega}U^1(k,i)$ then $\widehat{f}\ne g$ by (C).

If $u\subseteq\mathbf{Fun}_M$ then let $\widehat{u}=\{\widehat{f}:f\in u\}$. If $S\subseteq\mathbf{Seq}_M$ then let $\widehat{S}=\{\widehat{s}:s\in S\}$.

If U is a system then define a system $\widehat{U}$, such that

$$\widehat{U}(k,i)=U(k,i),\qquad\text{in case } k<n;$$

$$\widehat{U}(\mathbb{h}(k),i)=\widehat{U(k,i)}=\{\widehat{f}:f\in U(k,i)\},\qquad\text{in case } k\ge n.$$

If $p\in{}^*\mathbf{P}_M$ then let $\widehat{p}\in{}^*\mathbf{P}_M$ be defined so that

$$\widehat{p}(k,i)=p(k,i),\qquad\text{in case } k<n;$$

$$\widehat{p}(\mathbb{h}(k),i)=\langle\widehat{\mathbf{S}_p(k,i)};\widehat{\mathbf{F}_p(k,i)}\rangle,\qquad\text{in case } k\ge n;$$

where $\widehat{\mathbf{S}_p(k,i)}=\{\widehat{s}:s\in\mathbf{S}_p(k,i)\}$ and $\widehat{\mathbf{F}_p(k,i)}=\{\widehat{s}:s\in\mathbf{F}_p(k,i)\}$ by the above. These are consistent definitions because $\mathbb{h}\in\mathbb{H}_n$ in Theorem 30.1.

(E) We have $\widehat{U}\restriction^{<n}=U\restriction^{<n}$ for any system U. The map $U\longmapsto\widehat{U}$ is a bijection of $\mathbf{sDS}_M$ onto itself and $\mathbf{sDS}_M[n']$ onto itself for any $n'\le n$.

(F) We have $\widehat{p}{\restriction}^{<n} = p{\restriction}^{<n}$ for any $p \in {}^*\mathbf{P}_M$. The map $p \longmapsto \widehat{p}$ is a $\leqslant$-preserving bijection of $\mathbf{P}[U]$ onto $\mathbf{P}[\widehat{U}]$.

If in addition $z \subseteq \omega \times \omega$ (not necessarily $z \in \mathbf{L}$), then if conditions $p, q \in {}^*\mathbf{P}_M$ satisfy $p{\restriction}z = q{\restriction}z$, then easily $\widehat{p} \restriction \widehat{z} = \widehat{q} \restriction \widehat{z}$, where $\widehat{z} = \mathbb{h} \cdot z = \{\langle \mathbb{h}(k), i\rangle : \langle k, i\rangle \in z\}$. This allows us to define $\widehat{r} := \widehat{p} \restriction \widehat{z}$ for every $r \in {}^*\mathbf{P}_M{\restriction}z$, where $p \in {}^*\mathbf{P}_M$ is any condition satisfying $r = p{\restriction}z$.

(G) If $z \subseteq \omega\times\omega$ then $p \longmapsto \widehat{p}$ is a $\leqslant$-preserving bijection of $\mathbf{P}[U]{\restriction}z$ onto $\mathbf{P}[\widehat{U}] \restriction \widehat{z}$.

If $z \subseteq \omega \times \omega$ and $\tau \in \mathbf{Nam}_z$ (Subsection 25) then let $\widehat{\tau} = \{\langle \widehat{p}, x\rangle : \langle p, x\rangle \in \tau\}$, and if φ is a $\mathcal{L}(z)$-formula then $\widehat{\varphi}$ is obtained by substituting $\widehat{\tau}$ for each name τ in φ.

(H) If $z \subseteq \omega \times \omega$, $z \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$, then the mapping $\tau \longmapsto \widehat{\tau}$ is a bijection of $\mathbf{Nam}_z$ onto $\mathbf{Nam}_{\widehat{z}}$ and a bijection of $\mathcal{L}(z)$-formulas onto $\mathcal{L}(\widehat{z})$-formulas.

Remark 32.1. The action of $\widehat{\ }$ is idempotent, so that *e.g.* $\widehat{\widehat{f}} = f$ for any $f \in \mathbf{Fun}_M$ *etc.* This is because we require that $\mathbb{b}^{-1} = \mathbb{b}$ and $\mathbb{B}_\alpha^{-1} = \mathbb{B}_\alpha$ for all $\alpha < \Omega$.

The action of $\widehat{\ }$ is constructible on $\mathbf{Seq}_M$, $\mathbf{Fun}_M$, systems, ${}^*\mathbf{P}_M$, since $\mathbb{h}$, $\mathbb{b}$, and the sequence of maps $\mathbb{B}_\alpha$ belong to $\mathbf{L}$ by (B), (C).

However if $z \in \mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$ then the action of $\widehat{\ }$ on ${}^*\mathbf{P}_M{\restriction}z$ and names in $\mathbf{Nam}_z$ belongs to $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$, since the extra parameter z does not necessarily belong to $\mathbf{L}$. □

It is not unusual that transformations of a forcing notion considered lead to this or another invariance. Lemma 32.2 is exactly of this type.

Lemma 32.2 (in $\mathbf{L}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$)**.** *Assume that* $U \in \mathbf{sDS}_M$*,* $\boldsymbol{w} = \boldsymbol{w}[\vec{\mathbf{s}}, \boldsymbol{\theta}]$*,* $n < \omega$*,* $\mathbb{h} \in \mathbb{H}_n$*,* $\widehat{\boldsymbol{w}} = \mathbb{h}\boldsymbol{w}$*,* $p \in \mathbf{P}[U]{\restriction}\boldsymbol{w}$*, and* Φ *is a closed formula of* $\mathcal{L}(\boldsymbol{w})$*,* $\#(\Phi) \le n + 1$*. Then* $p\,\mathbf{forc}_U^{\boldsymbol{w}}\,\Phi$ *iff* $\widehat{p}\,\mathbf{forc}_{\widehat{U}}^{\widehat{\boldsymbol{w}}}\,\widehat{\Phi}$*.*

Proof. We argue by induction on the structure of Φ. Routine cases of formulas (F2) and steps (F4) and (F5) of Subsection 26 by means of (D)–(H) are left to the reader. Thus we concentrate on atomic formulas of type (F3) and steps (F6) and (F7) in Subsection 26. We take care of only one direction of the equivalence of the lemma, as the other direction is entirely similar via the idempotency (see Remark 32.1).

Formulas of type (F3). Let Φ be $\alpha \in \tau$, where $\alpha < \Omega_M$ and $\tau \in \mathbf{Nam}_{\boldsymbol{w}}$. Assume that $p\,\mathbf{forc}_U^{\boldsymbol{w}}\,\alpha \in \tau$. Then by definition there is a condition $q \in \mathbf{P}[U]{\restriction}\boldsymbol{w}$ such that $p \leqslant q$ and $\langle q, \alpha\rangle \in \tau$. Then $\widehat{q}$ and $\widehat{p}$ belong to $\mathbf{P}[\widehat{U}]{\restriction}\widehat{\boldsymbol{w}}$, $\widehat{p} \leqslant \widehat{q}$, and $\langle \widehat{q}, \alpha\rangle \in \widehat{\tau}$. Thus we have $\widehat{p}\,\mathbf{forc}_{\widehat{U}}^{\widehat{\boldsymbol{w}}}\,\alpha \in \widehat{\tau}$, as required.

Step (F6). Let $\Phi := \exists\, X\, \Psi(X)$. Suppose that $p\,\mathbf{forc}_U^{\boldsymbol{w}}\,\Phi$. By definition there exists a name $\tau \in \mathbf{Nam}_{\boldsymbol{w}}$ such that $p\,\mathbf{forc}_U^{\boldsymbol{w}}\,\Psi(\tau)$, Then we have $\widehat{p}\,\mathbf{forc}_{\widehat{U}}^{\widehat{\boldsymbol{w}}}\,\widehat{\Psi(\tau)}$ by the inductive hypothesis. But $\widehat{\Psi(\tau)}$ coincides with $\widehat{\Psi}(\widehat{\tau})$, where $\widehat{\tau} \in \mathbf{Nam}_{\widehat{\boldsymbol{w}}}$ by (H). We conclude that $\widehat{p}\,\mathbf{forc}_{\widehat{U}}^{\widehat{\boldsymbol{w}}}\,\exists\, X\, \widehat{\Psi(X)}$, that is, $\widehat{p}\,\mathbf{forc}_{\widehat{U}}^{\widehat{\boldsymbol{w}}}\,\widehat{\Phi}$, as required.

Step (F7). Prove the lemma for an $\mathcal{L}(\boldsymbol{w})$ formula $\Phi := \neg\,\Psi$, assuming that the result holds for Ψ itself. Note that $\#(\Phi) \leqslant n + 1$, hence $m = \#(\Psi) \leqslant n$. Suppose that $\neg\,(p\,\mathbf{forc}_U^{\boldsymbol{w}}\,\neg\,\Psi)$. By definition there is a system $U' \in \mathbf{sDS}_M[m]$, extending U', and a condition $q \in \mathbf{P}[U']{\restriction}\boldsymbol{w}$, $q \leqslant p$, such that $q\,\mathbf{forc}_{U'}^{\boldsymbol{w}}\,\Psi$. Then we have $\widehat{q}\,\mathbf{forc}_{\widehat{U'}}^{\widehat{\boldsymbol{w}}}\,\widehat{\Psi}$ by the inductive hypothesis. However $\widehat{U'}$ belongs to $\mathbf{sDS}_M$, extends $\widehat{U}$, and satisfies $\widehat{U'}{\restriction}^{<n} = U'{\restriction}^{<n}$ by (E), and even belongs to $\mathbf{sDS}_M[m]$ by the choice

of U', and in addition $\widehat{q} \in \mathbf{P}[\widehat{U'}]\restriction\widehat{\boldsymbol{w}}$ and $\widehat{q} \leqslant \widehat{p}$ by (F). We conclude, by definition, that $\neg\,(\widehat{p}\,\mathbf{forc}^{\widehat{\boldsymbol{w}}}_{\widehat{U}}\,\neg\,\widehat{\Psi})$ as well, as required.

□

33. Finalization

We continue to argue under the assumptions and notation of (∗) of Theorem 24.1. The goal of this subsection is to accomplish the proof of Theorem 30.1 that was started in Subsection 31. We return to objects U^1, p_1 introduced in (A), (B), (C) of Subsection 31.

Let $q_1 = p_1\restriction\boldsymbol{w}$, so that $q_1 \in \mathbf{P}[U^1]\restriction\boldsymbol{w}$ and $q_1\ \mathbf{forc}^{\boldsymbol{w}}_{U^1}\ \varphi$ by (A). We have

$$\widehat{U^1} \in \mathbf{sDS}_M[n]\ \wedge\ \widehat{p_1} \in \mathbf{P}[\widehat{U^1}]\ \wedge\ \widehat{q_1} = \widehat{p_1}\restriction\widehat{\boldsymbol{w}} \in \mathbf{P}[\widehat{U^1}]\restriction\widehat{\boldsymbol{w}}\ \wedge\ \widehat{q_1}\ \mathbf{forc}^{\widehat{\boldsymbol{w}}}_{\widehat{U^1}}\ \varphi \tag{12}$$

by Lemma 32.2. (Here φ, as a parameterfree formula, coincides with $\widehat{\varphi}$.) Let system U be defined by $U(k,i) = U^1(k,i) \cup \widehat{U^1}(k,i)$ for all k, i.

Lemma 33.1. *The system U belongs to $\mathbf{sDS}_M[n]$ and extends both U^1 and $\widehat{U^1}$. Conditions p_1 and $\widehat{p_1}$ belong to $\mathbf{P}[U]$ and are compatible in $\mathbf{P}[U]$.*

Proof (Lemma). It follows by (D) (last claim) that U is a disjoint system. It follows by (E) that $U\restriction^{<n} = U^1\restriction^{<n} = \widehat{U^1}\restriction^{<n}$. Therefore U belongs to $\mathbf{sDS}_M[n]$ because so does U^1.

To prove the compatibility, it suffices to check that if $k, i < \omega$ then either $p_1(k,i) = \widehat{p_1}(k,i)$ or $\|p_1\|_{ki} \cap \|\widehat{p_1}\|_{ki} = \varnothing$. If $k < n$ then we have the 'either' case because by definition $p_1\restriction^{<n} = \widehat{p_1}\restriction^{<n}$. Suppose that $k \geq n$. Let $\widehat{k} = \mathbb{h}^{-1}(k)$; thus still $\widehat{k} \geq n$ (as $\mathbb{h} \in \mathbb{H}_n$), $k = \mathbb{h}(\widehat{k})$, and $\widehat{p_1}(k,i) = \langle \widehat{\mathbf{S}_p(\widehat{k},i)}; \widehat{\mathbf{F}_p(\widehat{k},i)} \rangle$. It follows that $\|\widehat{p_1}\|_{ki}$ is the $\mathbb{B}_0$-image, hence the $\mathbb{b}$-image of the set $\|p_1\|_{\widehat{k}i}$. However $\|p_1\|_{\widehat{k}i} \cup \|p_1\|_{ki} \subseteq \|p_1\|$. We conclude that $\|p_1\|_{ki} \cap \|\widehat{p_1}\|_{ki} = \varnothing$ by Claim (B) of Subsection 31, as required. □

To finalize the proof of Theorem 30.1, let, by Lemma 33.1, $r \in \mathbf{P}[U]\restriction\widehat{\boldsymbol{w}}$ satisfy both $r \leqslant p_1\restriction\widehat{\boldsymbol{w}}$ and $r \leqslant \widehat{p_1}\restriction\widehat{\boldsymbol{w}} = \widehat{q_1}$. However $\widehat{q_1}\ \mathbf{forc}^{\widehat{\boldsymbol{w}}}_{\widehat{U^1}}\ \varphi$ by (12). Then we have $r\ \mathbf{forc}^{\widehat{\boldsymbol{w}}}_{U}\ \varphi$ by Lemma 33.1 and Lemma 26.2. On the other hand, $p_1\restriction\widehat{\boldsymbol{w}}\ \mathbf{forc}^{\widehat{\boldsymbol{w}}}_{U^1}\ \neg\,\varphi$ by (A) of Subsection 31, therefore we have $r\ \mathbf{forc}^{\widehat{\boldsymbol{w}}}_{U}\ \neg\,\varphi$. It remains to remind that $\#(\varphi) = n$ and $U \in \mathbf{sDS}_M[n]$ by Lemma 33.1 — and we get a contradiction still by Lemma 26.2(ii). The contradiction completes the proof of Theorem 30.1. □

Theorem 30.1 just proved implies Theorem 24.1, see Subsection 30.
Theorem 24.1 ends the proof of Theorem 20.2 of Subsection 19, see Remark 24.2.
This finalizes the **proof of Theorem 4.1**, see Remark 20.4.

Section VII. Conclusions and discussion

In this study, the method of almost-disjoint forcing was employed to the problem of getting a model of **ZFC** in which the Tarski sentence $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ holds for all m from a pre-selected constructible set $\Upsilon \subseteq \omega$, and fails for all $m \geq 1$ not in Υ (Theorem 4.1). It follows that Tarski's problems (whether $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ holds) not only are independent of **ZFC**, but also independent from each other in a rather strong sense. As the second main result, Theorem 4.2, a model is defined, in which the set $\mathbf{D}_1 = \bigcup_m \mathbf{D}_{1m}$ of all type-theoretically definable reals is equal to the set $\mathbf{L} \cap \mathscr{P}(\omega)$ of all constructible reals, and hence $\mathbf{D}_1 \in \mathbf{D}_2$.

The problem of getting a model for $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$ for different numbers $m \geq 1$ appeared in Alfred Tarski's article [51]. Its the particular case $m = 1$ (analytical definability) the problem of getting models for $\mathbf{D}_{11} \in \mathbf{D}_{21}$, or stronger, $\mathbf{D}_{11} = \mathbf{L} \cap \mathscr{P}(\omega)$, was known in the early years of forcing, see *e.g.* problem 87 in Harvey Friedman's treatise *One hundred and two problems in mathematical logic* [14], and problems 3110, 3111, 3112 in an early survey [47] by A. R. D. Mathias. As mentioned in [14,47], the problem was solved by Leo Harrington, whose handwritten notes [24] at least contain a sketch of a model in which $\mathscr{P}(\omega) \cap \mathbf{L} =$ all $\varDelta^1_3$ reals

From this study, it is concluded that the hidden invariance technique (as outlined in Subsection 29) allows to solve the Tarski "$\mathbf{D}_{1m} \in \mathbf{D}_{2m}$" problem by providing generic **ZFC** models as required by our main results (Theorem 4.1 and Theorem 4.2). The hidden invariance technique has also been applied in our recent papers [30, 32, 33] for the problem of getting a set theoretic structure of this or another kind at a pre-fixed projective level. This is interesting in connection with the following all-important open problem by S. D. Friedman, [15, P. 209] and [16, P. 602]:

Problem 1. Assuming only the consistency of an inaccessible cardinal, is it consistent for each n that all $\mathbf{\Sigma}^1_n$ sets of reals be Lebesgue Measurable and have the Baire and Perfect Set properties, while there is a $\mathbf{\Delta}^1_{n+1}$ wellordering of the reals?[9] □

We finish with a couple of related questions.

Problem 2. If $x \subseteq \omega$ then let $\mathbf{D}_{pm}(x)$ be the set of all objects of type p, definable by a formula with x as the only parameter, and all variables over types $\leq m$. (Compare to Definition 2.1.) One may be interested in getting a model for

$$\forall x \subseteq \omega\, (\mathbf{D}_{1m}(x) \in \mathbf{D}_{2m}(x), \text{ or stronger, } \mathbf{D}_{1m}(x) = \mathscr{P}(\omega) \cap \mathbf{L}[x]). \tag{2.1}$$

This is somewhat similar to Problem 87′ in [14]: find a model of

$$\mathbf{ZFC} + \text{"for any reals } x, y, \text{ we have: } x \in \mathbf{L}[y] \implies x \text{ is } \varDelta^1_3 \text{ in } y\text{"}. \tag{2.2}$$

Problem (2.2) was known in the early years of forcing, see, *e.g.*, problem 3111 in [47] or (3) in [27, Section 6.1]. Problem (2.2) was positively solved by René David in [10], where the question is attributed to Harrington. The proof in [10] makes use of a tool now known as *David's trick*, see S. D. Friedman [15, Chapters 6 and 8].

So far it is unknown whether the result of David [10] generalizes to higher projective classes $\varDelta^1_n$, $n \geq 4$, or $\varDelta^1_\infty$, whether it can be strengthened towards $\iff$ instead of $\implies$, and whether it can lead to even partial solution of (2.1). This is a very interesting and perhaps difficult question. □

Problem 3. Coming back to Harrington's sketch of a model for the sentence

$$\text{the set } d_n = \mathscr{P}(\omega) \cap \varDelta^1_n \text{ of all } \varDelta^1_n \text{ reals is equal to } \mathscr{P}(\omega) \cap \mathbf{L}, \tag{3.1}$$

for $n = 3$ in [24, pp. 2–4], it's clear that, unlike $\mathbf{D}_{1m} \in \mathbf{D}_{2m}$, if (3.1) holds for some $n \geq 3$ then it definitely fails for any $n' \neq n$. But we can try to weaken (3.1) to just

$$d_n \in \varPi^1_n, \tag{3.2}$$

[9] It will be no less interesting to prove the consistency only of BP for $\mathbf{\Sigma}^1_n$ sets with the existence of a $\mathbf{\Delta}^1_{n+1}$ wellordering of the reals assuming only the consistency of **ZFC**. It is known that the LM and Perfect Set properties do need an inaccessible cardinal though.

and then ask whether there is a generic extension of $\mathbf{L}$ satisfying $\forall n\,(d_n \in \varPi^1_n)$. It is known that $d_1 \in \varPi^1_1 \smallsetminus \varSigma^1_1$, $d_2 \in \varSigma^1_2 \smallsetminus \varPi^1_2$, and if all reals are constructible then $d_n \in \varSigma^1_n \smallsetminus \varPi^1_n$ for all $n \geq 3$, so $\varPi^1_n$ looks rather suitable in (3.2). □

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