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On Borel orderable groups

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Abstract

We prove that any Borel Abelian ordered group B, having a countable subgroup G as the largest convex subgroup, and such that the quotient B/G is order isomorphic to $\mathbb{R}$, the reals, is Borel group-order isomorphic to the product $\mathbb{R} \times G$, ordered lexicographically. As a main ingredient of this proof, we show, answering a question of D. Marker, that all Borel cocycles $\mathbb{R}^2 \to \mathbb{Z}$ are Borel coboundaries. A Borel classification theorem for Borel ordered CCC groups is proved.

Keywords: Borel Abelian group; Borel isomorphism; Cocycle

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Introduction

A *Borel Abelian* group (or: BA group) is any Abelian group $G = \langle G; +\rangle$ such that G is a Borel subset of a Polish (complete metric separable) space $\mathcal{X}$ while the group operation is a Borel function from $\mathcal{X}^2$ to $\mathcal{X}$ (or equivalently: the set $\{\langle x, y, z\rangle:\ x + y = z\}$ is a Borel subset of $\mathcal{X}^3$). A BA *ordered* (BAO) group is any BA group $G = \langle G; +, <\rangle$, endowed with a Borel linear order $<$ on G, compatible with the group operation, so that $x < x'$ and $y < y'$ implies $x + y < x' + y'$.

The notions of *group isomorphism* (G-isomorphism), *order isomorphism* (O-isomorphism), and *group order isomorphism* (GO-isomorphism) have obvious meaning. We shall be interested in the case when the isomorphisms are Borel maps (i.e., those with Borel

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graphs). The phrases like: "groups G and G' are G-isomorphic", or "Borel G-isomorphic", or "Borel GO-isomorphic" are understood naturally.

We give [8] as a broad reference in matters of ordered groups.

Clearly G-isomorphic BA groups are not necessarily Borel G-isomorphic. For instance the additive groups of $\mathbb{R}$ [1] and $\mathbb{C}$ are G-isomorphic (as divisible torsion-free groups of the same cardinality) but not Borel G-isomorphic. An example given by Hjorth shows that even GO-isomorphic BAO groups are not necessarily Borel GO-isomorphic (see below). Thus the "Borel" classification of BAO groups should be quite different from the ordinary one. However, some particular cases still admit reasoning which leads to Borel isomorphisms.

Theorem 1. *Suppose that A is a BAO group,* GO*-isomorphic to a group of the form $R \times \mathbb{Z}$, where R is a Borel divisible subgroup of $\mathbb{R}$. Then A is Borel* GO*-isomorphic to $R \times \mathbb{Z}$.* [2]

The proof (Section 1) is rather easy: in this case, any isomorphism is Borel because every $\mathbb{Z}$-interval in A contains a unique element divisible by each natural n. It is an interesting question whether one can replace the condition that A is GO-isomorphic to $R \times \mathbb{Z}$ by a weaker requirement that A is order-isomorphic to $R \times \mathbb{Z}$ as an ordered set. An example (Section 6), based on a nonstandard model of arithmetic, shows that this can be false for instance in the case $R = \mathbb{Q}$ (the rationals). On the other hand, the case $R = \mathbb{R}$ admits the following theorem, which is essentially the main result of this paper:

Theorem 2. *Let B be a BAO group having a countable subgroup G as the largest proper convex subgroup. Suppose that B/G is* O*-isomorphic to $\mathbb{R}$. Then B is Borel* GO*-isomorphic to $\mathbb{R} \times G$ ordered lexicographically.*

The proof of this theorem (Sections 2–5) is not so elementary. We prove, using methods of descriptive set theory, that there is a Borel subgroup $B' \subseteq B$ which has exactly one element in common with every G-coset in B: this quickly leads to Theorem 2. (The first step is to find a Borel set $X \subseteq B$, not necessarily a subgroup, having exactly one element in common with every G-coset in B, which is already a nontrivial fact, based on a classification theorem for Borel equivalence relations, proved in [1].) To prove this selector theorem, we show that all Borel cocycles in $\mathbb{R} \times G$ are Borel coboundaries: this answers a question of Marker [7].

It would be interesting to figure out whether $\mathbb{R}$ can be replaced in Theorems 1 and 2 by another BAO group. Another possible direction of generalization of Theorem 2 is to consider uncountable Borel subgroups G, but this is bounded by a counterexample by Hjorth, see Section 6.

The case of Borel CCC groups (i.e., those which do not admit uncountable sets of pairwise disjoint open intervals [3]) admits a more comprehensive Borel classification,

[1] In this paper, $\mathbb{R}$ always means: the additive group of the reals.

[2] In this paper, all products of ordered groups are assumed to be ordered lexicographically. Subgroups of $\mathbb{R}$ are assumed to be ordered by the usual order of the reals.

[3] For Borel linear orders, CCC is equivalent to separability, see, e.g., Corollary 4.5 in [2].

mainly because for any such a group A and a convex subgroup $C \subseteq A$, the quotient A/C is countable. The next theorem (proved in Section 7) shows that BAO divisible CCC groups can be characterized in terms of certain countable products of Borel subgroups of $\mathbb{R}$. We have to give a few definitions.

For any ordered Abelian group C, $C^{\mathbb{Q}:\mathrm{WO}}$ will be the set of all maps $w \in C^{\mathbb{Q}}$ such that the non-zero domain $|w| = \{q \in \mathbb{Q}\colon\ w(q) \neq 0\}$ is well-ordered as a subset of $\mathbb{Q}$. Then $C^{\mathbb{Q}:\mathrm{WO}}$ is an Abelian ordered group, with componentwise addition and lexicographical order. In this case, a subgroup $W \subseteq C^{\mathbb{Q}:\mathrm{WO}}$ will be called *local-product* if for any $w \in W$ and $q_0 \in \mathbb{Q}$, the function $w' \in C^{\mathbb{Q}:\mathrm{WO}}$, defined by $w'(q_0) = w(q_0)$ while $w'(q) = 0$ for any $q \neq q_0$, belongs to W.[4]

Theorem 3. *Assume that $A = \langle A; +, < \rangle$ is a BAO divisible* CCC *group. Then A is Borel* GO*-isomorphic to one of the following*:

(i) *a Borel local-product subgroup W of $C^{\mathbb{Q}:\mathrm{WO}}$, where C is a countable divisible subgroup of $\mathbb{R}$, satisfying the following property*: *for any $q \in \bigcup_{w \in W} |w|$, the "projection" $\{w \restriction (-\infty, q]\colon\ w \in W\}$ is at most countable*;

(ii) *a lexicographical product of the form $W \times B$, where B is an uncountable Borel divisible subgroup of $\mathbb{R}$,[5] while W is a countable local-product divisible subgroup of $C^{\mathbb{Q}:\mathrm{WO}}$, C being a countable divisible subgroup of $\mathbb{R}$.*

Note that any group of type (i) or (ii) is clearly a CCC group. In addition, types (i) and (ii) are disjoint: indeed, any group of type (ii) contains an uncountable Archimedean convex subgroup $\{0\} \times B$, which is impossible for those of type (i). Examples for (ii) are trivial. As for (i), consider the subgroup $W \subseteq \mathbb{Q}^{\mathbb{Z}}$, which consists of those $\mathbb{Z}$-sequences $w = \{q_z\}_{z \in \mathbb{Z}}$ satisfying the property that the set $|w| = \{z\colon\ q_z \neq 0\} \subseteq \mathbb{Z}$ has only finitely many elements below any $z_0 \in \mathbb{Z}$.

1. Proof of Theorem 1

Thus let $A = \langle A; +, < \rangle$ be a BAO group, GO-isomorphic to $G \times \mathbb{Z}$, where G is a Borel divisible subgroup of $\mathbb{R}$, via a GO-isomorphism F. Prove that A is Borel GO-isomorphic to $G \times \mathbb{Z}$. We actually prove that F itself must be a Borel map.

For $x, y \in A$, let $x \approx y$ mean that $x - y \in \mathbb{Z}$. Then $\approx$ is a Borel equivalence relation. Note that the set $S = \{F(r, 0)\colon\ r \in G\} \subseteq A$ has exactly one point in common with each $\approx$-class. Thus, it suffices to check that S is a Borel set.

To see this note that the elements $x \in S$ are only those (among all $x \in A$) which are divisible in A by any natural n. This yields a Borel definition for S. $\square$

[4] Then, given a finite set $q_1 < q_2 < \cdots < q_k$ of rationals, $w \in W$, and any $c_i \in W(q_i) = \{w(q_i)\colon\ w \in W\}$, the function w', which differs from w only in its values $w'(q_i) = c_i$, $i = 1, \ldots, k$, belongs to W. Yet W is not necessarily a product of the form $\prod_{q \in \mathbb{Q}} W_q$.

[5] That is, a subgroup of the additive group of $\mathbb{R}$.

It would be interesting to figure out which conditions in this simple theorem are really necessary, in particular, the requirement that G is divisible.

On the other hand, the requirement, that A is GO-isomorphic to $G \times \mathbb{Z}$, apparently cannot be weakened to the following: A is O-isomorphic to $G \times \mathbb{Z}$ as an ordered set, even in the case $G = \mathbb{Q}$, see Section 6.

2. Borel selector theorem and the proof of Theorem 2

Our proof of Theorem 2 is based on the following theorem (the "Borel selector theorem" of the title).

Theorem 4. *Let B and G be as in Theorem* 2. *Then there is a Borel subgroup $B' \subseteq B$ which has exactly one element in common with each G-coset in B.*

(A *G-coset* is any set of the form $b + G$, where $b \in B$.) Let us show how this implies Theorem 2. We apply the following simple lemma.

Lemma 5. *Any archimedean BAO group B', order isomorphic to $\mathbb{R}$, is Borel* GO*-isomorphic to $\langle \mathbb{R}; + \rangle$ (i.e., the additive group of $\mathbb{R}$).*

Proof. Prove first that B' is divisible. Indeed, suppose that $n \geqslant 2$ and $a \in B'$ is, say, B'-positive but there is no $x \in B'$ such that $nx = a$ in B'. Then the sets $X = \{x \in B'\colon nx < a\}$ and $Y = \{y \in B'\colon ny > a\}$ form a partition of B' such that every $x \in X$ is $<$ than any $y \in Y$. Since B' is order isomorphic to $\mathbb{R}$, either X has a maximal element or Y has a minimal element. Consider the first case and let x be the largest element of X. (Clearly x is B-positive.) Then $nx < a < ny$ for any $y > x$ in B'. It follows that the difference $d = a - nx > 0$ in B' satisfies the requirement that $nz > d$ for any positive $z \in B'$. Now, using again the fact that B' is order isomorphic to $\mathbb{R}$, we present d in the form $d = d_1 + \cdots + d_n$, where each $d_i \in B'$ is (strictly) B-positive. To get a contradiction, it remains to take, as z, the B-least among $d_1, \ldots, d_n$.

Now fix any B-positive element $e \in B'$. Then $qe \in B'$ is well-defined in B' for any rational q. Furthermore the set $E = \{qe\colon q \in \mathbb{Q}\}$ is cofinal and coinitial in B' since the subgroup is Archimedean.

Prove that E is dense in B' (in the order sense). Indeed otherwise there are elements $0 < a < b$ in B' such that the interval $[a, b]$ does not intersect E. Then the difference $d = b - a$ satisfies $qe > d$ in B' for any rational $q > 0$. It follows that $mq < e$ in B' for any natural m, a contradiction since B' is Archimedean.

Now define $H(q) = qe$ for any rational q. If $x \in \mathbb{R}$ is irrational then let $H(x)$ be the only element of B' such that $H(x) > qe$ whenever $q < x$ is rational and $H(x) < qe$ whenever $q > x$ is rational. It follows from the above that H is a Borel GO-isomorphism $\mathbb{R} \xrightarrow{\text{onto}} B'$. $\square$

The subgroup B', given by Theorem 4, is a BAO group ordered similarly to $\mathbb{R}$. Moreover B' is archimedean since B has $\mathbb{Z}$ as the largest convex subgroup. It remains to apply Lemma 5. $\square$

3. Preliminaries for Theorem 4: reduction to cocycles

Let $B = \langle B; +_B \rangle$ and $G \subseteq B$ be as in Theorems 2 and 4.

Lemma 6. *There is a Borel set $X \subseteq B$ which has exactly one common element with each G-coset in B.*

Proof. Consider a Borel equivalence relation: $a \mathsf{E} b$ iff $a - b \in G$, on B. It follows from the Glimm–Effros dichotomy theorem of Harrington, Kechris, and Louveau [1], that E satisfies one (and only one) of the two following requirements:

(i) E is *smooth*, i.e., there is a Borel map $F : B \to \mathbb{R}$ such that we have $a \mathsf{E} b \Leftrightarrow F(a) = F(b)$ for all $a, b \in B$.

(ii) The Vitali equivalence relation E_0 on $2^{\mathbb{N}}$ [6] is Borel reducible to E, so that there is a Borel map $F : 2^{\mathbb{N}} \to B$ such that $x \mathsf{E}_0 y \Leftrightarrow F(x) \mathsf{E} F(y)$.

Note that (ii) would imply that there is a Borel linear ordering of the set of all E_0-classes (induced by the order of B), which is known to be impossible. [7] Thus we have (i). Now, as the E-equivalence classes (i.e., G-cosets) are countable, the lemma follows from a classical theorem of descriptive set theory. [8] $\square$

Let us fix such a Borel set X. For $a, b \in X$, let $a * b$ be the only element of X which belongs to the same G-coset in B as $a +_B b$. Then clearly $\langle X; * \rangle$ is a BAO group (perhaps not a subgroup of B), order isomorphic to B/G, hence, to $\mathbb{R}$. It follows that $\langle X; * \rangle$ is Borel GO-isomorphic to $\langle \mathbb{R}; + \rangle$ by Lemma 5. Let $i : \mathbb{R} \xrightarrow{\text{onto}} X$ be a Borel isomorphism.

From now on let $+$ and $-$ denote the real number addition and subtraction. For $x, y \in \mathbb{R}$, let $f(x, y) = i(x) +_B i(y) -_B i(x + y)$. Thus $f(x, y) \in B$ and, moreover, it follows from the choice of i and X that in fact $f(x, y) \in G$ because $i(x) +_B i(y)$ and $i(x + y)$ belong to the same G-coset of B. We also have $f(x, y) = f(y, x)$ and

$$f(x, y) +_B f(x + y, z) = f(x, y + z) +_B f(y, z) \quad \text{for all } x, y, z \in \mathbb{R}. \tag{1}$$

Thus f is a *cocycle* $\mathbb{R}^2 \to G$.

Given a map $h : \mathbb{R} \to G$, the function $f_h(x, y) = h(x) +_B h(y) -_B h(x + y)$ is clearly a cocycle (i.e., it satisfies (1) and $f_h(x, y) = f_h(y, x)$). Cocycles of the form f_h are called *coboundaries*.

[6] For $x, y \in 2^{\mathbb{N}}$, $x \mathsf{E}_0 y$ means that the set $\{n\colon x(n) \neq y(n)\}$ is finite.

[7] This fact was first observed perhaps by Sierpiński [9]. We refer the reader to Kanovei [3] for a simple proof.

[8] This theorem says the following. Let P be a Borel subset of the product $X \times Y$ of complete separable metric spaces X, Y. Suppose that for any $x \in X$ there is at most countably many $y \in Y$ such that $\langle x, y \rangle \in P$. Then P can be presented as a union of the form $P = \bigcup_n P_n$, where each P_n is a Borel set such that any $x \in X$ there is at most one $y \in Y$ satisfying $\langle x, y \rangle \in P_n$. See Kechris [6]. We apply it to the set $P = \{\langle x, y \rangle\colon y \in B \text{ and } x = F(y)\}$.

This vocabulary allows us to add some generality to our considerations.

Theorem 7. *Suppose that G is a countable Abelian group. Let $f:\mathbb{R}^2 \to G$ be a Borel cocycle.* (*That is, it satisfies 1 for $+_G$ and $f(x,y) = f(y,x)$.*) *Then $f = f_h$ for a Borel map $h:\mathbb{R} \to G$.*

Thus Borel cocycles are Borel-generated coboundaries. The question answered by this theorem for $G = \mathbb{Z}$ (the integers) was suggested to us by Marker [7].

To show that this implies Theorem 4, let $h:\mathbb{R} \to G$ be a Borel map given by Theorem 7: so that we have

$$i(x) +_B i(y) -_B i(x+y) = h(x) +_B h(y) -_B h(x+y) \quad \text{for all } x, y \in \mathbb{R}. \tag{2}$$

Define $H(x) = i(x) -_B h(x)$, for $x \in \mathbb{R}$. It is clear that $B' = \{H(x)\colon x \in \mathbb{R}\}$ is still a Borel subset of B having exactly one common element with each G-coset. Moreover, B' is a group because $H(x) +_B H(y) = H(x+y)$ by (2). □

4. Main lemmas for the proof of Theorem 7

Fix $G = \langle G; +_G, 0_G \rangle$ and f as in Theorem 7. Let $z \in \mathbb{R}$ effectively code the Borel map f. Fix a countable transitive set $\mathfrak{M}$, which contains z and G and models a large finite fragment Φ of ZFC. [9]

Let COH be the Cohen forcing, viewed as the set of all non-empty rational open intervals (a,b) in $\mathbb{R}$. (Smaller intervals are stronger conditions.) Fix a pair of rational intervals I and J of $\mathbb{R}$ such that I contains only positive reals and is shorter than J, and $I \times J$ COH^2-forces, over $\mathfrak{M}$, that $f(\dot{a}, \dot{b}) = \hat{g}$, for a fixed $\hat{g} \in G$, where $\dot{a}$ and $\dot{b}$ are the names for generic reals in the sense of COH^2. [10]

We need some additional notation. Define $f(x,y,z) = f(x,y) +_G f(x+y,z)$: this is invariant under any permutation within $\{x,y,z\}$ by (1). Define

$$f(x_1,\ldots,x_n,x_{n+1}) = f(x_1,\ldots,x_n) +_G f(x_1+\cdots+x_n, x_{n+1}), \tag{3}$$

by induction, so that $f(x_1,\ldots,x_n)$ is invariant under any permutation within the set $\{x_1,\ldots,x_n\}$. The meaning of this extended version of f is quite transparent:

$$f(x_1,\ldots,x_n) = i(x_1) +_B \cdots +_B i(x_n) -_B i(x_1+\cdots+x_n),$$

assuming f is defined by $f(x,y) = i(x) +_B i(y) -_B i(x+y)$, as in Section 3. Let, in addition, $f(z_1) = 0_G$ for any single z_1, for "arity" 1. It easily follows that

$$\begin{aligned} &f(x_1,\ldots,x_n,y_1,\ldots,y_k) \\ &\quad = f(x_1,\ldots,x_n) +_G f(y_1,\ldots,y_k) +_G f(x_1+\cdots+x_n, y_1+\cdots+y_k). \end{aligned} \tag{4}$$

[9] Let Φ contain first one million of the ZFC axioms and the schemata for Σ_{100} formulas.

[10] The use of forcing notation is mainly a figure of speech here. The given description of I, J has the following meaning. If a pair $\langle a,b \rangle \in I \times J$ does not belong to any closed nowhere dense subset of $I \times J$, having a code in $\mathfrak{M}$, then $f(a,b) = \hat{g}$.

(Let, for brevity, $\boldsymbol{x}$ denote the string $x_1,\dots,x_n$ and $s = x_1 + \cdots + x_n$. Argue by induction on k. For $k = 1$ apply (3). To carry out the step suppose that

$$f(\boldsymbol{x}, y_1,\dots,y_{k-1}) = f(\boldsymbol{x}) +_G f(y_1,\dots,y_{k-1}) +_G f(s, y_1 + \cdots + y_{k-1}).$$

Adding $f(s + y_1 + \cdots + y_{k-1}, y_k)$, we get $f(\boldsymbol{x}, y_1,\dots,y_k)$ on the left, and

$$f(\boldsymbol{x}) +_G f(y_1,\dots,y_{k-1}) +_G f(y_k, y_1 + \cdots + y_{k-1}) +_G f(s, y_1 + \cdots + y_k)$$

on the right by (1), which equals the right-hand side of (4) by (1).)

Lemma 8. *Let $x_1,\dots,x_n, y_1,\dots,y_n \in I$ be* COH*-generic*[11] *reals over $\mathfrak{M}$, such that $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. Then $f(x_1,\dots,x_n) = f(y_1,\dots,y_n)$.*

Proof. Argue by induction on n. We start with $n = 2$. Let $x, y, x', y' \in I$ be COH-generic over $\mathfrak{M}$, and $x + y = x' + y'$; prove that $f(x, y) = f(x', y')$.

Let us suppose that $x < x' < y' < y$. As I is shorter, there is a real $\alpha \in J$, COH-generic over $\mathfrak{M}[x, x', y, y']$,[12] such that $\alpha' = \alpha + (x' - x) \in J$. Note that each of the pairs $\langle x, \alpha'\rangle$, $\langle y, \alpha\rangle$, $\langle x', \alpha\rangle$, $\langle y', \alpha'\rangle$, is COH^2-generic over $\mathfrak{M}$. Therefore

$$f(x, y, \alpha, \alpha') = f(x, \alpha') +_G f(y, \alpha) +_G f(x + \alpha', y + \alpha) = 2\hat{g} +_G f(\gamma, \gamma'),$$

$$f(x', y', \alpha, \alpha') = f(x', \alpha) +_G f(y', \alpha') +_G f(x' + \alpha, y' + \alpha') = 2\hat{g} +_G f(\gamma, \gamma')$$

by (4), where $\gamma = x + \alpha' = x' + \alpha$ and $\gamma' = y + \alpha = y' + \alpha'$, so that $f(x, y, \alpha, \alpha') = f(x', y', \alpha, \alpha')$. However, on the other hand, we have

$$f(x, y, \alpha, \alpha') = f(x, y) +_G f(\alpha, \alpha') +_G f(x + y, \alpha + \alpha'), \quad \text{and}$$

$$f(x', y', \alpha, \alpha') = f(x', y') +_G f(\alpha, \alpha') +_G f(x' + y', \alpha + \alpha'),$$

so that $f(x, y) = f(x', y')$ because $x + y = x' + y'$.

We carry out the step. Assume that $x_1 + \cdots + x_n + x_{n+1} = y_1 + \cdots + y_n + y_{n+1}$. Consider first the case when $x_{n+1} = y_{n+1}$. Then $x_1 + \cdots + x_n = y_1 + \cdots + y_n$, hence $f(x_1,\dots,x_n) = f(y_1,\dots,y_n)$ by the assumption. On the other hand, by definition,

$$f(x_1,\dots,x_n, x_{n+1}) = f(x_1,\dots,x_n) +_G f(x_1 + \cdots + x_n, x_{n+1}),$$

and the same for $f(y_1,\dots,y_n, y_{n+1})$, as required.

Consider the general case. Assume that x_1 and y_1 are the smallest while x_{n+1} and y_{n+1} the largest among respectively x_i, y_i. Let, for instance, $x_1 < y_1$. Let $\varepsilon > 0$ be a real, COH-

[11] A real is COH-generic over $\mathfrak{M}$ if it does not belong to any closed nowhere dense set of reals having a code in $\mathfrak{M}$. To define this in a more classical way would mean to specify a complicated list of countably many relevant nowhere dense closed sets.

[12] $\mathfrak{M}[x_1,\dots,x_n]$ will denote a countable transitive model of the fragment of ZFC introduced in footnote 11, containing the reals $x_1,\dots,x_n$ and all sets in $\mathfrak{M}$. We do not bother here that $\mathfrak{M}[x_1,\dots,x_n]$ is not uniquely defined and may contain more ordinals than $\mathfrak{M}$ does. Note that if a real x is COH-generic over $\mathfrak{M}[x_1,\dots,x_n]$ then each pair $\langle x, x_i\rangle$ is COH^2-generic over $\mathfrak{M}$. It is not so clear how to carry out this argument classically in forcing-free terms.

generic over $\mathfrak{M}[x_1, y_1, \ldots, x_{n+1}, y_{n+1}]$, satisfying $\varepsilon < y_1 - x_1$, and such that $y_{n+1} + \delta$ still belongs to I, where $\delta = y_1 - x_1 - \varepsilon$. Define x_i' and y_i' so that

$$x_1' = x_1 + \varepsilon, \quad x_{n+1}' = x_{n+1} - \varepsilon, \quad y_1' = y_1 - \delta, \quad y_{n+1}' = y_{n+1} + \delta,$$

(these reals are COH-generic over $\mathfrak{M}$ by the choice of ε), while $x_k' = x_k$ and $y_k' = y_k$ for $2 \leqslant k \leqslant n$. Thus, $x_2 = x_2'$ and $y_2' = y_2$, so, by the particular case,

$$f(x_1, \ldots, x_{n+1}) = f(x_1', \ldots, x_{n+1}') \quad \text{and} \quad f(y_1, \ldots, y_{n+1}) = f(y_1', \ldots, y_{n+1}').$$

Similarly, $f(y_1', \ldots, y_{n+1}') = f(x_1', \ldots, x_{n+1}')$, because $y_1' = x_1'$ by definition. □

Lemma 9. *Assume that $1 \leqslant k < n$, $1 \leqslant k' < n'$, and reals $x_1, \ldots, x_n, y_1, \ldots, y_k \in I$ and $x_1', \ldots, x_{n'}', y_1', \ldots, y_{k'}' \in I$ are* COH*-generic over $\mathfrak{M}$. Suppose further that*

$$x_1 + \cdots + x_n = y_1 + \cdots + y_k = s \quad \textit{and} \quad x_1' + \cdots + x_{n'}' = y_1' + \cdots + y_{k'}' = s'.$$

Then $(n' - k')[f(x_1, \ldots, x_n) -_G f(y_1, \ldots, y_k)] = (n - k)[f(x_1', \ldots, x_{n'}') -_G f(y_1', \ldots, y_{k'}')]$.

(If $g \in G$ and $m \in \omega$ then mg denotes the G-sum of m copies of g.)

Proof. If z is a string of reals (perhaps, containing only one term) then $z^{[m]}$ will denote the concatenation of m-many copies of z. Let $\boldsymbol{x}$ denote the string $x_1, \ldots, x_n$. Let $\boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}'$ have analogous meaning. Note that $f(\boldsymbol{x}^{[n'-k']}, \boldsymbol{y}'^{[n-k]}) = f(\boldsymbol{x}'^{[n-k]}, \boldsymbol{y}^{[n'-k']})$ by Lemma 8. (The strings to which f is applied have $nn' - kk'$ terms and the sum equal to $(n' - k')s + (n - k)s'$ each.) It follows from (4) that the left-hand side and the right-hand side of the last equality are equal respectively to

$$f\big(\boldsymbol{x}^{[n'-k']}\big) +_G f\big(\boldsymbol{y}'^{[n-k]}\big) +_G f\big((n' - k')s, (n - k)s'\big);$$
$$f\big(\boldsymbol{x}'^{[n-k]}\big) +_G f\big(\boldsymbol{y}^{[n'-k']}\big) +_G f\big((n - k)s', (n' - k')s\big);$$

so that

$$f\big(\boldsymbol{x}^{[n'-k']}\big) +_G f\big(\boldsymbol{y}'^{[n-k]}\big) = f\big(\boldsymbol{x}'^{[n-k]}\big) +_G f\big(\boldsymbol{y}^{[n'-k']}\big). \tag{$*$}$$

It follows from (4), by induction on m, that $f(\boldsymbol{x}^{[m]}) = mf(\boldsymbol{x}) +_G f(s^{[m]})$ and $f(\boldsymbol{y}^{[m]}) = mf(\boldsymbol{y}) +_G f(s^{[m]})$ for any m; hence

$$f\big(\boldsymbol{x}^{[n'-k']}\big) -_G f\big(\boldsymbol{y}^{[n'-k']}\big) = (n' - k')\big(f(\boldsymbol{x}) -_G f(\boldsymbol{y})\big).$$

Similarly, $f(\boldsymbol{x}'^{[n-k]}) -_G f(\boldsymbol{y}'^{[n-k]}) = (n - k)(f(\boldsymbol{x}') -_G f(\boldsymbol{y}'))$. We conclude, by $(*)$, that $(n' - k')(f(\boldsymbol{x}) -_G f(\boldsymbol{y})) = (n - k)(f(\boldsymbol{x}') -_G f(\boldsymbol{y}'))$, as required. □

5. Proof of Theorem 7

We are going to prove that $f = f_h$, i.e., $f(x, y) = h(x) +_G h(y) -_G h(x + y)$, where a Borel "shift" $h : \mathbb{R} \to G$ is a superposition of three more elementary Borel maps.

There is a big enough natural m such that there exist reals $x, y \in I$, COH-generic over $\mathfrak{M}$ and satisfying $my = (m+1)x$. By Lemma 9, the element $q' = f(x^{[m+1]}) -_G f(y^{[m]}) \in G$ (hence $\in \mathfrak{M}$) does not depend on the choice of m, x, y, and we have $f(x_1, \dots, x_n) -_G f(y_1, \dots, y_k) = (n-k)q'$ whenever $1 \leqslant k \leqslant n$ and the reals $x_i, y_j \in I$ are COH-generic over $\mathfrak{M}$ and satisfy $x_1 + \dots + x_n = y_1 + \dots + y_k$.

Step 1. Put $h_1(x) = -_G q'$, $\forall x$. Let $f_1(x, y) = f(x, y) +_G f_{h_1}(x, y) = f(x, y) -_G q'$.

Corollary 10. *Assume that reals $x_1, \dots, x_n, y_1, \dots, y_k \in I$ are* COH*-generic over $\mathfrak{M}$, and $x_1 + \dots + x_n = y_1 + \dots + y_k$. Then $f_1(x_1, \dots, x_n) = f_1(y_1, \dots, y_k)$.*

Proof. Let, for instance, $k < n$. Note that $f_{h_1}(z_1, \dots, z_m) = -_G (m-1)q'$, hence $f_1(x_1, \dots, x_n) -_G f_1(y_1, \dots, y_k) = f(x_1, \dots, x_n) -_G f(y_1, \dots, y_k) -_G (n-k)q' = 0_G$. $\square$

Recall that $I = (a, b)$, a rational interval in $\mathbb{R}$, lies to the right of 0. Define $nI = (na, nb)$. There is a real $C > b > 0$ such that $[C, +\infty) \subseteq \bigcup_n nI$.

Let $x \geqslant C$. Then $x = x_1 + \dots + x_n$ for some reals $x_1, \dots, x_n \in I$, COH-generic over $\mathfrak{M}$. We consistently define, using Corollary 10, $F(x) = f_1(x_1, \dots, x_n)$. Clearly (the graph of) F is analytic, therefore $F : [C, +\infty) \to G$ is a Borel function.

Step 2. Put $h_2(x) = F(x)$ for $x \geqslant C$ and $h_2(x) = 0_G$ for $x < C$. In particular $h_2(x) = 0_G$ for $x \in I$. Let $f_2(x, y) = f_1(x, y) +_G f_{h_2}(x, y)$. Easily $f_2(x_1, \dots, x_n) = 0_G$ for all COH-generic reals $x_1, \dots, x_n \in I$ such that $x_1 + \dots + x_n \geqslant C$.

Lemma 11. $f_2(x, y) = 0_G$ *for all* $x, y \geqslant C$.

Proof. Let $x = x_1 + \dots + x_n$ and $y = y_1 + \dots + y_k$, where $x_i, y_j \in I$ are COH-generic over $\mathfrak{M}$. It follows from (4) that

$$f_2(x_1, \dots, x_n, y_1, \dots, y_k) = f_2(x_1, \dots, x_n) +_G f_2(y_1, \dots, y_k) +_G f_2(x, y).$$

But $f_2(x_1, \dots, x_n, y_1, \dots, y_k) = f_2(x_1, \dots, x_n) = f_2(y_1, \dots, y_k) = 0_G$ by the above. $\square$

Step 3. Let $C_x = \max\{C, C - x\}$. Define $h_3(x) = -_G f_2(x, C_x)$, so that

$$f_{h_3}(x, y) = -_G f_2(x, C_x) -_G f_2(y, C_y) +_G f_2(x + y, C_{x+y}), \tag{$\star$}$$

and put $f_3(x, y) = f_2(x, y) +_G f_{h_3}(x, y)$.

Lemma 12. $f_3(x, y) = 0_G$ *for all* x, y.

Proof. For any z, we have $f_3(x, y) = f_3(x, z) +_G f_3(x + z, y) -_G f_3(x + y, z)$. By ($\star$), this transforms straightforwardly to

$$\begin{aligned} &f_2(x, z) +_G f_2(x + z, y) -_G f_2(x + y, z) -_G f_2(x, C_x) \\ &\quad -_G f_2(y, C_y) +_G f_2(x + y, C_{x+y}). \end{aligned}$$

Take $z = \max\{C_x, C_{x+y}, C_y - x\}$. Then, in particular,

$$f_2(x, z) -_G f_2(x, C_x) = f_2(x + z, C_x) - f_2(x + C_x, z) = 0$$

by Lemma 11. Each of the other two pairs gives 0 analogously. □

To accomplish the proof of Theorem 7, note that the map h_1 is obviously Borel, h_2 is Borel because F is Borel (see above), so that f_2 and h_3 are Borel, too. However f is equal to $-f_{h_3}$ by Lemma 12, so that f is a Borel-generated coboundary. □

6. Two counterexamples

This section presents two counterexamples which show that Theorem 1 cannot be easily generalized in certain directions.

A counterexample order isomorphic to $\mathbb{Q} \times \mathbb{Z}$

Proposition 13. *There is an abelian ordered group A, such that $\mathbb{Z}$ is the only proper convex subgroup of A and $A/\mathbb{Z}$ GO-isomorphic to $\mathbb{Q}$ (hence A is O-isomorphic to $\mathbb{Q} \times \mathbb{Z}$ as an ordered set), but not G-isomorphic to $\mathbb{Q} \times \mathbb{Z}$.*

Proof. We make use of a nonstandard model M of Peano arithmetic. Adding the negative part $-M$ appropriately, we obtain an Abelian group $G = M \cup -M$. For $x, y \in G$, define $x \approx y$ iff $x - y \in \mathbb{Z}$. Note that there exists an $\approx$-class X such that none of $x \in X$ is divided by 2^n for all finite n. (Indeed, fix an infinitely large $m \in M$. The $\approx$-class X of the number $x \in M$, closest to the fraction $2^m/3$, is as required.) To see that $A = \bigcup_{q \in \mathbb{Q}} qX$ is not group isomorphic to $\mathbb{Q} \times \mathbb{Z}$ note that the product $\mathbb{Q} \times \mathbb{Z}$ contains, in each $\mathbb{Z}$-interval $\{q\} \times \mathbb{Z}$, an element $x = \langle q, 0\rangle$ divided in $\mathbb{Q} \times \mathbb{Z}$ by any number 2^n, $n \in \mathbb{N}$, while on the other hand X, which is a $\mathbb{Z}$-interval in A, does not contain any element x of this kind. □

A counterexample with uncountable convex subgroup

The following example [13] shows that Theorem 2 fails, generally speaking, for uncountable Borel convex subgroups G. We consider $\mathbb{R}^2$ as the product of two copies of the additive group of the reals. Define $\mathrm{pr}_X A = \{x\colon \exists y\ (\langle x, y\rangle \in A)\}$ and $\mathrm{pr}_Y A = \{y\colon \exists x\ (\langle x, y\rangle \in A)\}$ for any set $A \subseteq \mathbb{R}^2$.

Proposition 14. *There is a Borel subgroup A of $\mathbb{R}^2$ such that*

(i) $\mathrm{pr}_X A = \mathbb{R}$;
(ii) *for any real c, A does not completely include the line $y = cx$.*

Proof. Let $Y \subseteq \mathbb{R}$ be an uncountable closed set such that $q_1 y_1 + \cdots + q_n y_n \neq 0$ whenever $q_1, \ldots, q_n \in \mathbb{Q} \setminus \{0\}$ while $y_1, \ldots, y_n$ are pairwise different elements of Y. (In particular

[13] Communicated by G. Hjorth in May 1998 and presented here with his permission.

$0 \notin Y$.) Let F be a Borel 1–1 map of $\mathbb{R}$ onto Y. Define A to be the $\mathbb{Q}$-closure of the graph of F, that is, the set of all points of the form

$$\langle q_1x_1 + \cdots + q_nx_n,\ q_1F(x_1) + \cdots + q_nF(x_n)\rangle \in \mathbb{R}^2,$$

where $q_1, \ldots, q_n \in \mathbb{Q}$ while $x_1, \ldots, x_n \in \mathbb{R}$. Clearly A is a Borel group satisfying (i). Let us show that (ii) also holds. First of all A does not contain any point of the form $\langle x, 0\rangle$, except for $\langle 0, 0\rangle$. Now let $c \neq 0$. If A entirely includes the line $y = cx$ then $\mathrm{pr}_Y A = \mathbb{R}$. Then clearly Y is a Borel basis of $\mathbb{R}$ as a vectorspace over $\mathbb{Q}$, which is impossible.[14] □

Assume that A is such a group. Then $A_0 = \{y\colon \langle 0, y\rangle \in A\}$ is a Borel subgroup of $\mathbb{R}$ since A is a group. We assert that A is GO-isomorphic to $\mathbb{R} \times A_0$ viewed as a lexicographically ordered Borel group: then in particular, A_0 is the only proper convex subgroup of A. To prove the assertion it suffices to define an additive map (homomorphism) $f : \mathbb{R} \to \mathbb{R}$ such that $\langle x, f(x)\rangle \in A$ for any x. (Then the map sending any $\langle x, y\rangle \in \mathbb{R} \times A_0$ to $\langle x, f(x) + y\rangle$ is an isomorphism of $\mathbb{R} \times A_0$ onto A, as required.) To define such a map f, let us first of all choose a set $B \subseteq \mathbb{R}$ which is a Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vectorspace. The values $f(b)$ for $b \in B$ can be chosen arbitrarily. Then, any $x \in \mathbb{R} \setminus B$ admits a unique presentation in the form $x = r_1b_1 + \cdots + r_mb_m$, where r_i are rationals and $b_i \in B$. In this case define $f(x) = r_i f(b_1) + \cdots + r_m f(b_m)$.

However, A and $\mathbb{R} \times A_0$ are not Borel isomorphic even as groups! Indeed, assume that $F : A \xrightarrow{\text{onto}} \mathbb{R} \times A_0$ is a Borel group isomorphism. Then $F(\langle x, 0\rangle) = \langle f(x), g(x)\rangle$ for any x, where $f, g : \mathbb{R} \to \mathbb{R}$ are Borel homomorphisms (i.e., $f(x+y) = f(x) + f(y)$ and similarly for g), and, by (ii), there is no c such that $g(x) = cf(x)$ for all x. In this case, there is a real c such that the sets

$$X^+ = \{x > 0\colon\ f(x) > cg(x)\} \quad \text{and} \quad X^- = \{x > 0\colon\ f(x) < cg(x)\}$$

are non-empty. Of those at least one set is co-meager on an interval $[a, b]$, where $0 < a < b$. Let this be, e.g., X^+. A simple argument shows that each real $z > 0$ has the form $z = rx + qy$, where r, q are positive rationals while $x, y \in [a, b]$, so that $z \in X^+$ as well. It follows that X^- is empty, a contradiction.

7. CCC **groups**

It turns out that the difference between (i) and (ii) of Theorem 3 can be traced down to the structure of *galaxies*—convex subgroups of A, the given group, of the form $\bigcup_n [-nx, nx]$, where $x \in A$. By the CCC assumption, A cannot contain a countable galaxy other than $\{0\}$ (unless A itself is countable)—and then the type of A is (i) in the case when there is no minimal galaxy, and (ii) otherwise. (In the "otherwise" case, B is just the minimal non-$\{0\}$ galaxy in A.)

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[14] If Y contains a rational r then the $\mathbb{Q}$-closure of $Y \setminus \{r\}$ is a Borel selector for the Vitali equivalence relation, which is impossible. If Y does not contain a rational then $1 = q_1y_1 + \cdots + q_ny_n$ for some $y_i \in Y$ and rationals $q_i \neq 0$. Replace q_1 by 1 in Y, getting the first case.

The assumption that A is divisible cannot be dropped. Indeed, there is (Section 6) an Abelian ordered group A, order isomorphic but not group isomorphic to $\mathbb{Q} \times \mathbb{Z}$. If it were of type (i) (but non-divisible), then, as A has only one proper convex subgroup, A would be a subgroup of $C \times C$ for a countable group $C \subseteq \mathbb{R}$. But this easily leads to isomorphism between A and $\mathbb{Q} \times \mathbb{Z}$, which is a contradiction.

Another simple argument shows that $C \subseteq \mathbb{R}$ cannot be one and the same countable group for any A in (i) or (ii). As a counterexample, take, as A, a countable divisible subgroup of $\mathbb{R}$, not GO-isomorphic to any subgroup of C.

Beginning the proof of Theorem 3, let us assume that $A = \langle A; +, < \rangle$ is a BAO CCC group. As A is divisible, any convex subgroup $H \subseteq A$ and the corresponding quotient A/H are divisible (Abelian ordered) groups. Let, for H a convex subgroup of A, *H-coset* or *coset of size H* mean a subset of A of the form $a + H$, where $a \in A$. *Coset* will mean H-coset for some convex subgroup $H \subsetneq A$.

Lemma 15. *For any coset X, a representative $r(X) \in X$ can be chosen so that*
(a) $r(X) + r(Y) = r(X + Y)$ *for any two cosets X, Y of equal size*;
(b) *if $X' \subseteq X$ and $r = r(X) \in X'$ then $r(X') = r$.*

Proof. A *partial representative function*, or PRF, is any function F such that
(i) the domain $\mathcal{X} = \operatorname{dom} F$ consists of cosets and $F(X) \in X$ for any X;
(ii) if $X \in \mathcal{X}$, $X \subseteq Y$, and Y is a coset then $Y \in \mathcal{X}$;
(iii) if $X \in \mathcal{X}$ then any coset $Y \subsetneq X$, such that $F(X) \in Y$, belongs to $\mathcal{X}$, too, and $F(X) = F(Y)$;
(iv) if $X, Y \in \mathcal{X}$ have equal size and q, s are rationals then the coset $Z = qX + sY$ belongs to $\mathcal{X}$ and $F(Z) = qF(X) + sF(Y)$.

It clearly suffices to prove that, if F is a PRF and $\mathcal{X} = \operatorname{dom} F$ does not contain a coset K, then we can extend F so that the extended domain contains K.

Choose $F(K) \in K$ arbitrarily. Let $\mathcal{K}^+$ be the set of all cosets L such that *either* $L \subseteq K$ and $F(K) \in L$ or $K \subseteq L$. Let $\mathcal{K} = \mathcal{K}^+ \setminus \mathcal{X}$. Note that $\mathcal{K}^+$ is linearly ordered by $\subseteq$, while $\mathcal{K}$ is an initial segment of $\mathcal{K}^+$ by (ii), containing K. Now define $F(L) = F(K)$ for all $L \in \mathcal{K}$.

Let $\mathcal{X}'$ (the extended domain) be the set of all cosets $Z = qX + sL$, where cosets $X \in \mathcal{X}$ and $L \in \mathcal{K}$ have equal size. Put $F(Z) = qF(X) + sF(L)$.

We prove that the extended F satisfies (ii) and (iii). (That (i) and (iv) hold is clear. Recall that A, hence all convex subgroups of A, are divisible.)

(ii) Suppose that $Z = qX + sL$, where $X \in \mathcal{X}$ and $L \in \mathcal{K}$ have the same size while q, s are rationals. Assume that $Z \subsetneq Z'$, where Z' is a coset; prove that $Z' \in \mathcal{X}'$. Let X' and L' be cosets of the same size as Z', satisfying $X \subsetneq X'$ and $L \subsetneq L'$; clearly X', L' exist, are unique, belong to respectively $\mathcal{X}$ and $\mathcal{K}^+$ (by (ii) for $\mathcal{X}$), and $Z' = qX' + sL'$. If now $L' \notin \mathcal{K}$ then $L' \in \mathcal{X}$ and $Z \in \mathcal{X}$ by (ii) for $\mathcal{X}$. Otherwise $Z \in \mathcal{X}'$ by definition.

(iii) Let again $Z = qX + sL$, where X and L are as above, while $Z' \subsetneq Z$ is a coset and $F(Z) \in Z'$. Prove that $Z' \in \mathcal{X}'$ and $F(Z') = F(Z)$. By definition $F(Z) = qF(X) + sF(L)$. Let X' and L' be the cosets of the same size as Z', containing respectively $F(X)$ and $F(L)$, hence, satisfying $X' \subsetneq X$, $L' \subsetneq L$, $X' \in \mathcal{X}'$, $L' \in \mathcal{K}$, $F(X') = F(X)$, and, by definition,

$F(L') = F(L)$. Moreover, in this case clearly $F(Z') = qF(X') + sF(L')$, so that $Z' \in \mathcal{X}'$ and $F(Z') = F(Z)$. $\square$

Using the lemma, let us fix a representative $r(X) \in X$ for any coset X so that (a) and (b) are satisfied. Then, given a convex subgroup H of A, the *H-coordinate* $c_H(x) = x - r(x + H)$ belongs to H for any $x \in A$. Note that $c_H(x) = 0$ and $r(x + H) = x$ for all x in the particular case $H = \{0\}$.

Recall that a *galaxy* is a convex subgroup of the form $\operatorname{Gal} x = \bigcup_{n\in\mathbb{N}}[-nx, nx]$. The set $\mathcal{G}$ of all galaxies $G \subseteq A,\ G \neq \{0\}$, is at most countable by the CCC assumption. (However there can be continuum-many convex subgroups which are not galaxies: all of them are increasing countable unions of galaxies.)

For any galaxy $G \subseteq A$, there is a largest convex subgroup of A strictly smaller than G: it will be denoted by G_- (possibly $G_- = \{0\}$), so that $G_- \subsetneq G$.

Lemma 16. *If $G \in \mathcal{G}$ is not $\subseteq$-least in $\mathcal{G}$ then the quotient G/G_- is GO-isomorphic to a countable divisible subgroup of $\mathbb{R}$. If G is the $\subseteq$-least in $\mathcal{G}$ then $G/G_- = G$ is Borel GO-isomorphic to a Borel divisible subgroup of $\mathbb{R}$.*

Proof. The first part is clear as G/G_- is a countable Archimedean group. Consider the second part. Now, $G_- = \{0\}$, hence $G/G_- = G$ is an Archimedean BAO group. Let us prove that G is Borel GO-isomorphic to a Borel subgroup of the reals.

Fix $a \in G,\ a > 0$ in G. For any $x \in G$, let $Q_x = \{q \in \mathbb{Q}\colon\ qa < x\}$. Then Q_x is a proper (as G is Archimedean) initial segment in $\mathbb{Q}$. Put $F(x) = \sup Q_x$.

Then $F : G \to \mathbb{R}$ is a Borel map. Moreover, as G is Archimedean, F is 1–1, hence the image $\operatorname{ran} F$ is a Borel subset of $\mathbb{R}$. Finally it is a routine exercise to check that F is a GO-isomorphism. $\square$

Order $\mathcal{G}$ by inverse inclusion, so that $G \prec G'$ iff $G' \subsetneq G$.

Consider $\Pi = \prod_{G\in\mathcal{G}}(G/G_-)$, a BA product group with componentwise addition. Thus elements of Π are functions w defined on $\mathcal{G}$ and satisfying $w(G) \in G/G_-$ for all $G \in \mathcal{G}$. For any $w \in \Pi$, let $|w| = \{G \in \mathcal{G}\colon\ w(G) \neq 0\}$. We shall be especially interested in the subgroup $\Pi^{\mathrm{WO}} = \{w \in \Pi\colon\ |w| \text{ is well-ordered by } \prec\}$ of Π. Note that, unlike Π, Π^{WO} is an ordered (lexicographically) coanalytic but, generally speaking, non-Borel subgroup of Π.

For any $x \in A$, define $w_x \in \Pi$ as follows: $w_x(G) = c_G(x) + G_-$ for any galaxy $G \in \mathcal{G}$. Thus $w_x(G) \in G/G_-$ for any G, so that $w_x \in \Pi$.

Lemma 17. *The map $x \mapsto w_x$ is a Borel GO-isomorphism of A onto a local-product subgroup of Π^{WO}.*

Proof. It follows from (a) that $c_G(x) + c_G(y) = c_G(x + y)$ for any galaxy G. Therefore $w_x(G) + w_y(G) = w_{x+y}(G)$ for any $G \in \mathcal{G}$, so that $w_x + w_y = w_{x+y}$.

We prove that $x \mapsto w_x$ is 1–1. Let $x \neq y \in G$. Consider the galaxy $G = \mathrm{Gal}(x - y)$. Then $x - y \in G \setminus G_-$, so that clearly $c_G(x) - c_G(y) = x - y \notin G_-$, hence $w_x(G) \neq w_y(G)$. The proof that $x \mapsto w_x$ is order-preserving is similar.

We prove that $w_x \in \Pi^{\mathrm{WO}}$ for any x. Suppose on the contrary that there is a sequence $G_0 \subsetneq G_1 \subsetneq G_2 \subsetneq \cdots$ of galaxies $G_k \in \mathcal{G}$ such that $c_{G_k}(x) \notin G_k^-$—hence $c_{G_{k+1}} \notin G_k$, for all k. Then $G = \bigcup_k G_k$ is a convex group. By definition $c_G(x) = x - r(x + G) \in G$, thus $\in G_k$ for some k. It follows that $r(x + G) \in x + G_k$, hence, $\in x + G_{k+1}$, so that $r(x + G) = r(x + G_{k+1})$ by (b). Now $c_{G_{k+1}}(x) = x - r(x + G_{k+1}) = x - r(x + G) = c_G(x) \in G_k$, which is a contradiction.

We prove that the map is Borel. It suffices to check that $x \mapsto w_x(G)$ is a Borel map for any galaxy $G \neq \{0\}$. By the CCC assumption, A/G is countable, hence, the map $x \mapsto c_G(x)$ is Borel. If now $G_- = \{0\}$ then $w_x(G) = c_G(x)$. If $G_- \neq \{0\}$ then the quotient G/G_- is countable, so that the map $w_x(G) = c_G(x) + G_-$ takes only countably many values and is easily seen to be Borel.

Finally let us show that the range $W = \{w_x\colon\ x \in A\}$ is a local-product group. By definition it suffices, given $G \in \mathcal{G}$ and $X \in G/G_-$, to find $x \in A$ such that $w_x(G) = X$ while $w_x(H) = H_-$ for any galaxy $H \neq G$. Let $x = r(X)$. Then $x + G = G$, so easily $r(x + G) = 0$ by (a). It follows that $c_G(x) = x - r(x + G) = x = r(X)$ and $w_x(G) = x + G_- = X$. If $H \subseteq G_-$ is a galaxy then $r(x + H) = r(X)$ by (b), therefore $c_H(x) = 0$ and $w_x(H) = H_-$, as required. If a galaxy H satisfies $G \subsetneq H$, then $x \in H_-$ and easily $w_x(H) = H_-$. $\square$

Now, to prove Theorem 3, we have to verify that the group $W = \{w_x\colon\ x \in A\} \subseteq \Pi^{\mathrm{WO}}$ satisfies either (i) or (ii) of Theorem 3.

Case 1. There is no $\prec$-maximal, hence $\subseteq$-minimal, galaxy in $\mathcal{G}$. This leads us to (i). Indeed, fix $G \in \mathcal{G}$ and define, for any $w \in \Pi$, the restriction $w{\restriction}_{\prec G} = w \restriction \{G' \in \mathcal{G}\colon\ G' \prec G\}$. Then $W{\restriction}_{\prec G} = \{w{\restriction}_{\prec G}\colon\ w \in W\}$ cannot be uncountable because W clearly contains a set of $W{\restriction}_{\prec G}$-many disjoint open intervals (since W is local-product, see above). It remains to note that, in this case, every quotient G/G_- (where $G \in \mathcal{G}$) is a countable divisible subgroup of $\mathbb{R}$, by Lemma 16. Take as C the group closure of their union in $\mathbb{R}$.

Case 2. H is a $\prec$-maximal, hence $\subseteq$-minimal, galaxy in $\mathcal{G}$. Then $H/H_- = H$ is a Borel divisible subgroup of $\mathbb{R}$ by Lemma 16. Assume that H is uncountable. (If it is countable we get (i) as in case 1.) Then, identifying any $w \in W$ with the pair $\langle w{\restriction}_{\prec H}, w(H)\rangle$, we get a Borel GO-isomorphism between W and $W' = (W{\restriction}_{\prec H}) \times H$, which easily leads to (ii) of Theorem 3. $\square$

Clearly the possibility of characterization modulo *Borel* isomorphism follows from the CCC assumption. The argument, generally speaking, does not work in the non-CCC case. More exactly, the only part affected in the reasoning is that the map $x \mapsto w_x$ is Borel. We should prove the following: if A is a BAO divisible group and $H \subseteq A$ a convex Borel subgroup then there is a *Borel* choice of a representative $r(X) \in X$ for any $X \in A/H$, satisfying (a). Hjorth's counterexample in Section 6 shows that this is not always possible. At the moment, only the case of a countable H and A/H isomorphic to $\mathbb{R}$ admits a positive solution (Theorem 2).

As for non-Borel isomorphisms, our arguments easily prove that

(i) *Every Abelian ordered divisible group A is* GO*-isomorphic to a local-product group $W \subseteq \mathbb{R}^{\xi:\mathrm{WO}}$, for a linear order ξ of cardinality* $\operatorname{card}\xi \leqslant \operatorname{card} A$.

(ii) *In addition, if A is Borel then ξ can be chosen among orders 2^{α}, $\alpha < \omega_1$.* (By a theorem in [2], any Borel linear order is Borel order isomorphic to a Borel subset of 2^{α}, viewed as a lexicographical order, for some $\alpha < \omega_1$.)

Final remarks. The methods developed for the proof of Theorem 2 have been used in [4,5] to prove some other results related to the additive group of the reals, in particular:

(1) Suppose that G is a countable subgroup of the additive group of the reals, and a Baire measurable map $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) - f(x) - f(y) \in G$ for all x, y. Then there is a real c such that $f(x) - cx \in G$ for all x.

(2) Suppose that $\oplus$ is a Borel Abelian group operation on $\mathbb{R}$, such that the difference $(x \oplus y) - (x + y)$ takes only countably many values. Then $\langle \mathbb{R}; \oplus \rangle$ is Borel isomorphic to $\langle \mathbb{R}; + \rangle$.

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